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ON CLOSED SUBSETS OF ROOT SYSTEMS

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ABSTRACT. Let R be a root system (in the sense of Bourbaki) in a finite dimensional real inner product space V. A subset $P \subset R$ is *closed* if $\alpha, \beta \in P$ and $\alpha + \beta \in R$ imply that $\alpha + \beta \in P$. In this paper we shall classify, up to conjugacy by the Weyl group W of R, all closed sets $P \subset R$ such that $R \setminus P$ is also closed. We also show that if $\theta: R \longrightarrow R'$ is a bijection between two root systems such that both θ and θ^{-1} preserve closed sets, and if R has at most one irreducible component of type A_1, then θ is an isomorphism of root systems.

1. **Partitions of root systems into two closed sets.** Closed subsets of root systems play an important role in the study of subalgebras of finite dimensional semisimple Lie algebras and in the theory of reductive algebraic groups. The problem of decomposing a root system into a union of two closed subsets has been studied by Malyshev [4] in connection with the classification problem for complex homogeneous spaces. We would like to point out that the parabolic subsets of the (infinite) root systems of the affine Lie algebras have been classified by Futorny [3]. It would be interesting to extend our results to these infinite root systems.

The intersection of closed sets is closed, and so given any $P \subset R$, there exists the smallest closed set containing P. This closed set is called the *closure* of P, and it will be denoted by $[P]$. A set P is said to be *invertible* if both P and $R \setminus P$ are closed. This definition and the notation $[P]$ are due to Malyshev [4]. We shall denote by I the set of all invertible subsets of R. A *parabolic* set is a closed set P such that $P \cup (-P) = R$. A *horocyclic* set is the complement of a parabolic set. It is easy to see that horocyclic sets are also closed, and so parabolic and horocyclic sets are invertible. We shall see soon that, in general, the converse is not valid (see Theorem 4 below).

Let Π be a base of R, and R^+ the corresponding set of positive roots. For each subset $\Delta \subset \Pi$ we shall denote by R_Δ the root system consisting of all $\alpha \in R$ which are linear combinations of Δ. It is well known that Δ is a base of R_Δ. We write R^+_Δ for the set of positive roots of R_Δ, with respect to Δ. Note that $R^+_\Delta = R^+ \cap R_\Delta$. For $\Delta \subset \Pi$ we set $P_\Delta := R^+ \cup R_\Delta$. It is well known that the P_Δ's are representatives of W-orbits of parabolic subsets of R (see [1], Chapter 6, §1, Proposition 21).

Let $P \subset R$ be closed. Then $P_s := P \cap (-P)$ is a root system and we say that P_s is the *Levi component* of P. We shall refer to $P_u := P \setminus P_s$ as the *radical* of P. If $P \subset R$ is

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closed then a subset Q of P is called an *ideal* of P if

$$\alpha \in P,\ \beta \in Q, \quad \alpha+\beta \in P \Rightarrow \alpha+\beta \in Q.$$

The following lemma is well known (and easy to prove).

LEMMA 1. *If $P \subset R$ is closed then P_u is an ideal of P.*

The inner product of $\alpha, \beta \in V$ will be written as $(\alpha \mid \beta)$. Two linearly independent roots $\alpha, \beta \in R$ are said to be *strongly orthogonal* if $\alpha+\beta$ and $\alpha-\beta$ are not roots. In that case α and β are orthogonal, *i.e.* $(\alpha \mid \beta) = 0$.

If $P \in I$ and $Q = R \setminus P$ then define $\bar{P} := P \cup Q_s$. (This agrees with the corresponding definition in [4].) Note that $Q_u = -P_u$. It is clear that if $w \in W$ and $P \in I$ then $w(P) \in I$ and

$$w(P_s) = w(P)_s, \quad w(P_u) = w(P)_u, \quad w(\bar{P}) = \overline{w(P)}.$$

LEMMA 2. *If $P \in I$ and $Q = R \setminus P$ then*
(a) P_s and Q_s are strongly orthogonal;
(b) $\bar{P}$ and $\bar{Q}$ are parabolic.

PROOF. (a) Let $\alpha \in P_s$ and $\beta \in Q_s$. Assume that $\alpha+\beta \in R$, say $\alpha+\beta \in P$. Now $-\alpha \in P$ and so $\beta = (\alpha+\beta)+(-\alpha) \in P$, a contradiction. Thus $\alpha+\beta \notin R$. Similarly one shows that $\alpha-\beta \notin R$.

(b) We claim that $\bar{P} = P \cup Q_s$, is closed. In order to prove this claim it suffices to show that if $\alpha \in P$, $\beta \in Q_s$, and $\alpha+\beta \in R$ then $\alpha+\beta \in P$. But since $-\beta \in Q$, if $\alpha+\beta \in Q$ then $\alpha \in Q$, a contradiction. So $\alpha+\beta \in P$.

Since $Q_u = -P_u$, it is now clear that $\bar{P}$ is a parabolic set. The assertion for $\bar{Q}$ follows by applying the above argument to Q. ∎

If $P \in I$ and $Q = R \setminus P$ then it is clear that

$$(\bar{P})_s = P_s \cup Q_s \quad \text{and} \quad (\bar{P})_u = P_u.$$

We shall write $\bar{P}_s$ instead of $(\bar{P})_s$, and $\bar{P}_u$ instead of $(\bar{P})_u$.

The proper parabolic (resp. non-empty horocyclic) subsets of R can be characterized as the intersections of R with closed (resp. open) half-spaces of V (see [5], Corollary 1.1.2.11). In the next theorem we give a similar property of arbitrary invertible sets. In the case of reduced root systems, part (c) of this result is essentially contained in the paper [4] of Malyshev. If $\alpha \in R$ then $\alpha^\vee$ denotes the corresponding co-root in the dual space V^* of V, and s_α denotes the corresponding reflection of V.

THEOREM 3. *Let $P \in I$, $Q = R \setminus P$, and $\xi = \sum_{\alpha \in P} \alpha$. Then*
(a) $\bar{P}_s \perp \xi$;
(b) $P_u = \{\alpha \in R \mid (\alpha \mid \xi) > 0\}$;
(c) if V_0 is the subspace spanned by $\bar{P}_s$ then $R \cap V_0 = \bar{P}_s$.

PROOF. (a) Let $\alpha \in \bar{P}_s$. If $\beta \in P_u$ then $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha \in R$. Since $\bar{P}$ is closed and $\bar{P}_u = P_u$, Lemma 1 implies that $s_\alpha(\beta) \in P_u$. Thus $s_\alpha(P_u) = P_u$. As $\xi = \sum_{\beta \in P_u} \beta$ it follows that $s_\alpha(\xi) = \xi$, *i.e.* $(\alpha \mid \xi) = 0$.

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(b) Let $\alpha \in P_u$. Since $\bar{P}$ is a parabolic set, we can choose a base Π of R such that the corresponding set of positive roots, R^+, contains $\bar{P}_u$ and $\Pi \cap \bar{P}_s$ is a base of $\bar{P}_s$ (see [1], Chapter 6, No. 1.7). Hence $\bar{P}_s^+ := R^+ \cap \bar{P}_s$ is the corresponding set of positive roots of $\bar{P}_s$ and we have $R^+ = \bar{P}_s^+ \cup P_u$. Consequently $\xi = \delta - \gamma$ where $\delta = \sum_{\beta \in R^+} \beta$, $\gamma = \sum_{\beta \in \bar{P}_s^+} \beta$. Let W_0 be the Weyl group of $\bar{P}_s$, considered as a subgroup of the Weyl group W of R. We can choose $w_0 \in W_0$ such that $\big(\alpha \,|\, w_0(\gamma)\big) \leq 0$. Then $w_0(\xi) = \xi$ by (a) and so

$$\begin{aligned}(\alpha \,|\, \xi) &= \big(\alpha \,|\, w_0(\xi)\big)\\ &= \big(\alpha \,|\, w_0(\delta)\big) - \big(\alpha \,|\, w_0(\gamma)\big)\\ &\geq \big(w_0^{-1}(\alpha) \,|\, \delta\big).\end{aligned}$$

Since $P_u = \bar{P}_u$ is an ideal of $\bar{P}$ and $\alpha \in P_u$, we have $w_0^{-1}(\alpha) \in P_u \subset R^+$ and so $\big(w_0^{-1}(\alpha) \,|\, \delta\big) > 0$.

Hence we have shown that $\alpha \in P_u$ implies that $(\alpha \,|\, \xi) > 0$. If $\alpha \in \bar{P}_s$ then $(\alpha \,|\, \xi) = 0$ by (a). Finally if $\alpha \in Q_u = -P_u$ then $(\alpha \,|\, \xi) < 0$.

(c) If $H = \{x \in V \mid (x \,|\, \xi) = 0\}$, by (a) and (b), we have $\bar{P}_s \subset V_0 \subset H$ and $H \cap R = \bar{P}_s$. Consequently $V_0 \cap R = \bar{P}_s$. ∎

Let $\mathcal{P}$ be the set of all ordered pairs (Δ, Δ') where $\Delta' \subset \Delta \subset \Pi$ and Δ' is orthogonal to $\Delta \setminus \Delta'$. To such a pair we associate the set

$$P(\Delta, \Delta') := R_{\Delta'} \cup (R^+ \setminus R_\Delta^+).$$

It is clear that $P := P(\Delta, \Delta')$ belongs to I and that

$$P_s = R_{\Delta'}, \quad P_u = R^+ \setminus R_\Delta^+, \quad \text{and} \quad \bar{P} = P_\Delta.$$

If l is the rank of R then the number of W-orbits of parabolic subsets of R is 2^l. As mentioned earlier, the sets P_Δ, $\Delta \subset \Pi$, are representatives of these orbits. In the next theorem we exhibit a system of representatives of W-orbits in I which contains the above mentioned representatives of parabolic orbits.

THEOREM 4. *The sets $P(\Delta, \Delta')$ introduced above are representatives of W-orbits in I.*

PROOF. Let $P \in I$ and $Q = R \setminus P$. Since $\bar{P}$ is parabolic, we can choose $w \in W$ such that $w(\bar{P}) = P_\Delta$ for some $\Delta \subset \Pi$. Then the sets

$$\Delta' = \Pi \cap w(P)_s \quad \text{and} \quad \Delta'' = \Pi \cap w(Q)_s$$

are strongly orthogonal by Lemma 2(a) and form a partition of Δ. Furthermore we have

$$w(P) = P(\Delta, \Delta').$$

Now assume that (Δ, Δ') and (Γ, Γ') are in $\mathcal{P}$ and that

$$w\big(P(\Delta, \Delta')\big) = P(\Gamma, \Gamma')$$

for some $w \in W$. Since $\overline{P(\Delta, \Delta')} = P_\Delta$ and $\overline{P(\Gamma, \Gamma')} = P_\Gamma$, we conclude that $w(P_\Delta) = P_\Gamma$, and consequently $\Delta = \Gamma$ and $w(R_\Delta) = R_\Delta$. Note that $P(\Delta, \Delta')$ and $P(\Delta, \Gamma')$ have the same radical, namely

$$P(\Delta, \Delta')_u = P(\Delta, \Gamma')_u = R^+ \setminus R^+_\Delta. \tag{1}$$

Their respective Levi components are $P(\Delta, \Delta')_s = R_{\Delta'}$ and $P(\Delta, \Gamma')_s = R_{\Gamma'}$. Since $w\big(P(\Delta, \Delta')_s\big) = P(\Delta, \Gamma')_s$, we have $w(R_{\Delta'}) = R_{\Gamma'}$. As $w(R_\Delta) = R_\Delta$, there exists w_0 in the Weyl group W_0 of R_Δ such that $w_0 w(R^+_\Delta) = R^+_\Delta$. The set (1) is stable under w because $w\big(P(\Delta, \Delta')\big) = P(\Delta, \Gamma')$, and stable under w_0 by Lemma 1 applied to P_Δ. It follows that $w_0 w(R^+) = R^+$, because $R^+ = R^+_\Delta \cup P(\Delta, \Delta')_u$. Consequently $w_0 w = 1$, *i.e.* $w = w_0^{-1} \in W_0$. Since $\Delta' \perp (\Delta \setminus \Delta')$, we have $w(R_{\Delta'}) = R_{\Delta'}$ and so $R_{\Gamma'} = w(R_{\Delta'}) = R_{\Delta'}$. Hence $\Gamma' = \Delta'$. ∎

The set $P(\Delta, \Delta')$ is parabolic (resp. horocyclic) if and only if $\Delta' = \Delta$ (resp. $\Delta' = \emptyset$). We also have $P(\Delta, \Delta) = P_\Delta$.

2. **Counting Weyl group orbits.** Given any $\Delta \subset \Pi$ we can consider the full subdiagram of the Dynkin diagram of Π with vertex set Δ. By a *connected component* of Δ we shall mean a connected component of the above-mentioned sub-diagram.

The number of W-orbits, $N(R)$, in I is given by the following formula:

$$N(R) = \sum_{k \geq 0} 2^k N_k(R) \tag{2}$$

where $N_k(R)$ is the number of subsets $\Delta \subset \Pi$ having exactly k connected components. Indeed, given $\Delta \subset \Pi$ with k connected components, then there are 2^k choices for a subset $\Delta' \subset \Delta$ such that $\Delta' \perp (\Delta \setminus \Delta')$.

An easy computation shows that

$$N_k(A_n) = \binom{n+1}{2k}.$$

It is clear that $N_k(B_n)$, $N_k(C_n)$, and $N_k(BC_n)$ are given by the same formula. For D_n we have the following easily established recursive formula:

$$N_k(D_n) = 2N_k(D_{n-1}) - N_k(D_{n-2}) + N_{k-1}(D_{n-2})$$

which is valid for $n \geq 4$ and $k \geq 0$. Here we use the conventions that $N_{-1}(D_n) = 0$, $D_2 = 2A_1$ and $D_3 = A_3$. An initial part of the table of integers $N_k(D_n)$ is shown below:

$n \backslash k$	0	1	2	3	4	5
2	1	2	1	0	0	0
3	1	6	1	0	0	0
4	1	11	3	1	0	0
5	1	17	11	3	0	0
6	1	24	30	8	1	0
7	1	32	66	24	5	0

TABLE 1

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By using Table 1 and formula (2) one finds that

$$N(D_n) = 9, 17, 43, 103, 249, 601$$

for $n = 2, \ldots, 7$ respectively.

The complete table of integers $N_k(E_n)$ is given below:

$n \setminus k$	0	1	2	3	4	5
6	1	25	27	11	0	0
7	1	34	60	30	3	0
8	1	44	118	76	17	0

TABLE 2

By using Table 2 and formula (2) one finds that

$$N(E_n) = 247, 597, 1441$$

for $n = 6, 7, 8$ respectively. In the remaining exceptional cases we have: $N(G_2) = N(A_2) = 7$ and $N(F_4) = N(A_4) = 41$.

We can determine the stabilizer in W of an invertible set P. It suffices to consider the case where P is one of the representatives $P(\Delta, \Delta')$. Knowing this stabilizer is important in calculating the cardinality of I.

THEOREM 5. *Let $(\Delta, \Delta') \in \mathcal{P}$. Then the stabilizer of $P(\Delta, \Delta')$ in W is the Weyl group W_Δ of the root system R_Δ.*

PROOF. Denote by W_0 the stabilizer of $P = P(\Delta, \Delta')$ in W. If $\alpha \in \Delta$ then $\Delta' \perp (\Delta \setminus \Delta')$ implies that the reflection s_α preserves $P_s = R_{\Delta'}$. Since s_α also preserves $P_u = R^+ \setminus R^+_\Delta$, it follows that s_α preserves P, *i.e.* $s_\alpha \in W_0$. Consequently $W_\Delta \subset W_0$.

By Theorem 2.5.8 of [2], the elements $w \in W$ such that $w(\Delta) \subset R^+$ are left coset representatives of W_Δ in W. Consequently, in order to establish the equality $W_\Delta = W_0$ it suffices to show that if $w \in W_0$ and $w(\Delta) \subset R^+$ then $w = 1$.

Thus assume that $w \in W_0$ and $w(\Delta) \subset R^+$. As $R_\Delta = \bar{P}_s$, $w(P) = P$ implies that $w(R_\Delta) = R_\Delta$. It follows that $w(\Delta) \subset R_\Delta \cap R^+ = R^+_\Delta$. Since also $w(P_u) = P_u$ and $P_u = R^+ \setminus R^+_\Delta$ we conclude that $w(R^+) = R^+$. Hence $w = 1$. ■

Malyshev [4] has shown that if $P \subset R$ is closed, then the set $P^\circ := R \setminus [R \setminus P]$ is invertible. Thus (if P is closed) P° is the largest invertible subset of P. An interesting consequence is that if P and Q are closed subsets and $P \cup Q = R$ then there exists an invertible set $S \subset P$ such that $R \setminus S \subset Q$. Namely, one can choose $S = P^\circ$.

It appears that the proof of the lemma on p. 420 of [4] is incomplete. Namely, it is not clear how the inclusion $P^r + [R \setminus P] \subset [R \setminus P]$ follows from $P^r + (R \setminus P) \subset R \setminus P$. One of us (J.-Y. H.) has filled this gap, but the proof is not short and will not be given here.

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3. **Bijections preserving closed sets.** In this section we show that if $\theta\colon R \longrightarrow R'$ is a bijection between root systems such that both θ and θ^{-1} preserve closed sets, and R has at most one irreducible component of type A_1, then θ is an isomorphism of root systems. More precisely, if V and V' are the real vector spaces spanned by R and R', respectively, then θ extends to an isomorphism of vector spaces $V \longrightarrow V'$.

It will be convenient to refer to the pairs (R, V) and (R', V') as root systems.

Our proof will use the following lemma.

LEMMA 6. *Let (R, V) and (R', V') be root systems and $\theta\colon R \longrightarrow R'$ a bijection such that*

(a) $\theta(-\alpha) = -\theta(\alpha)$ for all $\alpha \in R$;

(b) $\alpha, \beta, \alpha + \beta \in R \Rightarrow \theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$;

(c) $\alpha', \beta', \alpha' + \beta' \in R' \Rightarrow \theta^{-1}(\alpha' + \beta') = \theta^{-1}(\alpha') + \theta^{-1}(\beta')$.

Then θ extends to an isomorphism $V \longrightarrow V'$ of vector spaces.

PROOF. This is a simple consequence of [1, Chapter 6, §1, Proposition 19, Corollary 2]. Indeed, this corollary and the hypotheses (a) and (b) imply that θ extends to a linear map $\varphi\colon V \longrightarrow V'$. Since (a) implies that $\theta^{-1}(-\alpha') = -\theta^{-1}(\alpha')$ for all $\alpha' \in R'$, the same argument is applicable to θ^{-1}. Hence θ^{-1} extends to a linear map $\varphi'\colon V' \longrightarrow V$. It is clear that φ and φ' are inverses of each other. ■

The example $R = A_1 + A_1 + A_1$, $R' = A_2$ shows that the hypothesis (c) of Lemma 6 cannot be omitted.

We can now prove the main result of this section.

THEOREM 7. *Let (R, V) and (R', V') be root systems and $\theta\colon R \longrightarrow R'$ a bijection such that both θ and θ^{-1} preserve closed subsets. Then*

(a) for $X \subset R$, $\theta([X]) = [\theta(X)]$;

(b) $\alpha, 2\alpha \in R \Rightarrow \theta(k\alpha) = k\theta(\alpha)$, $k = \pm 1, \pm 2$;

(c) if $W \subset V$ is a 2-dimensional subspace and $S = R \cap W$ is an irreducible root system of rank 2, then there is a linear map $W \longrightarrow V'$ which agrees with θ on S;

(d) if R has at most one irreducible component of type A_1, θ extends to an isomorphism $V \longrightarrow V'$.

PROOF. (a) Since $\theta([X])$ is closed and contains $\theta(X)$, we have

$$\theta([X]) \supset [\theta(X)].$$

By replacing θ with θ^{-1} and X with $\theta(X)$, we obtain

$$\theta^{-1}\big([\theta(X)]\big) \supset [X],$$

i.e.

$$[\theta(X)] \supset \theta([X]).$$

(b) As $[\{\alpha\}] = \{\alpha, 2\alpha\}$, (a) implies that $[\{\theta(\alpha)\}] = \{\theta(\alpha), \theta(2\alpha)\}$, and so $\theta(2\alpha) = 2\theta(\alpha)$. Similarly, $\theta(-2\alpha) = 2\theta(-\alpha)$. Since $[\{\alpha, -2\alpha\}] = \{\pm\alpha, \pm 2\alpha\}$, (a) implies that

$$[\{\theta(\alpha), 2\theta(-\alpha)\}] = \{\theta(\pm\alpha), 2\theta(\pm\alpha)\}.$$

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Consequently $\theta(\alpha)$ and $\theta(-\alpha)$ are linearly dependent, and $\theta(-\alpha) = -\theta(\alpha)$ follows.

(c) Let $\{\alpha,\beta\}$ be a base of S. The root $\gamma := s_\alpha(\beta)$ belongs to the set $[\{\alpha,\beta\}] = S^+$ of positive roots of S.

We claim that $\theta(\alpha)$ and $\theta(\beta)$ are linearly independent. Assume the contrary. By applying (b) to θ^{-1}, we infer that $2\theta(\alpha)$ and $\theta(\alpha)/2$ are not in R'. Consequently $\theta(\beta) = -\theta(\alpha)$ and the set $\{\theta(\alpha),\theta(\beta)\} = \{\pm\theta(\alpha)\}$ is closed. Now (a) implies that $\theta(S^+) = \{\pm\theta(\alpha)\}$, a contradiction. Thus our claim is proved.

As $\gamma \notin [\{\alpha\}]$, (a) implies that $\theta(\gamma) \notin [\{\theta(\alpha)\}]$. Similarly, $\theta(\gamma) \notin [\{\theta(\beta)\}]$. As $\theta(\gamma) \in [\{\theta(\alpha),\theta(\beta)\}]$, it follows that $\theta(\alpha)$, $\theta(\beta)$ and $\theta(\gamma)$ are pairwise linearly independent. Let W' be the 2-dimensional subspace of V' spanned by $\theta(\alpha)$ and $\theta(\beta)$. As $\theta(S^+) = [\{\theta(\alpha),\theta(\beta)\}] \subset W'$, the set $S' := R' \cap W'$ is an irreducible root system of rank 2. By replacing $\{\alpha,\beta\}$ with the base $s_\alpha(\{\alpha,\beta\}) = \{-\alpha,\gamma\}$, and by observing that $\beta \in [\{-\alpha,\gamma\}]$, we conclude that $\theta(-\alpha)$ lies in the plane spanned by $\theta(\beta)$ and $\theta(\gamma)$, and so $\theta(-\alpha) \in S'$.

Assume that $\theta(\alpha)$ and $\theta(-\alpha)$ are linearly independent. By applying (b) to θ^{-1}, we infer that $\theta(\alpha)/2 \notin S'$. Hence there exists a base $\{\alpha',\beta'\}$ of S' with $\alpha' = \theta(\alpha)$ and $\theta(-\alpha) \in [\{\alpha',\beta'\}]$. By the above claim applied to θ^{-1}, we see that $\alpha = \theta^{-1}(\alpha')$ and $\delta := \theta^{-1}(\beta')$ are linearly independent. By applying (a) to θ^{-1}, we obtain $-\alpha \in \theta^{-1}([\{\alpha',\beta'\}]) = [\{\alpha,\delta\}]$, a contradiction. Hence $\theta(\alpha)$ and $\theta(-\alpha)$ must be linearly dependent. Now (b) implies that $\theta(-\alpha) = -\theta(\alpha)$. Since $\{\alpha,\beta\}$ is an arbitrary base of S, we have $\theta(-\delta) = -\theta(\delta)$ for all $\delta \in S$. As $\theta(S^+) \subset S'$, it follows now that $\theta(S) \subset S'$. By applying this result to θ^{-1}, we conclude that $\theta^{-1}(S') \subset S$. Hence $\theta(S) = S'$.

Since the irreducible root systems of rank 2: A_2, B_2, BC_2 and G_2 have cardinalities 6, 8, 12 and 12, respectively, it follows from $|S| = |S'|$ and (b) that S and S' are isomorphic.

The assertion (c) can now be easily verified by considering each of the four types of irreducible root systems A_2, B_2, BC_2, G_2 of rank 2. For instance, let S (and S') be of type G_2. If $\alpha \in S$ is a short root, there are exactly five roots $\beta \neq \alpha$ such that $\{\alpha,\beta\}$ is closed. If α is a long root then there are seven such roots β. It follows that θ maps short roots to short roots. Now let $\{\alpha,\beta\}$ be a base of S with α short. Then $\alpha' = \theta(\alpha)$ is short and $\beta' = \theta(\beta)$ a long root and $\{\alpha',\beta'\}$ is a base of S'. The short roots in S^+ are $\alpha, \alpha+\beta$, and $2\alpha+\beta$. Since $2\alpha+\beta \in [\{\alpha,\alpha+\beta\}]$, we must have $\theta(\alpha+\beta) = \alpha'+\beta'$ and $\theta(2\alpha+\beta) = 2\alpha'+\beta'$. The long roots in S^+ are β, $3\alpha+\beta$, and $3\alpha+2\beta$. Since $3\alpha+2\beta \in [\{\beta,3\alpha+\beta\}]$, we must have $\theta(3\alpha+\beta) = 3\alpha'+\beta'$ and $\theta(3\alpha+2\beta) = 3\alpha'+2\beta'$. As $\theta(-\gamma) = -\theta(\gamma)$ for all $\gamma \in S$, $\theta|_S$ extends to an isomorphism $W \longrightarrow W'$.

(d) We need only show that the hypotheses of Lemma 6 are satisfied. We show first that $\theta(-\alpha) = -\theta(\alpha)$ for all $\alpha \in R$. This is true by (b) if $2\alpha \in R$ or $\alpha/2 \in R$. It is also true if α is contained in an irreducible subsystem $S \subset R$ of rank 2, by (c). This accounts for all roots of R except those which lie in irreducible components of type A_1. Since there is at most one such component, the claim is true for all $\alpha \in R$.

Next we claim that if $\alpha, \beta, \alpha+\beta \in R$ then $\theta(\alpha+\beta) = \theta(\alpha)+\theta(\beta)$. This follows from (b) if α and β are linearly dependent, and from (c) if they are independent.

Since (b) and (c) also hold for θ^{-1}, we also have $\theta^{-1}(\alpha' + \beta') = \theta^{-1}(\alpha') + \theta^{-1}(\beta')$ when $\alpha', \beta', \alpha' + \beta' \in R'$. ∎

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