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ON THE ABSOLUTE SUMMABILITY FACTOR OF FOURIER SERIES

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The purpose of this paper is to give a general theorem on the absolute Riesz summability factor of Fourier series which implies Matsumoto's Theorem [*Tôhoku Math. J.* 8 (1956), 114-124] and to deduce some results from the theorem.

1.

Let $\sum a_n$ be an infinite series and s_n its nth partial sum. Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as} \quad n \to \infty .$$

If the sequence

$$(1.1) \qquad t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad (n = 0, 1, 2, \ldots)$$

is of bounded variation, that is, $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is said to be summable $|R, P_n, 1|$.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier

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series of $f(t)$ is given by

(1.2) $$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t) \text{ , say,}$$

and

$$\int_{-\pi}^{\pi} f(t)dt = 0 \text{ .}$$

We use the following notations:

$$\phi_x(t) = \phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\} \text{ ;}$$

$$\Phi(t) = \int_0^t |\phi(u)|du \text{ ;}$$

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2 \sin t/2} \text{ ;}$$

$$L_0(t) = 1,\ L_1(t) = \log t,\ L_p(t) = L_1\left(L_{p-1}(t)\right) = \log \dots \log t \quad (p \text{ times}) \text{ ,}$$

$$L_p^{(\varepsilon)}(t) = L_1(t) \dots L_{p-1}(t)\left(L_p(t)\right)^{1+\varepsilon} \quad (\varepsilon \ge 0,\ p = 1, 2, \dots) \text{ ,}$$

where, if the right hand sides are not determined as positive numbers, we replace them by 1's .

Let $\{\lambda_n\}$ be a monotone decreasing sequence. We put $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Also we define a function $\lambda(t)$ continuous in the interval $(0, \infty)$ such that $\lambda(n) = \lambda_n$ for $n = 1, 2, \dots$ and $\lambda(t)$ is linear for every non-integral t . Similarly $p(t)$ is defined by the sequence $\{p_n\}$ and we put

$$P(t) = \int_0^t p(u)du \text{ .}$$

A denotes a positive absolute constant that is not always the same.

2.

Concerning the absolute Riesz summability factor of Fourier series, Matsumoto [6] proved the following theorem.

THEOREM A. *The* $|R, P_n, 1|$ *summability of the series* $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ *at* $t = x$ *is a local property where*

P_n	$\exp n^{\delta}$	$\exp(\log n)^{\delta}$	$\exp(\log\log n)^{\delta}$
λ_n	$\dfrac{1}{n^{\delta}(\log n)^{1+\varepsilon}}$	$\dfrac{1}{(\log(n+1))^{\delta+\varepsilon}}$	$\dfrac{1}{(\log\log(n+1))^{\delta+\varepsilon}}$
δ	$0 < \delta \le 1$	$0 < \delta$	$0 < \delta$
$\chi(1/t)$	$\dfrac{t}{\log 1/t}$	$\dfrac{t}{\log 1/t}$	$\dfrac{t}{\log 1/t \log\log 1/t}$

More precisely, if

$$\Phi(t) = \int_0^t |\phi(u)|du = O\{\chi(1/t)\} ,$$

then the series $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ *is summable* $|R, P_n, 1|$, *at* $t = x$.

In this paper we shall generalize Theorem A in the following form.

THEOREM. *Let* $\{\lambda_n\}$ *be a monotone decreasing sequence and* $\{p_n\}$ *be a positive sequence such that* $\{p_n/P_n\}$ *is decreasing.*

If

(2.1) $$\int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du < \infty$$

and

(2.2) $$\int_0^{\pi} \frac{\Phi(t)\lambda(1/t)}{t^2}\, dt < \infty ,$$

then the series $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ *is summable* $|R, P_n, 1|$, *at* $t = x$.

The conditions (2.1) and (2.2) have been obtained by referring to the theorems on the absolute Nörlund summability of Fourier series, which are due to Izumi and Izumi [2], [3], Kolhekar [4] and Leindler [5].

In our theorem we put $\lambda(t) = 1/t^{\delta}(\log t)^{1+\varepsilon}$, $\chi(t) = 1/t \log t$ and $P(t) = \exp t^{\delta}$ $(0 < \delta \leq 1)$. Then we have

$$\int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du \leq A \int_0^{\pi} \frac{\log \log 1/u}{u(\log 1/u)^{1+\varepsilon}}\, du < \infty$$

and

$$\int_0^{\pi} \frac{\Phi(t)\lambda(1/t)}{t^2}\, dt \leq A \int_0^{\pi} \frac{dt}{t^{1-\delta}(\log 1/t)^{2+\varepsilon}} < \infty .$$

The other cases are similarly showed. Thus we see that our theorem is a generalization of Theorem A.

3.

We need the following lemma for the proof of our theorem.

LEMMA. *Under the same assumptions as those of the theorem, we have*

(i) $$\int_0^{\pi} \frac{\Phi(t)p(1/t)\lambda(1/t)}{t^3 P(1/t)}\, dt \leq A \int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du < \infty$$

(ii) $$\int_0^{\pi} \frac{|\phi(t)|}{t}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du$$
$$\leq A \int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du < \infty , \text{ and}$$

(iii) $$\int_0^{\pi} \frac{|\phi(t)|\lambda(1/t)}{t}\, dt \leq A \int_0^{\pi} \frac{\Phi(t)\lambda(1/t)}{t^2}\, dt < \infty .$$

Proof. Let $\varepsilon > 0$. Then we obtain by an integration by parts

$$\int_{\varepsilon}^{\pi} \frac{\Phi(t)p(1/t)\lambda(1/t)}{t^3 P(1/t)}\, dt = \frac{\Phi(\pi)}{\pi} \int_0^{\pi} \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du - \frac{\Phi(\varepsilon)}{\varepsilon} \int_0^{\varepsilon} \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du$$
$$- \int_{\varepsilon}^{\pi} \frac{|\phi(t)|}{t}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du + \int_{\varepsilon}^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du .$$

Thus we have

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$$\int_\varepsilon^\pi \frac{\Phi(t)p(1/t)\lambda(1/t)}{t^3P(1/t)}\,dt + \int_\varepsilon^\pi \frac{|\phi(t)|}{t}\,dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)}\,du$$

$$\le \frac{\Phi(\pi)}{\pi}\int_0^\pi \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)}\,du + \int_0^\pi \frac{\Phi(t)}{t^2}\,dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)}\,du .$$

Therefore we can obtain cases *(i)* and *(ii)* as $\varepsilon \to 0$.

Noting that $\frac{d\lambda(t)}{dt} < 0$, case *(iii)* is similarly proved.

4.

Proof of the theorem. Let $\bar{t}_n$ denote the Riesz means of the series $\sum \lambda_n a_n$. Then we have, by (1.1),

(4.1) $\bar{t}_n - \bar{t}_{n-1}$

$$= \frac{p_n}{P_nP_{n-1}} \sum_{k=1}^{n-1} P_{k-1}\lambda_k a_k$$

$$= \frac{p_nP_{n-2}\lambda_{n-1}s_{n-1}}{P_nP_{n-1}} + \frac{p_n}{P_nP_{n-1}} \sum_{k=1}^{n-2} \left(P_{k-1}\lambda_k - P_k\lambda_{k+1}\right)s_k - \frac{p_nP_0\lambda_1 s_0}{P_nP_{n-1}} .$$

Let $\bar{t}_n(x)$ denote the Riesz means of the series $\sum \lambda_n A_n(x)$. Then we obtain, by (4.1),

$$\frac{\pi}{2}\sum_{n=1}^{\infty} |\bar{t}_n(x) - \bar{t}_{n-1}(x)| \le \sum_{n=1}^{\infty} \frac{p_nP_{n-2}\lambda_{n-1}}{P_nP_{n-1}} \int_0^\pi |\phi(t)||D_{n-1}(t)|dt$$

$$+ \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \sum_{k=1}^{n-2} |P_{k-1}\lambda_k - P_k\lambda_{k+1}| \int_0^\pi |\phi(t)||D_k(t)|dt$$

$$= I + J ,$$

say, where $s_k(x) = \frac{2}{\pi}\int_0^\pi \phi(t)D_k(t)dt$.

Since $D_n(t) = O(n)$ or $O(1/t)$, we have

$$I \le A \sum_{n=1}^{\infty} \frac{np_n\lambda_{n-1}}{P_n} \int_0^{\pi/n} |\phi(t)|dt + A \sum_{n=1}^{\infty} \frac{p_n\lambda_{n-1}}{P_n} \int_{\pi/n}^{\pi} \frac{|\phi(t)|}{t}\,dt$$

$$= I_1 + I_2 ,$$

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say. Since $\{\lambda_n\}$ and $\{p_n/P_n\}$ are decreasing, we have, by the lemma,

$$\begin{aligned}
I_1 &= A \sum_{n=1}^{\infty} \frac{np_n\lambda_{n-1}}{P_n} \sum_{k=n}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)|dt \\
&\le A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)|dt \sum_{n=1}^{k} \frac{np_n\lambda_n}{P_n} \\
&\le A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} |\phi(t)|dt \int_t^{\pi} \frac{p(1/u)\lambda(1/u)}{u^3P(1/u)} du \\
&= A \int_0^{\pi} |\phi(t)|dt \int_t^{\pi} \frac{p(1/u)\lambda(1/u)}{u^3P(1/u)} du \\
&= A \int_0^{\pi} \frac{p(1/u)\lambda(1/u)}{u^3P(1/u)} \Phi(u)du \\
&\le A \int_0^{\pi} \frac{|\Phi(t)|}{t^2} dt \int_0^{t} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} du < \infty
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= A \sum_{n=1}^{\infty} \frac{p_n\lambda_{n-1}}{P_n} \sum_{k=1}^{n-1} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} dt \\
&\le A \sum_{k=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} dt \sum_{n=k+1}^{\infty} \frac{p_n\lambda_{n-1}}{P_n} \\
&\le A \sum_{n=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} dt \sum_{n=k+1}^{\infty} \frac{p(n/\pi)\lambda(n/\pi)}{P(n/\pi)} \quad \text{for } n > \pi/(\pi-1) \\
&\le A \sum_{n=1}^{\infty} \int_{\pi/k+1}^{\pi/k} \frac{|\phi(t)|}{t} dt \int_0^{t} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} du \\
&= A \int_0^{\pi} \frac{|\phi(t)|}{t} dt \int_0^{t} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} du \\
&\le A \int_0^{\pi} \frac{|\Phi(t)|}{t^2} dt \int_0^{t} \frac{p(1/u)\lambda(1/u)}{u^2P(1/u)} du < \infty .
\end{aligned}$$

Thus, by I_1 and I_2, we see that I is finite.

Since $P_{k-1}\lambda_k - P_k\lambda_{k+1} = -p_k\lambda_k + P_k\Delta\lambda_k$, we have

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$$J \le \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} p_k \lambda_k \int_0^{\pi} |\phi(t)||D_k(t)|dt + \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} P_k \Delta\lambda_k \int_0^{\pi} |\phi(t)||D_k(t)|dt$$

$$= J_1 + J_2 ,$$

say. The finiteness of J_1 is proved by the same estimation as I because we obtain

$$J_1 \le \sum_{k=1}^{\infty} \frac{p_k \lambda_k}{P_k} \int_0^{\pi} |\phi(t)||D_k(t)|dt .$$

Next, we have,

$$J_2 \le \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} P_k \Delta\lambda_k \int_0^{\pi/k} |\phi(t)||D_k(t)|dt + \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} P_k \Delta\lambda_k \int_{\pi/k}^{\pi} |\phi(t)||D_k(t)|dt$$

$$= J_{21} + J_{22} ,$$

say. Since $\{\lambda_k\}$ is decreasing, we have, by the lemma,

$$\begin{aligned}
J_{21} &\le A \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} k P_k \Delta\lambda_k \sum_{j=k}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \\
&= A \sum_{k=1}^{\infty} k P_k \Delta\lambda_k \sum_{n=k+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{j=k}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \\
&\le A \sum_{k=1}^{\infty} k \Delta\lambda_k \sum_{j=k}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \\
&= A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \sum_{k=1}^{j} k \Delta\lambda_k \\
&\le A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \sum_{k=1}^{j} \lambda_k \\
&\le A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} |\phi(t)|dt \int_t^{\pi} \frac{\lambda(1/u)}{u^2} du \\
&= A \int_0^{\pi} |\phi(t)|dt \int_t^{\pi} \frac{\lambda(1/u)}{u^2} du \\
&= A \int_0^{\pi} \frac{\lambda(1/u)}{u^2} \Phi(u)du < \infty
\end{aligned}$$

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and

$$J_{22} \le A \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-2} P_k \Delta\lambda_k \sum_{j=1}^{k-1} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|}{t}\, dt$$

$$= A \sum_{k=1}^{\infty} P_k \Delta\lambda_k \sum_{n=k+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{k-1} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|}{t}\, dt$$

$$\le A \sum_{k=1}^{\infty} \Delta\lambda_k \sum_{j=1}^{k-1} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|}{t}\, dt$$

$$= A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|}{t}\, dt \sum_{k=j+1}^{\infty} \Delta\lambda_k$$

$$= A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|}{t}\, \lambda_{j+1} dt$$

$$\le A \sum_{j=1}^{\infty} \int_{\pi/j+1}^{\pi/j} \frac{|\phi(t)|\lambda(1/t)}{t}\, dt$$

$$= A \int_0^{\pi} \frac{|\phi(t)|\lambda(1/t)}{t}\, dt$$

$$\le A \int_0^{\pi} \frac{\lambda(1/t)}{t^2}\, \Phi(t) dt < \infty .$$

Thus, by J_1 and J_2 we see that J is finite. Therefore, by the above estimations, our theorem is completely proved.

5.

In this section we consider some applications of our theorem.

COROLLARY. *Let $\{p_n\}$ be a positive sequence such that $\{p_n/P_n\}$ is decreasing. Suppose that $\chi(t)$ is a positive decreasing function satisfying the condition*

(5.1) $$p(t)^{-1}P(t)\chi(t) = O(1) \quad \text{for} \quad t > 0$$

and $\mu(t)$ is a positive decreasing function such that

(5.2) $$\sum_{n=1}^{\infty} \frac{\alpha_n \mu_n}{n} < \infty ,$$

where $\{\alpha_n\}$ is a sequence defined by

$$\alpha_n = \int_{1/n+1}^{\pi} \frac{\chi(1/t)}{t^2}\, dt .$$

If $\Phi(t) = O\{\chi(1/t)\}$ *, then*

(5.3) $$\sum_{n=1}^{\infty} \frac{P_n \mu_n}{n p_n} A_n(t)$$

is summable $|R, P_n, 1|$ *, at* $t = x$.

Proof. Putting $\lambda(t) = P(t)\mu(t)/tp(t)$ in the theorem, we have, by (5.2),

$$\int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^t \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du$$

$$= \int_0^{\pi} \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du \int_u^{\pi} \frac{\Phi(t)}{t^2}\, dt$$

$$\le A + \int_0^1 \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du \int_u^{\pi} \frac{\Phi(t)}{t^2}\, dt$$

$$\le A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{p(1/u)\lambda(1/u)}{u^2 P(1/u)}\, du \int_{1/n+1}^{\pi} \frac{\chi(1/t)}{t^2}\, dt$$

$$\le A + A \sum_{n=1}^{\infty} \alpha_n \int_{1/n+1}^{1/n} \frac{\mu(1/u)}{u}\, du$$

$$\le A + A \sum_{n=1}^{\infty} \frac{\alpha_n \mu_n}{n} < \infty .$$

Similarly we have, by (5.1),

$$\int_0^{\pi} \frac{\Phi(t)\lambda(1/t)}{t^2}\, dt \le A + \int_0^1 \frac{\Phi(t)\lambda(1/t)}{t^2}\, dt$$

$$\le A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{\chi(1/t)P(1/t)\mu(1/t)}{tp(1/t)}\, dt$$

$$\le A + A \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{\mu(1/t)}{t}\, dt$$

$$\le A + A \sum_{n=1}^{\infty} \frac{\mu_n}{n} < \infty$$

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by virtue of the facts that $\{\alpha_n\}$ is increasing and $\sum_{n=1}^{\infty} \alpha_n \mu_n / n$ converges. Hence we prove the corollary.

Here we shall make a list of the interesting examples of P_n, λ_n and $\chi(t)$ in the corollary. If we put $\lambda_n = P_n \mu_n / n p_n$ and $\mu_n = 1/L_l^{(\varepsilon)}(n)$, then we have the following list:

	P_n	λ_n	$\chi(1/t)$	δ
(i)	$\exp n^{\delta}$	$\dfrac{1}{n^{\delta} L_s^{(\varepsilon)}(n)}$	$\dfrac{t}{L_s^{(0)}(1/t)}$	$0 < \delta \le 1$
(ii)	n^{δ}	$\dfrac{1}{L_s^{(\varepsilon)}(n)}$	$\dfrac{t}{L_s^{(0)}(1/t)}$	$0 < \delta$
(iii)	$\exp L_s(n)^{\delta}$	$\dfrac{L_s^{(0)}(n)}{L_s(n)^{\delta} L_{s+p}^{(\varepsilon)}(n)}$	$\dfrac{t}{L_{s+p}^{(0)}(1/t)}$	$0 < \delta$
(iv)	$L_s(n)^{\delta}$	$\dfrac{L_s^{(0)}(n)}{L_{s+p}^{(\varepsilon)}(n)}$	$\dfrac{t}{L_{s+p}^{(0)}(1/t)}$	$0 < \delta$
(v)	$\exp \dfrac{n}{L_s(n)^{\delta}}$	$\dfrac{L_s(n)^{\delta}}{n L_{s+p}^{(\varepsilon)}(n)}$	$\dfrac{t}{L_{s+p}^{(0)}(1/t)}$	$0 < \delta$
(vi)	$\dfrac{n}{L_s(n)^{\delta}}$	$\dfrac{1}{L_{s+p}^{(\varepsilon)}(n)}$	$\dfrac{t}{L_{s+p}^{(0)}(1/t)}$	$0 < \delta$

where s is a positive integer and p is a non-negative integer.

By Mohanty's lemma [7], we see that cases (i), (iii) and (v) are also deduced from cases (ii), (iv) and (vi), respectively. The positive number ε in $L_s^{(\varepsilon)}(n)$ or $L_{s+p}^{(\varepsilon)}(n)$ is indispensable from the theorems due to Matsumoto [6] and Dikshit [1].

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