SUMMATION OF A SERIES OF PRODUCTS OF E-FUNCTIONS

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(Received 9th May, 1961)

1. Introductory. In previous papers [**1, 2, 3**] the sums of a number of series of products of E-functions have been found. For the definitions and properties of the E-functions the reader is referred to [**4**, pp. 348–358]. In § 3 a further series of this type is given. The proof is based on an integral of an E-function with respect to its parameters, to be established in § 2. Similar integrals were given in [**5**] and [**6**].

The following formulae will be made use of in the proofs.

If $p \leqq q$, $z \neq 0$, [**4**, p. 352]

$$E(p\,;\alpha_r:q\,;\rho_s:z)=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}\,F\begin{pmatrix}p\,;\alpha_r:-1/z\\ q\,;\rho_s\end{pmatrix}. \tag{1}$$

If $R(\rho_{q+1})>R(\alpha_{p+1})>0$, [**4**, p. 395]

$$\int_0^1 \lambda^{\alpha_{p+1}-1}(1-\lambda)^{\rho_{q+1}-\alpha_{p+1}-1}E(p\,;\alpha_r:q\,;\rho_s:z/\lambda)\,d\lambda$$
$$=\Gamma(\rho_{q+1}-\alpha_{p+1})E(p+1\,;\alpha_r:q+1\,;\rho_s:z). \tag{2}$$

If $R(\alpha_{p+1})>0$, [**4**, p. 394]

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1}E(p\,;\alpha_r:q\,;\rho_s:z/\lambda)\,d\lambda=E(p+1\,;\alpha_r:q\,;\rho_s:z). \tag{3}$$

If $|z|<1$, [**7**, p. 100]

$$F\begin{pmatrix}\alpha,\beta: & z\\ \alpha+\beta+\frac12\end{pmatrix}F\begin{pmatrix}\frac12-\alpha,\frac12-\beta:z\\ \frac32-\alpha-\beta\end{pmatrix}=F\begin{pmatrix}\alpha-\beta+\frac12,\beta-\alpha+\frac12,\frac12:z\\ \alpha+\beta+\frac12,\frac32-\alpha-\beta\end{pmatrix}. \tag{4}$$

If $|\operatorname{amp} z|<\pi$, [**4**, p. 374]

$$E(p\,;\alpha_r:q\,;\rho_s:z)=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)}z^\zeta\,d\zeta, \tag{5}$$

where the contour of integration is taken up the η-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1,\alpha_2,\ldots,\alpha_p$ to the right of the contour. Zero and negative integral values of the parameters are excluded. If $p<q+1$ the contour is bent to the left at both ends. When $p>q+1$ the formula is valid for $|\operatorname{amp} z|<\frac12(p-q+1)\pi$.

By applying (1) to (4) it can be deduced that

$$E\begin{pmatrix}\alpha,\beta: & z\\ \alpha+\beta+\frac12\end{pmatrix}E\begin{pmatrix}\frac12-\alpha,\frac12-\beta:z\\ \frac32-\alpha-\beta\end{pmatrix}=\cos(\alpha-\beta)\pi$$
$$\times\pi^{-3/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\tfrac12-\alpha)\Gamma(\tfrac12-\beta)E\begin{pmatrix}\alpha-\beta+\frac12,\beta-\alpha+\frac12,\frac12:z\\ \alpha+\beta+\frac12,\frac32-\alpha-\beta\end{pmatrix}. \tag{6}$$

Finally, [4, p. 351]

$$E(\alpha, \beta :: z) = \Gamma(\alpha)\Gamma(\beta) z^{\frac{1}{2}(\alpha+\beta-1)} e^{\frac{1}{2}z} W_{\frac{1}{2}(1-\alpha-\beta),\ \frac{1}{2}(\beta-\alpha)}(z). \tag{7}$$

2. Integration of an *E*-function with respect to its parameters. The formula to be proved is

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}-\zeta)} z^{\zeta} E\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta, \alpha_1-\zeta, \dots, \alpha_p-\zeta : z\\ \frac{3}{2}-\alpha-\beta, \rho_1-\zeta, \dots, \rho_q-\zeta\end{matrix}\right) d\zeta$$

$$= \pi^{-3/2}\cos(\alpha-\beta)\pi\ \Gamma(\alpha)\Gamma(\beta)\Gamma(\tfrac{1}{2}-\alpha)\Gamma(\tfrac{1}{2}-\beta) E\left(\begin{matrix}\frac{1}{2}, \alpha-\beta+\frac{1}{2}, \beta-\alpha+\frac{1}{2}, \alpha_1, \dots, \alpha_p : z\\ \alpha+\beta+\frac{1}{2}, \frac{3}{2}-\alpha-\beta, \rho_1, \dots, \rho_q\end{matrix}\right), \tag{8}$$

where $p \geqq q$, $|\operatorname{amp} z| < \frac{1}{2}(p-q+2)\pi$, $R(\rho_n-\alpha_n) > 0$ $(n = 1, 2, \dots, q)$, $R(\alpha_n) > 0$ $(n = 1, 2, \dots, p)$, α and β being such that the *E*-functions exist. The contour of integration is the same as in (5) with loops, if necessary, to ensure that α and β are to the right of the contour.

From (2) and (3) it follows that the left-hand side of (8) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}-\zeta)} z^{\zeta}\left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-\zeta-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n$$

$$\times\prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-\zeta-1}d\lambda_n\, E\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta : z/\lambda_1\lambda_2\dots\lambda_p\\ \frac{3}{2}-\alpha-\beta\end{matrix}\right)d\zeta.$$

Here change the order of the factors and get

$$\left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n$$

$$\times\prod_{n=q+1}^{p-1}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n\int_0^{\infty} e^{-\lambda_p}\lambda_p^{\alpha_p-1}E\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta : z/(\lambda_1\dots\lambda_p)\\ \frac{3}{2}-\alpha-\beta\end{matrix}\right)d\lambda_p$$

$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}-\zeta)}\left(\frac{z}{\lambda_1\dots\lambda_p}\right)^{\zeta}d\zeta.$$

On substituting from (5) for the last integral the expression becomes

$$\left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n\prod_{n=q+1}^{p-1}\int_0^{\infty}e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n$$

$$\times\int_0^{\infty}e^{-\lambda_p}\lambda_p^{\alpha_p-1}E\left(\begin{matrix}\alpha, \beta & : z/\lambda_1\dots\lambda_p\\ \alpha+\beta+\frac{1}{2} & \end{matrix}\right)E\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta : z/\lambda_1\dots\lambda_p\\ \frac{3}{2}-\alpha-\beta\end{matrix}\right)d\lambda_p.$$

Now substitute from (6) and, on integrating, using (2) and (3), formula (8) is obtained. The restrictions on the ρ's can be removed as the paths of integration from 0 to 1 can be replaced by contours which start from 0, pass round the point 1 and return to 0.

3. A series of products of *E*-functions. It will now be shown that the formula (8) can be applied to obtain the series

$$\sum_{r=0}^{\infty} \frac{z^{-2r}}{r!(\gamma\,;\,r)} E\left(\begin{matrix}\alpha+r, \beta+r, \gamma+r : z \\ \alpha+\beta+\frac{1}{2}+r\end{matrix}\right) E\left(\begin{matrix}\frac{1}{2}-\alpha+r, \frac{1}{2}-\beta+r, \gamma+r : z \\ \frac{3}{2}-\alpha-\beta+r\end{matrix}\right)$$

$$= \pi^{-3/2}\cos(\alpha-\beta)\pi\, \Gamma(\alpha)\Gamma(\beta)\Gamma(\tfrac{1}{2}-\alpha)\Gamma(\tfrac{1}{2}-\beta)\Gamma(\gamma) E\left(\begin{matrix}\alpha-\beta+\frac{1}{2}, \beta-\alpha+\frac{1}{2}, \gamma, \frac{1}{2} : z \\ \alpha+\beta+\frac{1}{2}, \frac{3}{2}-\alpha-\beta\end{matrix}\right), \qquad (9)$$

where $|\operatorname{amp} z| < \frac{3}{2}\pi$, $R(\gamma) > 0$, $0 < R(\alpha) < \frac{1}{2}$, $0 < R(\beta) < \frac{1}{2}$.

To prove this substitute from (5) for the *E*-functions on the left of (9) and get

$$\sum_{r=0}^{\infty} \frac{z^{-2r}}{r!(\gamma\,;\,r)} \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\Gamma(\alpha+r-\zeta)\Gamma(\beta+r-\zeta)\Gamma(\gamma+r-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}+r-\zeta)} z^{\zeta}\, d\zeta$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(Z)\Gamma(\frac{1}{2}-\alpha+r-Z)\Gamma(\frac{1}{2}-\beta+r-Z)\Gamma(\gamma+r-Z)}{\Gamma(\frac{3}{2}-\alpha-\beta+r-Z)} z^{Z}\, dZ.$$

Here replace ζ and Z by $\zeta+r$ and $Z+r$, and interchange the order of summation and integration, so getting

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)\Gamma(\gamma-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}-\zeta)} z^{\zeta}\, d\zeta$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(Z)\Gamma(\frac{1}{2}-\alpha-Z)\Gamma(\frac{1}{2}-\beta-Z)\Gamma(\gamma-Z)}{\Gamma(\frac{3}{2}-\alpha-\beta-Z)} z^{Z} F\left(\begin{matrix}\zeta, Z\,;\,1 \\ \gamma\end{matrix}\right) dZ.$$

On applying Gauss's theorem this becomes

$$\frac{\Gamma(\gamma)}{2\pi i} \int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\frac{1}{2}-\zeta)} z^{\zeta} E\left(\begin{matrix}\frac{1}{2}-\alpha, \frac{1}{2}-\beta, \gamma-\zeta : z \\ \frac{3}{2}-\alpha-\beta\end{matrix}\right) d\zeta\,;$$

and, from (8), with $p = 1, q = 0$, the result follows.

Note. On replacing γ in (9) by $\alpha+\beta+\frac{1}{2}$ or $\frac{3}{2}-\alpha-\beta$ and applying (7), series involving Whittaker functions can be obtained.

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