ELLIPTIC COHOMOLOGY IS UNIQUE UP TO HOMOTOPY

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Abstract

Homotopy theory folklore tells us that the sheaf defining the cohomology theory Tmf of topological modular forms is unique up to homotopy. Here we provide a proof of this fact, although we claim no originality for the statement. This retroactively reconciles all previous constructions of Tmf.

1. Introduction

Generalised cohomology theories are powerful tools in algebraic topology, not only in their immediate utility in topology and geometry, but also in their ability to unify various areas of mathematics. The two most prominent examples of such are singular cohomology and topological *K*-theory, which possess connections to algebra, functional analysis, differential geometry and physics.

In this article, we discuss the cohomology theory Tmf of *topological modular forms*, a higher order analogue of singular cohomology and topological *K*-theory. This theory Tmf possesses connections to number theory and arithmetic geometry through modular forms and the moduli stack of elliptic curves, as well as to string topology and physics through a map from the string bordism groups to the Tmf-cohomology of spheres; see [Beh20, DFHH14] for general introductions. Moreover, the recent advances in equivariant topological modular forms [GM20, Lur19], applications to various computations in homotopy theory [GHMR05, WX17] and conjectural

connections to topological field theories [ST11] all suggest that we are still now only scratching the surface of Tmf.

At present, there is a singular definition of Tmf, and that is as the global sections $\Gamma(\mathcal{M}_{\mathrm{Ell}}, \mathcal{O}^{\mathrm{top}})$. Here, and elsewhere in this article, $\mathcal{M}_{\mathrm{Ell}}$ is the compactification of the moduli stack of elliptic curves and $\mathcal{O}^{\mathrm{top}}$ is the Goerss–Hopkins–Miller–Lurie sheaf; see [Beh20] for more background. For transparency, we remind the reader that the word *sheaf* is used in a homotopy-theoretic sense, as it takes values in the ∞-category of $\mathbf{E}_\infty$-rings—the homotopically meaningful environment to study cohomology theories with a highly structured multiplication. The first published construction of this sheaf is due to Behrens [DFHH14, Section 12], and is based on the unpublished work of Goerss, Hopkins and Miller.

One can find variations of this construction in the literature, some using logarithmic geometry on the moduli stack $\mathcal{M}_{\mathrm{Ell}}$ [HL16], and others using oriented derived elliptic curves and p-divisible groups [Lur18]. This leads us to a fundamental question:

Are these different constructions in any way compatible?

This question would be easily answered if one could show that $\mathcal{O}^{\mathrm{top}}$ is the 'unique' sheaf with a certain property, assuming this property holds for each competing construction. Moreover, such uniqueness could also be used to further justify the significance of $\mathcal{O}^{\mathrm{top}}$; we leave it to the reader to come to their own conclusions on that front. The property we would like to consider is that of an *elliptic cohomology theory*.

DEFINITION 1.1. Let E be a generalised elliptic curve over a ring R with irreducible geometric fibres, which is equivalent data to a morphism of stacks $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$. We say that a homotopy commutative ring spectrum $\mathcal{E}$ is an *elliptic cohomology theory* for E (or $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$) if we have the following data:

(1) $\mathcal{E}$ is weakly 2-periodic, meaning the homotopy group $\pi_2\mathcal{E}$ is a projective $\pi_0\mathcal{E}$-module of rank one and for every integer n, the canonical map of $\pi_0\mathcal{E}$-modules

$$\pi_2\mathcal{E} \underset{\pi_0\mathcal{E}}{\otimes} \pi_n\mathcal{E} \to \pi_{n+2}\mathcal{E}$$

is an isomorphism;

(2) the groups $\pi_k\mathcal{E}$ vanish for all odd integers k, so in particular $\mathcal{E}$ is complex orientable;

(3) there is a chosen isomorphism of rings $\pi_0\mathcal{E} \simeq R$; and

(4) there is a chosen isomorphism of formal groups $\widehat{E} \simeq \widehat{\mathbf{G}}_\mathcal{E}$ over R, between the formal group of E and the classical Quillen formal group of $\mathcal{E}$; see [Lur18, Section 4].

We say a collection of such $\mathcal{E}$ is *natural* if the isomorphisms in parts (3) and (4) above are natural with respect to a specified subcategory of affine schemes over $\mathcal{M}_{\mathrm{Ell}}$. (Other

variations of this definition can be found elsewhere in the literature; see [DFHH14, Section 12.6] or [Lur09a, Definition 1.2].)

The following is a simple uniqueness statement for $\mathscr{O}^{\text{top}}$ as a functor valued in homotopy commutative ring spectra—a much weaker structure than that of an $\mathbf{E}_\infty$-ring.

PROPOSITION 1.2. *The functor* $\text{h}\mathscr{O}^{\text{top}}\colon \mathcal{U}^{\text{op}} \to \text{CAlg}(\text{hSp})$, *from the small affine étale site* $\mathcal{U}$ *of* $\mathcal{M}_{\text{Ell}}$ *to the 1-category of homotopy commutative ring spectra, is uniquely defined up to isomorphism by the property that it defines natural elliptic cohomology theories on* $\mathcal{U}$.

The proof of the above statement follows from the fact that each section $\mathscr{O}^{\text{top}}(R)$ is Landweber exact; see [DFHH14, Section 12 and Remark 1.4] for some discussion. A remarkable fact about $\mathscr{O}^{\text{top}}$ is that the property that it defines natural elliptic cohomology theories actually characterises this sheaf with values in the ∞-category CAlg of $\mathbf{E}_\infty$-rings. The following is stated (without proof) in [Lur09a, Theorem 1.1] and [Goe10, Theorem 1.2].

THEOREM 1.3. *The sheaf of* $\mathbf{E}_\infty$-*rings* $\mathscr{O}^{\text{top}}$ *on the small étale site of* $\mathcal{M}_{\text{Ell}}$ *is uniquely defined up to homotopy by the property that it defines natural elliptic cohomology theories on the small affine étale site of* $\mathcal{M}_{\text{Ell}}$. *The same holds for the restriction* $\mathscr{O}^{\text{top}}_{\text{sm}}$ *of* $\mathscr{O}^{\text{top}}$ *to the small étale site of* $\mathcal{M}^{\text{sm}}_{\text{Ell}}$, *the moduli stack of smooth elliptic curves. (Along the way, we prove similar uniqueness statements for completions of* $\mathscr{O}^{\text{top}}$ *at a prime, its rationalisations as well as localisations as the height* 1 *and height* 2 *Morava K-theories.)*

The difference between Proposition 1.2 and Theorem 1.3 is two-fold: Theorem 1.3 states not only that $\mathscr{O}^{\text{top}}$ is uniquely defined up to homotopy as a sheaf of $\mathbf{E}_\infty$-rings (as opposed to a diagram in CAlg(hSp)), but that statement is made in an ∞-category (as opposed to the 1-category CAlg(hSp)).

The utility of Theorem 1.3 is evident. For example, and as indicated above, it retroactively shows that the various constructions of $\mathscr{O}^{\text{top}}$ found in [DFHH14, Section 12], [HL16] and [Dav21, Section 1] (and also [Lur18, Section 7], [Dav20, Section 5.3], [Dav22a, Section 6.1] and [Dav22b, Section 1] over the moduli stack of smooth elliptic curves) all agree up to homotopy. Importantly, Theorem 1.3 constructs noncanonical (see Remark 2.2) equivalences of $\mathbf{E}_\infty$-rings between all available definitions of Tmf; a conclusion which does **not** follow directly from Proposition 1.2. The author also finds the proof long and complex enough to warrant a publicly available write-up.

1.1. Outline. To prove Theorem 1.3, we first reduce the question to one of the connectedness of a certain moduli space; see Section 2. In Section 3, we formulate and prove a statement about spaces of natural transformations which we often use; we suggest that the reader initially skips this section and only returns when they deem it necessary. Our proof of Theorem 1.3 (reformulated as Theorem 2.1) occurs in

Section 4, and follows Behrens' construction of Tmf rather closely: first, we work with the separate chromatic layers, before gluing things together in both a transchromatic sense and then an arithmetic sense. The $K(1)$-local case in this section requires a statement about p-adic Adams operations on p-adic K-theory, which is the focus of our final technical Section 5.

1.2. Conventions. We assume that the reader is familiar with the language of ∞-categories as well as the techniques used in the construction of $\mathscr{O}^{\mathrm{top}}$ as described by Behrens [DFHH14, Section 12]. Furthermore, we advise the reader keeps a copy of *idem* in their vicinity. Write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for the mapping space between any two objects X, Y in an ∞-category $\mathcal{C}$, and $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ for its zeroth truncation, or equivalently, for $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$. Let us also suppress the notation distinguishing a 1-category from its nerve, considered as an ∞-category, and the same for a 2-category, such as the small étale site over $\mathcal{M}_{\mathrm{Ell}}$; see [Lur, Tag 007J].

2. A reformulation

Let us now make a statement to help us prove Theorem 1.3.

THEOREM 2.1. *Write $\mathcal{U}$ (respectively $\mathcal{U}_{\mathrm{sm}}$) for the (2-) category of affine schemes with étale maps to $\mathcal{M}_{\mathrm{Ell}}$ (respectively $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$). Then the spaces*

$$\mathcal{Z} = \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}) \underset{\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{hSp}))}{\times} \{\mathrm{h}\mathscr{O}^{\mathrm{top}}\}$$

$$\mathcal{Z}^{\mathrm{sm}} = \mathrm{Fun}(\mathcal{U}_{\mathrm{sm}}^{\mathrm{op}}, \mathrm{CAlg}) \underset{\mathrm{Fun}(\mathcal{U}_{\mathrm{sm}}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{hSp}))}{\times} \{\mathrm{h}\mathscr{O}^{\mathrm{top}}\}$$

are connected.

REMARK 2.2. As mentioned in [Lur18, Remark 7.0.2], the moduli space $\mathcal{Z}^{\mathrm{sm}}$ is **not** contractible. In other words, Theorem 2.1 states that $\mathscr{O}^{\mathrm{top}}$ is unique as a CAlg-valued presheaf of elliptic cohomology theories on $\mathcal{U}^{\mathrm{sm}}$ only up to homotopy, and does not claim anything more about the contractibility of this space. We would like to guide the reader to an explanation of this fact given by Tyler Lawson on `mathoverflow.net`; see [Law].

PROOF OF THEOREM 1.3 FROM THEOREM 2.1. The ∞-category of sheaves of $\mathbf{E}_{\infty}$-rings on the étale site of $\mathcal{M}_{\mathrm{Ell}}$ is equivalent, by restriction and right Kan extension, to the ∞-category of sheaves of $\mathbf{E}_{\infty}$-rings on the affine étale site of $\mathcal{M}_{\mathrm{Ell}}$; see [Dav22a, Lemma 6.1.10] for a similar argument following the 'comparison lemma' of [Hoy14, Lemma C.3]. Note that the latter is an ∞-subcategory of $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg})$, and that if a functor $F \colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ defines natural elliptic cohomology theories and there is an equivalence $F \simeq G$, then G also defines natural elliptic cohomology theories. These two observations show that it suffices to prove the space $\mathcal{Z}'$ is connected, where $\mathcal{Z}'$ is the component of $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg})^{\simeq}$ spanned by those functors which define natural elliptic cohomology theories. There is a map $\mathcal{Z} \to \mathcal{Z}'$ as both $\mathscr{O}^{\mathrm{top}}$ and any presheaf of $\mathbf{E}_{\infty}$-rings, equivalent to $\mathscr{O}^{\mathrm{top}}$ as a diagram of homotopy commutative ring

spectra, define natural elliptic cohomology theories. The map $\mathcal{Z} \to \mathcal{Z}'$ induces an equivalence on π_0 as [DFHH14, Section 12 and Remark 1.6] states that any functor $\mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(h\mathrm{Sp})$ which defines natural elliptic cohomology theories is isomorphic to $h\mathscr{O}^{\mathrm{top}}$. (The argument outlined in [DFHH14, Section 12 and Remark 1.4] is stated for hSp, but the statement can be modified for CAlg(hSp). Indeed, the crucial fact is that there are no phantom maps between Landweber exact spectra, meaning the functor from hSp to the 1-category of generalised cohomology theories is fully faithful on such spectra. The analogous fact in the multiplicative context follows, meaning the functor from CAlg(hSp) to the 1-category of multiplicative cohomology theories is fully faithful on Landweber exact homotopy commutative ring spectra.) Theorem 2.1 then implies that the moduli space $\mathcal{Z}'$ is also connected. The same argument can be made for $\mathcal{Z}^{\mathrm{sm}}$. $\square$

REMARK 2.3. Write $\mathcal{U}_{\mathbf{Q}}$ for the small affine étale site of $\mathcal{M}_{\mathrm{Ell}} \times \mathrm{Spec}\, \mathbf{Q}$ and for each prime p, write $\mathcal{U}_p$ for the small affine étale site of $\mathcal{M}_{\mathrm{Ell}} \times \mathrm{Spf}\, \mathbf{Z}_p$. The construction of $\mathscr{O}^{\mathrm{top}}$, as found in [DFHH14, Section 12] for example, proceeds first with a rational construction $\mathscr{O}_{\mathbf{Q}}^{\mathrm{top}}$ over $\mathcal{U}_{\mathbf{Q}}$, and a p-complete construction $\mathscr{O}_p^{\mathrm{top}}$ over $\mathcal{U}_p$. The methods of our proof for Theorem 2.1 show that the moduli spaces $\mathcal{Z}_{\mathbf{Q}}$ and $\mathcal{Z}_p$, of realisations of $h\mathscr{O}_{\mathbf{Q}}^{\mathrm{top}}$ and $h\mathscr{O}_p^{\mathrm{top}}$ over these aforementioned sites, are also connected. This means that analogues of Theorem 1.3 also hold for both $\mathscr{O}_{\mathbf{Q}}^{\mathrm{top}}$ and $\mathscr{O}_p^{\mathrm{top}}$. The same hold for the p-complete and rational version of $\mathscr{O}_{\mathrm{sm}}^{\mathrm{top}}$ for similar reasons. Moreover, following the 'arithmetic compatibility' discussed in the proof of Theorem 2.1, it follows that the localisations $\mathscr{O}^{\mathrm{top}}[\mathcal{P}^{-1}]$ and $\mathscr{O}_{\mathrm{sm}}^{\mathrm{top}}[\mathcal{P}^{-1}]$ satisfy their own version of Theorem 1.3, where $\mathcal{P}$ is any set of primes.

The following is a short remark on the homotopy groups of elliptic cohomology theories which is important later.

REMARK 2.4. Let $\mathcal{E}$ be an elliptic cohomology theory for some $E \colon \mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$. It follows that there is a natural isomorphism $\pi_{2k}\mathcal{E} \simeq \omega_E^{\otimes k}$ for all integers k, where ω_E is the dualising line for the formal group $\widehat{E}$; see [Lur18, Section 4.2]. Indeed, as the odd homotopy groups of $\mathcal{E}$ vanish, we see $\mathcal{E}$ possesses a complex orientation that yields the classical Quillen formal group $\widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$ over $\pi_0\mathcal{E}$; see [Lur18, Example 4.1.2], for example. From this, we see $\mathcal{E}$ is *complex periodic*, meaning it has a complex orientation and is weakly 2-periodic (see [Lur18, Section 4.1]), and [Lur18, Example 4.2.19] then implies that $\pi_2\mathcal{E}$ is isomorphic to the dualising line for the formal group $\widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$. Part (4) of Definition 1.1 states that $\pi_2\mathcal{E}$ is naturally isomorphic to ω_E, and part (1) gives us the claim above.

3. Spaces of natural transformations

To prove Theorem 2.1, we show that any two functors $\mathscr{O}$ and $\mathscr{O}'$ in $\mathcal{Z}$ can be connected by a path in $\mathcal{Z}$. In particular, we would like effective tools for studying

spaces of natural transformations between functors of ∞-categories. The following is known to experts, and a model categorical interpretation can be found in [DKS89].

PROPOSITION 3.1. *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. Suppose that for all objects X, Y in $\mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{D}}(FX, GY)$ is discrete, meaning the natural map*

$$\mathrm{Map}_{\mathcal{D}}(FX, GY) \to \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(\mathrm{h}FX, \mathrm{h}GY)$$

is an equivalence of spaces. Then the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ is also discrete, so the natural map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{h}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G)$$

is an equivalence of spaces, where an h *before a functor denotes post-composition with the unit map $\mathcal{D} \to \mathrm{h}\mathcal{D}$ of the homotopy category-nerve adjunction of* [Lur09b, *Proposition 1.2.3.1].*

PROOF. By [GHN17, Proposition 5.1], the space of natural transformations from F to G is naturally equivalent to the limit of the diagram

$$\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \xrightarrow{T} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F^{\mathrm{op}} \times G} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Map}_{\mathcal{D}}(-,-)} \mathcal{S}, \tag{3-1}$$

where $\mathrm{Tw}(\mathcal{C})$ is the *twisted arrow category* of $\mathcal{C}$ (see [GHN17, Definition 2.2]), and the $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is the natural right fibration (see *idem*). (We stick to the notation and conventions of [GHN17], which is a particular choice out of a possible two; see [GHN17, Warning 2.4].) The limit of Equation (3-1) is, by definition, the *end* of the composition $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. Consider the following not *a priori* commutative diagram of ∞-categories:

$$\tag{3-2}$$

Above, the vertical functors are the obvious ones, and hence the left and middle squares commute. Additionally, $\mathcal{S}_{\leq 0} \subseteq \mathcal{S}$ denotes the ∞-category of discrete spaces. Our hypotheses dictate that the dashed arrow above exists, which we now denote by P, such that the upper-right and lower-left triangles commute. As the inclusion of ∞-subcategories $\mathcal{S}_{\leq 0} \subseteq \mathcal{S}$ preserves limits, we note it suffices to compute the limit of Equation (3-1) as the limit of $P \circ T$ inside $\mathcal{S}_{\leq 0}$. As this limit lands in $\mathcal{S}_{\leq 0}$, which is equivalent to the nerve of the 1-category of sets, we see the limit of $P \circ T$ can be calculated as the limit of the lower-horizontal composition of Equation (3-2). We then obtain the following natural equivalences, twice employing [GHN17, Proposition 5.1],

first for general ∞-categories, and again in the classical 1-categorical case:

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \simeq \lim_{\mathrm{Tw}(\mathcal{C})} M(F^{\mathrm{op}} \times G)T \simeq \lim_{\mathrm{Tw}(\mathcal{C})} PT$$

$$\simeq \lim_{\mathrm{Tw}(\mathrm{h}\mathcal{C})} H(\mathrm{h}F^{\mathrm{op}} \times \mathrm{h}G)T' \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{h}\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G).$$

The final (discrete) space above is naturally equivalent to $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G)$ from the natural equivalence of ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathrm{h}\mathcal{D}) \simeq \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D})$. $\square$

4. The proof of Theorem 2.1

Let $\mathscr{O}\colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ be an object of $\mathcal{Z}$. Hence, it comes equipped with an equivalence $\mathrm{h}\phi\colon \mathrm{h}\mathscr{O}^{\mathrm{top}} \to \mathrm{h}\mathscr{O}$ of functors $\mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp})$. To see $\mathcal{Z}$ is connected, it suffices to show $\mathrm{h}\phi$ can be lifted to an equivalence $\phi\colon \mathscr{O}^{\mathrm{top}} \to \mathscr{O}$ of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$. Fix such an $\mathrm{h}\phi$ for the remainder of this proof. Let us work section-wise, so we also fix an object $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$ inside $\mathcal{U}$, and write

$$\mathrm{h}\phi\colon \mathcal{E}^{\mathrm{top}} := \mathrm{h}\mathscr{O}^{\mathrm{top}}(R) \to \mathrm{h}\mathscr{O}(R) =: \mathcal{E}$$

for the given natural equivalence of homotopy commutative ring spectra. To naturally lift this map to one of $\mathbf{E}_\infty$-rings, we work through the layers of chromatic homotopy theory. This means we first work $K(2)$-locally, $K(1)$-locally and then $K(0)$-locally, where $K(n)$ denotes the *nth Morava K-theory spectrum at a prime p*, before gluing these cases together with a p-complete statement followed by an arithmetic statement.

In what follows, we write $(-)_{K(n)}$ for the $K(n)$-localisation of $\mathbf{E}_\infty$-rings, and this notation is also used for the post-composition of a presheaf into CAlg with the $K(n)$-localisation functor.

K(2)-local case. Writing $\widehat{(-)}$ for base-change over $\mathrm{Spf}\,\mathbf{Z}_p$, we define $\mathrm{Spf}\,R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$ as the base-change of $\mathrm{Spf}\,\widehat{R} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$ over $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$, where the latter is the completion of $\widehat{\mathcal{M}}_{\mathrm{Ell}}$ at the moduli stack $\mathcal{M}_{\mathrm{Ell},\mathbf{F}_p}^{\mathrm{ss}}$ of supersingular elliptic curves over $\mathbf{F}_p$. This pullback $\mathrm{Spf}\,R^{\mathrm{ss}}$ is affine by [DFHH14, Section 12 and Remark 8.7]. Write E^{ss} for the elliptic curve defined by $\mathrm{Spf}\,R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$. Serre–Tate and Lubin–Tate theories yield another description of R^{ss}. Indeed, as $\mathcal{M}_{\mathrm{Ell},\mathbf{F}_p}^{\mathrm{ss}}$ is zero-dimensional and smooth over $\mathrm{Spec}\,\mathbf{F}_p$, it follows that $\mathrm{Spec}\,R^{\mathrm{ss}}/I$ is étale over $\mathbf{F}_p$, where I is the finitely generated ideal generating the topology on R^{ss}. This implies R^{ss}/I splits as a finite product $\prod_i \kappa_i$ where each κ_i is a finite field of characteristic p. This provides a splitting of E_0, the reduction of E^{ss} over R/I, into $E_0 \simeq \coprod E_0^i$. Writing $R_i \simeq W(\kappa_i)[\![u_1]\!]$ for the universal deformation ring of the pair $(\kappa_i, \widehat{E_0^i})$ with associated universal formal group $\widehat{E}_{R_i}^{\mathrm{ss}}$, we obtain a natural equivalence $R^{\mathrm{ss}} \simeq \prod_i R_i$ as $E^{\mathrm{ss}}\colon \mathrm{Spf}R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$ was étale; see [DFHH14, Section 12 and Corollary 4.3].

By [DFHH14, Section 12 and Proposition 4.4], the $K(2)$-localisations $\mathcal{E}_{K(2)}^{\mathrm{top}}$ and $\mathcal{E}_{K(2)}$ are elliptic cohomology theories for R^{ss}, and also split into products $\mathcal{E}_i^{\mathrm{top}}$ and $\mathcal{E}_i$ in the homotopy category $\mathrm{hCAlg}_{K(2)}$. It follows from [GH04, Section 7] (also see

[Lur18, Remark 5.0.5] or [PV19, Section 7]) that these $K(2)$-local $\mathbf{E}_\infty$-rings $\mathcal{E}_{K(2)}^{\text{top}}$ and $\mathcal{E}_{K(2)}$ are naturally equivalent to the product of Lubin–Tate $\mathbf{E}_\infty$-rings associated to the formal groups $\widehat{E}_{\kappa_i}^{\text{ss}}$ over the (finite and hence also) perfect fields κ_i. By *idem*, we see that morphisms between these Lubin–Tate $\mathbf{E}_\infty$-rings are defined by the associated morphisms on the pairs $(\kappa_i, \widehat{E}_{\kappa_i}^{\text{ss}})$. As $\mathrm{h}\phi_{K(2)}$ yields an equivalence on π_0 as well as an equivalence on associated Quillen formal groups, we see $\mathrm{h}\phi_{K(2)}$ lifts to a morphism $\phi_{K(2)}: \mathcal{E}_{K(2)}^{\text{top}} \to \mathcal{E}_{K(2)}$ of $K(2)$-local $\mathbf{E}_\infty$-rings, which is unique up to contractible choice. This uniqueness allows us to use Proposition 3.1 to conclude that this collection of morphisms of $\mathbf{E}_\infty$-rings define a natural morphism $\phi_{K(2)}: \mathscr{O}_{K(2)}^{\text{top}} \to \mathscr{O}_{K(2)}$ of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$.

$K(1)$-local case. Consider the $K(1)$-localisation of the map $\mathrm{h}\phi$ of homotopy commutative ring spectra $\mathrm{h}\phi_{K(1)}: \mathcal{E}_{K(1)}^{\text{top}} := \mathrm{h}\mathscr{O}_{K(1)}^{\text{top}}(R) \to \mathrm{h}\mathscr{O}_{K(1)}(R) =: \mathcal{E}_{K(1)}$. Recall from [DFHH14, Section 12.6], that the p-adic K-theory of an $\mathbf{E}_\infty$-ring has the structure of a θ-$\pi_*\,\mathrm{KU}_p$-algebra, functorially in maps of $\mathbf{E}_\infty$-rings. (Recall that for a spectrum X, one defines its *p-adic K-theory* as the homotopy groups of the localisation $\mathrm{K}_*^\wedge X = \pi_* L_{K(1)}(X \otimes \mathrm{KU}_p)$.) Let us write $\mathcal{M}_{\text{Ell}}^{\text{ord}}$ for the moduli of generalised elliptic curves over p-complete rings with ordinary reduction modulo p (see [DFHH14, Section 12(1.1)]), and $\mathcal{M}_{\text{Ell}}^{\text{ord}}(p^\infty)$ for the moduli stack of generalised elliptic curves E over p-complete rings and level structure given by an isomorphism $\widehat{\mathbf{G}}_m \simeq \widehat{E}$ of formal groups.

Let us work globally for a moment. Recall how the global sections of $\mathscr{O}_{K(1)}^{\text{top}}$, written as $\mathrm{Tmf}_{K(1)}$, are constructed in [DFHH14, Section 12]. For odd primes p, we define $\mathrm{Tmf}_{K(1)}$ as the $\mathbf{F}_p^\times$-homotopy fixed points of $\mathrm{Tmf}(p)^{\text{ord}}$, where this $K(1)$-local $\mathbf{E}_\infty$-ring is such that its p-adic K-theory W is given by pulling back the span of formal stacks

$$\mathcal{M}_{\text{Ell}}^{\text{ord}}(p) \to \mathcal{M}_{\text{Ell}}^{\text{ord}} \leftarrow \mathcal{M}_{\text{Ell}}^{\text{ord}}(p^\infty)$$

and $\mathcal{M}_{\text{Ell}}^{\text{ord}}(p)$ is the moduli of generalised elliptic curves E over p-complete rings with ordinary reduction modulo p and level structure given by an isomorphism $\mu_p \simeq \widehat{E}[p]$ of finite group schemes—both of the stacks on the left and right above are affine by [DFHH14, Section 12 and Lemma 5.2], and written as $\mathrm{Spf}\,V_1$ and $\mathrm{Spf}\,V_\infty^\wedge$, respectively. The $\mathbf{F}_p^\times$-structure on $\mathrm{Tmf}(p)^{\text{ord}}$ comes from the fact that W is an $\mathbf{F}_p^\times$-torsor over $V_\infty^\wedge$. The Goerss–Hopkins obstruction theory used to realise $\mathrm{Tmf}(p)^{\text{ord}}$ as an $\mathbf{F}_p^\times$-equivariant $K(1)$-local $\mathbf{E}_\infty$-ring with p-adic K-theory W also implies that such an $\mathbf{F}_p^\times$-equivariant $K(1)$-local $\mathbf{E}_\infty$-ring is unique up to homotopy; see [DFHH14, Section 12 and Theorem 7.1] for the obstruction theory and the proof of [DFHH14, Section 12 and Theorem 7.7] for justification that this works $\mathbf{F}_p^\times$-equivariantly. Pulling back $\mathscr{O}_{K(1)}$ to $\mathcal{M}_{\text{Ell}}^{\text{ord}}$, we see that $\mathcal{F}(p) := \mathscr{O}_{K(1)}(\mathcal{M}_{\text{Ell}}^{\text{ord}}(p))$ defines an elliptic cohomology theory for $\mathcal{M}_{\text{Ell}}^{\text{ord}}(p)$ by assumption, and by [DFHH14, Section 12 and Proposition 6.1] we see the p-adic K-theory of $\mathcal{F}(p)$ is isomorphic to W as $\mathbf{Z}_p^\times$-equivariant $\mathbf{Z}_p$-algebras. To see that this isomorphism is indeed an isomorphism of $\mathbf{F}_p^\times$-equivariant θ-algebras, we need to see that the algebraic p-adic Adams operation

ψ^p_{alg} on W agrees with the topological p-adic Adams operation ψ^p_{top} on the p-adic K-theory of $\mathcal{F}(p)$. This is done in the proof of Lemma 5.1 after Equation (5-2), to be stated and proven in Section 5 below. The uniquess of $\text{Tmf}(p)^{\text{ord}}$ up to homotopy gives us an equivalence of $\mathbf{F}^\times_p$-equivariant $K(1)$-local $\mathbf{E}_\infty$-rings $\text{Tmf}(p)^{\text{ord}} \simeq \mathcal{F}(p)$. Taking $\mathbf{F}^\times_p$-homotopy fixed points gives us an equivalence of $\mathbf{E}_\infty$-rings between $\text{Tmf}_{K(1)}$ and the global section of $\mathcal{O}_{K(1)}$ (note that $\mathcal{O}$ is an étale sheaf as it defines natural elliptic cohomology theories, so there is a well-defined notion of global sections). We now obtain a $K(1)$-local $\mathbf{E}_\infty$-$\text{Tmf}_{K(1)}$-algebra structure on the sections of $\mathcal{O}_{K(1)}$.

Suppose now that $p = 2$. By [DFHH14, Section 16], there are pushout diagrams of $K(1)$-local $\mathbf{E}_\infty$-rings

$$
\begin{array}{ccc}
\mathbf{P}(\mathbf{S}[-1])_{K(1)} \xrightarrow{\ 0\ } \mathbf{S}_{K(1)} & \qquad & \mathbf{P}(\mathbf{S})_{K(1)} \xrightarrow{\ 0\ } \mathbf{S}_{K(1)} \\
\downarrow{\scriptstyle\zeta} \qquad\quad \downarrow & & \downarrow{\scriptstyle\theta(f)-h(f)} \qquad\quad \downarrow \\
\mathbf{S}_{K(1)} \xrightarrow{\quad\quad} T_\zeta & & T_\zeta \xrightarrow{\quad\quad} \text{Tmf}_{K(1)}
\end{array}
$$

for specified elements $\zeta \in \pi_{-1}\mathbf{S}_{K(1)}$ and $\theta(f) - h(f) \in \pi_0 T_\zeta$, where $\mathbf{P}$ denotes the free $\mathbf{E}_\infty$-ring functor from Sp. It is shown in [DFHH14, Section 16] (after the proof of Remark 7.3) that all $K(1)$-local $\mathbf{E}_\infty$-elliptic cohomology theories have $\theta(f) = h(f)$ in their homotopy groups. This, combined with the fact that π_{-1} of an elliptic cohomology theory vanishes, implies that $\mathcal{O}_{K(1)}$ naturally takes values in $K(1)$-local $\mathbf{E}_\infty$-$\text{Tmf}_{K(1)}$-algebras.

We return to the local picture, equipped with the knowledge that $\mathcal{O}_{K(1)}$ takes values in $K(1)$-local $\mathbf{E}_\infty$-$\text{Tmf}_{K(1)}$-algebras. For another object $\text{Spec}\, R' \to \mathcal{M}_{\text{Ell}}$ inside $\mathcal{U}$, consider the map induced by the p-adic K-theory functor

$$
\text{Map}_{\text{CAlg}^{K(1)}_{\text{Tmf}_{K(1)}}}(\mathcal{E}^{\text{top}}_{K(1)}, \mathcal{E}'_{K(1)}) \to \text{Hom}_{\theta\text{Alg}_{(V^\wedge_\infty)_*}}(\text{K}^\wedge_* \mathcal{E}^{\text{top}}_{K(1)}, \text{K}^\wedge_* \mathcal{E}'_{K(1)}), \tag{4-1}
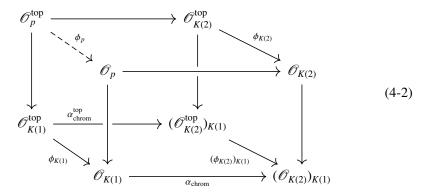$$

where $\mathcal{E}'_{K(1)} := \mathcal{O}_{K(1)}(R')$. As in the construction of $\mathcal{O}^{\text{top}}_{K(1)}$ found in [DFHH14, Section 12] before Proposition 7.16, the map of Equation (4-1) is an equivalence of spaces. Despite the fact that each $\text{h}\phi_{K(1)}$ is currently just a morphism of homotopy commutative ring spectra, Lemma 5.1 will guarantee that its zeroth p-adic K-theory is a morphism of θ-algebras. Moreover, this induced map of p-adic K-theories will be a morphism of $(V^\wedge_\infty)_*$-θ-algebras from the natural identifications of the preceeding paragraphs. As $\mathbf{Z}$-graded p-adic K-theory obtains a θ-algebra structure from that in degree zero, the p-adic K-theory of $\text{h}\phi_{K(1)}$ defines an element inside the codomain of Equation (4-1) when $R' = R$. By Proposition 3.1, we can therefore lift $\text{h}\phi_{K(1)}: \text{h}\mathcal{O}^{\text{top}}_{K(1)} \to \text{h}\mathcal{O}_{K(1)}$ to a morphism $\phi_{K(1)}: \mathcal{O}^{\text{top}}_{K(1)} \to \mathcal{O}_{K(1)}$ of presheaves of ($K(1)$-local $\mathbf{E}_\infty$-$\text{Tmf}_{K(1)}$-algebras, so in particular of) $\mathbf{E}_\infty$-rings on $\mathcal{U}$.

$K(0)$-local case. The Morava K-theory spectrum $K(0)$ is equivalent to $\mathbf{Q}$, the Eilenberg–MacLane spectrum of the rational numbers. We can actually lift $\text{h}\phi_{\mathbf{Q}}$ globally, meaning we are not working section-by-section. Consider post-composing the functors $\mathcal{O}^{\text{top}}$ and $\mathcal{O}$ with the localisation functor $\text{CAlg} \to \text{CAlg}_{\mathbf{Q}}$, and denote

the resulting presheaves with the subscript $\mathbf{Q}$. By construction (also see [HL16, Proposition 4.47]), the functor $\mathscr{O}_{\mathbf{Q}}^{\mathrm{top}}$ is formal and by [Mei21, Proposition 4.8], the sheaf $\mathscr{O}_{\mathbf{Q}}$ is also formal. This yields the following chain of equivalences lifting $\mathrm{h}\phi_{\mathbf{Q}}$:

$$\phi_{\mathbf{Q}} \colon \mathscr{O}_{\mathbf{Q}}^{\mathrm{top}} \xrightarrow{\simeq} \pi_* \mathscr{O}_{\mathbf{Q}}^{\mathrm{top}} \xrightarrow{\pi_* \mathrm{h}\phi_{\mathbf{Q}}, \simeq} \pi_* \mathscr{O}_{\mathbf{Q}} \xleftarrow{\simeq} \mathscr{O}_{\mathbf{Q}}$$

Transchromatic compatibility.   We now have morphisms fitting into the following not *a priori* commutative solid diagram of presheaves of p-complete $\mathbf{E}_\infty$-rings on $\mathcal{U}$:
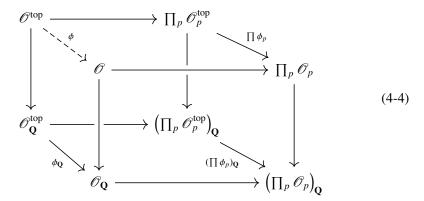

(4-2)

The right face commutes by the naturality of the unit of the $K(1)$-localisation functor. We also claim that the lower face commutes. In other words, we claim that for each $\mathrm{Spec}\, R \to \mathcal{M}$ inside $\mathcal{U}$, there is a natural path $\gamma(R)$ between $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(2)})_{K(1)} \circ \alpha_{\mathrm{chrom}}^{\mathrm{top}}$ as maps of $\mathbf{E}_\infty$-Tmf-algebras. Note that the $\mathbf{E}_\infty$-Tmf-algebra structure on $(\mathscr{O}_{K(2)})_{K(1)}$ can come from either one of these maps (and *a posteriori*, these two choices will agree up to homotopy). As we see from the discussion following [DFHH14, Section 12 and Lemma 8.8], the p-adic K-theory functor induces an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{Tmf}_{K(1)}}}(\mathcal{E}_{K(1)}^{\mathrm{top}}, (\mathcal{E}_{K(2)})_{K(1)}) \xrightarrow{\simeq} \mathrm{Hom}_{\theta\mathrm{Alg}_{(V_\infty^\wedge)_*}}(\mathrm{K}_*^\wedge \mathcal{E}_{K(1)}^{\mathrm{top}}, \mathrm{K}_*^\wedge (\mathcal{E}_{K(2)})_{K(1)}), \qquad (4\text{-}3)$$

where $(V_\infty^\wedge)_*$ denotes the p-adic K-theory of Tmf; see [DFHH14, Section 12.5] and [Beh20, Section 6]. As $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(2)})_{K(1)} \circ \alpha_{\mathrm{chrom}}^{\mathrm{top}}$ are isomorphic as functors into $\mathrm{CAlg}(\mathrm{hSp})$ and hence also in the codomain of Equation (4-3), we see these morphisms are homotopic as morphisms of $\mathbf{E}_\infty$-rings by the above equivalence. From the equivalence in Equation (4-3), we can employ Proposition 3.1 to obtain a homotopy between $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(2)})_{K(1)} \circ \alpha_{\mathrm{chrom}}^{\mathrm{top}}$ as morphisms of presheaves of $K(1)$-local $\mathbf{E}_\infty$-Tmf$_{K(1)}$-algebras from $\mathscr{O}_{K(1)}^{\mathrm{top}}$ to $(\mathscr{O}_{K(2)})_{K(1)}$. Using the fact that the front and back faces of Equation (4-2) are Cartesian, we obtain a natural morphism of presheaves of p-complete $\mathbf{E}_\infty$-rings $\phi_p \colon \mathscr{O}_p^{\mathrm{top}} \to \mathscr{O}_p$ on $\mathcal{U}$ which agrees with $\mathrm{h}\phi_p$ inside the category of presheaves functors from $\mathcal{U}$ to $\mathrm{CAlg}(\mathrm{hSp})$, as indicated by the dashed morphism in Equation (4-2).

Arithmetic compatibility.   Currently, we have morphisms $\phi_{\mathbf{Q}}$ and ϕ_p fitting into the not *a priori* commutative solid diagram of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$:

$$(4\text{-}4)$$

Similar to the transchromatic compatibilities, the right face naturally commutes, so we are left to argue why the lower face commutes. To study the lower face, let us first work on the open substacks $\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}]$ and $\mathcal{M}_{\mathrm{Ell}}[\Delta^{-1}]$ of $\mathcal{M}_{\mathrm{Ell}}$, which themselves form a cover of $\mathcal{M}_{\mathrm{Ell}}$; see [DFHH14, Section 12.9]. We then follow an analogous argument to the transchromatic situation above; see the discussion following [DFHH14, Section 12 and Lemma 9.4] which shows the discreteness of the desired mapping spaces. Indeed, as the two homotopies witnessing the commutativity of the lower face of Equation (4-4) restricted to the substacks $\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}]$ and $\mathcal{M}_{\mathrm{Ell}}[\Delta^{-1}]$ agree on their intersection $\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}, \Delta^{-1}]$ (as the mapping spaces in question are discrete), these homotopies then glue to a homotopy on $\mathcal{M}_{\mathrm{Ell}}$. This yields a homotopy witnessing the commutativity of the lower face of Equation (4-4). As the front and back faces of Equation (4-4) are Cartesian, we obtain our final natural equivalence of presheaves of $\mathbf{E}_\infty$-rings $\phi\colon \mathcal{E}^{\mathrm{top}} \to \mathcal{E}$ on $\mathcal{U}$, lifting $\mathrm{h}\phi$.

Therefore, $\mathcal{Z}$ is connected. The same argument can be made for $\mathcal{Z}^{\mathrm{sm}}$; see Remark 5.6 for why Lemma 5.1 simplifies in this case.

5. Compatibility of θ-algebra structures

The above proof of Theorem 2.1 is contingent on Lemma 5.1 below, whose proof we find rather delicate. Recall from [DFHH14, Section 12.6] that the *p*-adic *K*-theory of an $\mathbf{E}_\infty$-ring has the structure of a θ-algebra, and this structure is functorial in morphisms of $\mathbf{E}_\infty$-rings.

LEMMA 5.1. *Fix a prime p. Let $\mathscr{O}$ be an object of $\mathcal{Z}$ and $\mathrm{h}\phi\colon \mathrm{h}\mathscr{O}^{\mathrm{top}} \xrightarrow{\simeq} \mathrm{h}\mathscr{O}$ be the given equivalence of diagrams of homotopy commutative ring spectra. Then for any étale $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$, the map induced by*

$$\mathrm{h}\phi_{K(1)}\colon \mathcal{F}_{K(1)}^{\mathrm{top}} := \mathrm{h}\mathscr{O}_{K(1)}^{\mathrm{top}}(R) \to \mathrm{h}\mathscr{O}_{K(1)}(R) =: \mathcal{F}_{K(1)}$$

on the zeroth p-adic K-theory ring is a morphism of θ-algebras.

In general, it is not true that a morphism of homotopy commutative ring spectra should induce a morphism of θ-algebras upon taking their p-adic K-theory, even if the homotopy commutative ring spectra involved come equipped with some $\mathbf{E}_\infty$-structures.

However, in the situation above, the sections of the $K(2)$-localisation of the sheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}$ have a prescribed $\mathbf{E}_\infty$-structure given by Lubin–Tate spectra (also called Morava E-theories); see the $K(2)$-local case in the proof of Theorem 2.1 above. In particular, this means that an equivalence of homotopy commutative ring spectra $\mathcal{F}_{K(2)}^{\mathrm{top}} \simeq \mathcal{F}_{K(2)}$ can be refined to an equivalence of $\mathbf{E}_\infty$-rings, as both objects are naturally Lubin–Tate spectra. It follows that the $K(1)$-localisation of this equivalence of $\mathbf{E}_\infty$-rings induces a morphism of θ-algebras on p-adic K-theory. The comparison map in the chromatic fracture square between the $K(1)$-localisation of $\mathscr{O}$ and the $K(1)$-localisation of its $K(2)$-localisation is also a map of $\mathbf{E}_\infty$-rings. If we can show this map induces an injection on p-adic K-theory, then it would be clear that the equivalence of homotopy commutative ring spectra $\mathcal{F}_{K(1)}^{\mathrm{top}} \simeq \mathcal{F}_{K(1)}$ induces a morphism of θ-algebras on p-adic K-theory, which would lead us to Lemma 5.1.

This is first done for an explicit étale morphism into $\mathcal{M}_{\mathrm{Ell}}$, which has the properties that it covers $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$ and each of its connected components is an integral domain. Some descent and deformation theory is then used to obtain this result for a general étale morphism.

PROOF. To show $\lambda: \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} \to \mathrm{K}_0^\wedge \mathcal{F}$, the map induced by h$\phi$ on p-adic K-theory, is a morphism of θ-algebras, one must check it commutes with the stable p-adic Adams operations ψ^ℓ for every $\ell \in \mathbf{Z}_p^\times$ as well as the action of the operator θ. The stable p-adic Adams operations ψ^ℓ are constructed on the spectrum KU_p, so we automatically have compatibility with them for any map of spectra. It is shown shortly that both rings above are étale over the ring $V_\infty^\wedge$, and hence they are $V_\infty^\wedge$-torsion free, where we remind the reader that $V_\infty^\wedge$ is the ring of *generalised p-adic modular forms* discussed after the proof of [DFHH14, Section 12 and Lemma 5.2] or in the $K(1)$-local discussions of [Beh20]. In particular, this implies that both $\mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}}$ and $\mathrm{K}_0^\wedge \mathcal{F}$ are $\mathbf{Z}_p$-torsion free, in which case the operator θ is an equivalent datum to the p-adic Adams operator ψ^p; see [GH04, Remark 2.2.5]. It therefore suffices to show that the following diagram of $\mathbf{Z}_p$-algebras commutes:

$$\begin{array}{ccc} \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} & \xrightarrow{\ \lambda\ } & \mathrm{K}_0^\wedge \mathcal{F} \\ \downarrow{\scriptstyle \psi_{\mathrm{top}}^p} & & \downarrow{\scriptstyle \psi_{\mathrm{top}}^p} \\ \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} & \xrightarrow{\ \lambda\ } & \mathrm{K}_0^\wedge \mathcal{F} \end{array} \tag{5-1}$$

Let us write R^{ord} for the base-change of $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ over $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}} \to \widehat{\mathcal{M}}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{Ell}}$, where $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$ is the moduli stack of generalised elliptic curves over p-complete rings whose reduction modulo p is ordinary. By [DFHH14, Section 12 and Proposition 7.16], we see $\mathcal{F}_{K(1)}^{\mathrm{top}}$ is an elliptic cohomology theory for $\operatorname{Spf} R^{\mathrm{ord}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$, and we can also
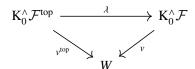
consider $\mathcal{F}_{K(1)}$ as an elliptic cohomology theory using $h\phi_{K(1)}$. Define W using the Cartesian diagram of formal stacks

$$
\begin{array}{ccc}
\mathrm{Spf}\, W & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^{\infty}) \\
\downarrow & & \downarrow \\
\mathrm{Spf}\, R^{\mathrm{ord}} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}
\end{array}
$$

where $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^{\infty})$ is the formal stack of generalised elliptic curves E with ordinary reduction modulo p with a given isomorphism of formal groups $\eta \colon \widehat{\mathbf{G}}_m \to \widehat{E}$; see [DFHH14, Section 12.5]. The stack $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^{\infty})$ is represented by the formal affine scheme $\mathrm{Spf}\, V_{\infty}^{\wedge}$ which is ind-étale over $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$; see the discussion after the proof of [DFHH14, Section 12 and Lemma 5.2]. This W also has the structure of a θ-algebra (see [DFHH14, Section 12.6]), and we denote the p-adic Adams operation on W by ψ_{alg}^{p}. By [DFHH14, Section 12 and Proposition 6.1], or rather its proof, we obtain isomorphisms of $\mathbf{Z}_p$-algebras $v^{\mathrm{top}} \colon \mathrm{K}_0^{\wedge}\mathcal{F}^{\mathrm{top}} \simeq W$ and $v \colon \mathrm{K}_0^{\wedge}\mathcal{F} \simeq W$, which are natural in complex orientation-preserving morphisms in $\mathrm{CAlg}(\mathrm{hSp})$. These isomorphisms are **not** *a priori* isomorphisms of θ-algebras; see [DFHH14, Section 12 and Definition 6.2]. As $\mathcal{F}$ obtains the structure of an elliptic cohomology theory for R^{ord} from the equivalence $h\phi_{K(1)}$, we see that the following diagram of isomorphisms of $\mathbf{Z}_p$-algebras commutes:

$$
\begin{array}{ccc}
\mathrm{K}_0^{\wedge}\mathcal{F}^{\mathrm{top}} & \xrightarrow{\ \lambda\ } & \mathrm{K}_0^{\wedge}\mathcal{F} \\
& \searrow^{v^{\mathrm{top}}} \quad \swarrow^{v} & \\
& W &
\end{array}
$$

By construction (see [DFHH14, Section 12 and Remark 6.3]), we see v^{top} is an isomorphism of θ-algebras. To show λ is a morphism of θ-algebras, it suffices to show v is a morphism of θ-algebras, or in other words, Equation (5-1) commutes if and only if the following diagram of $\mathbf{Z}_p$-algebras commutes:

$$
\begin{array}{ccc}
\mathrm{K}_0^{\wedge}\mathcal{F} & \xrightarrow{\ v\ } & W \\
\downarrow^{\psi_{\mathrm{top}}^{p}} & & \downarrow^{\psi_{\mathrm{alg}}^{p}} \\
\mathrm{K}_0^{\wedge}\mathcal{F} & \xrightarrow{\ v\ } & W
\end{array}
\tag{5-2}
$$

Let us now prove this is the case for a specific étale map $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$.

Choosing a particular étale morphism. Recall the moduli stack $\mathcal{M}_1^{\mathrm{sm}}(N)$ of smooth elliptic curves with $\Gamma_1(N)$-level structure. These objects are discussed at length in [DR73, Section IV.3] and [KM85], and summaries for homotopy theorists can be found in [Beh20, (6.3.8)] or [Mei22, Section 2.1], for example. Importantly, recall the map

$\mathcal{M}_1^{\mathrm{sm}}(N) \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[1/N]}^{\mathrm{sm}}$ is an étale cover and that for $N \geq 4$, the moduli stack $\mathcal{M}_1^{\mathrm{sm}}(N)$ is in fact affine. This implies that the morphism of stacks

$$\mathrm{Spec}\, A = \mathcal{M}_1^{\mathrm{sm}}(4) \sqcup \mathcal{M}_1^{\mathrm{sm}}(5) \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[1/2]}^{\mathrm{sm}} \sqcup \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[1/5]}^{\mathrm{sm}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}} \to \mathcal{M}_{\mathrm{Ell}}$$

is étale, and the restriction to $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$ is an étale cover. By base-change over $\mathrm{Spf}\, \mathbf{Z}_p$, we obtain an étale map $E : \mathrm{Spf}\,\widehat{A} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$. Following [DFHH14, Section 12 and Remark 8.7], write A_* for the graded ring defined by $A_{2k} = \omega_E^{\otimes k}(\mathrm{Spf}\,\widehat{A})$ where E is the elliptic curve over $\widehat{A}$ defined by the map of formal stacks above. Note that the Hasse invariant v_1 for E lives in $A_{2(p-1)}$. Let us also make the following definitions:

$$A_*^{\mathrm{ord}} = (A_*)[v_1^{-1}]_p^\wedge, \quad A_*^{\mathrm{ss}} = (A_*)_{(p,v_1)}^\wedge, \quad (A_*^{\mathrm{ss}})^{\mathrm{ord}} = (A_*^{\mathrm{ss}})[v_1^{-1}]_p^\wedge.$$

If we omit the subscript $*$, we are implicitly considering the ring in degree zero. By [DFHH14, Section 12 (8.6)], there is a canonical map $\alpha_* : A_*^{\mathrm{ord}} \to (A_*^{\mathrm{ss}})^{\mathrm{ord}}$ as v_1 is invertible in $(A_*^{\mathrm{ss}})^{\mathrm{ord}}$, and we now define W^{ss} using the diagram of stacks

$$
\begin{array}{ccc}
\mathrm{Spf}\, W^{\mathrm{ss}} & \xrightarrow{\widetilde{\alpha}} \mathrm{Spf}\, W \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^\infty) \\
\downarrow & \downarrow & \downarrow \\
\mathrm{Spf}(A^{\mathrm{ss}})^{\mathrm{ord}} & \xrightarrow{\alpha} \mathrm{Spf}\, A^{\mathrm{ord}} \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}
\end{array}
$$

where all squares are Cartesian. The ring W^{ss} obtains a θ-algebra structure from the above diagram, and in such a way that $\widetilde{\alpha} : W \to W^{\mathrm{ss}}$ is a morphism of θ-algebras; see the discussion after the proof of [DFHH14, Section 12 and Lemma 8.6]. We claim that $\widetilde{\alpha}$ comes from a map of $\mathbf{E}_\infty$-rings.
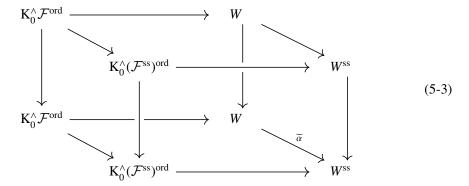
CLAIM 5.2. The zeroth p-adic K-theory of the canonical map of $\mathbf{E}_\infty$-rings

$$\alpha_{\mathrm{chrom}} : \mathcal{F}^{\mathrm{ord}} := \mathcal{O}_{K(1)}(A) \to (\mathcal{O}_{K(2)}(A))_{K(1)} =: (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$$

is isomorphic to $\widetilde{\alpha}$. □

PROOF OF CLAIM 5.2. Specialising from the case of $\mathcal{F}_{K(1)}$ discussed above, we see that $\mathcal{F}^{\mathrm{ord}}$ is an elliptic cohomology theory for the map $\mathrm{Spf}\, A^{\mathrm{ord}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$, and similarly by [DFHH14, Section 12 and Lemma 8.8], we see that $(\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$ is an elliptic cohomology theory for the map $\mathrm{Spf}(A^{\mathrm{ss}})^{\mathrm{ord}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$. The same is true for $\mathcal{A} = \mathcal{O}_{K(1)}^{\mathrm{top}}(A)$ and $(\mathcal{O}_{K(2)}^{\mathrm{top}}(A))_{K(1)} = \mathcal{A}'$, and in this case, we know that taking π_0 of $\alpha_{\mathrm{chrom}}^{\mathrm{top}} : \mathcal{A} \to \mathcal{A}'$ is isomorphic to $\alpha : A^{\mathrm{ord}} \to (A^{\mathrm{ss}})^{\mathrm{ord}}$ by construction; see [DFHH14, Section 12 and Remark 8.7]. The naturality of $h\phi : \mathcal{O}^{\mathrm{top}} \to \mathcal{O}$ and the chromatic fracture square imply that π_0 of the natural map of $\mathbf{E}_\infty$-rings $\alpha_{\mathrm{chrom}} : \mathcal{F}^{\mathrm{ord}} \to (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$ also realises α, and hence taking zeroth p-adic K-theory realises $\widetilde{\alpha}$. This proves Claim 5.2. □

Recall that $\mathcal{F}^{\mathrm{ord}} = \mathcal{O}_{K(1)}(A)$ for our choice of A above. Consider the diagram of $\mathbf{Z}_p$-algebras

$$
\begin{array}{ccc}
\mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{ord}} & \longrightarrow & W \\
& \mathrm{K}_0^\wedge (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}} \longrightarrow & W^{\mathrm{ss}} \\
\mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{ord}} & \longrightarrow & W \\
& \mathrm{K}_0^\wedge (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}} \xrightarrow{\quad\widetilde{\alpha}\quad} & W^{\mathrm{ss}}
\end{array}
\tag{5-3}
$$

where the maps are the obvious ones used above, and all the vertical morphisms are the unstable Adams operations: ψ_{top}^p on the left and ψ_{alg}^p on the right. Note that the upper and lower faces commute by Claim 5.2, the right face commutes as $\widetilde{\alpha}\colon W \to W^{\mathrm{ss}}$ is a morphism of θ-algebras and the left face commutes as α_{chrom} is a morphism of $\mathbf{E}_\infty$-rings. Most importantly, the front face also commutes. Indeed, from the arguments in the $K(2)$-local case of the proof of Theorem 2.1, we see $\mathcal{F}^{\mathrm{ss}}$ is naturally equivalent to a product of $K(2)$-local Lubin–Tate spectra recognising the given elliptic curve over $\mathrm{Spf}\, A^{\mathrm{ss}}$, and we can then apply [DFHH14, Section 12 and Theorem 6.10]; the hypotheses and proof of this theorem are dispersed between pages 21 and 24 of *idem*. The back face of Equation (5-3) is precisely Equation (5-2) for $R = A$.

CLAIM 5.3. The morphism $\widetilde{\alpha}\colon W \to W^{\mathrm{ss}}$ is injective.

Using this claim for now, to show that the back face of Equation (5-3) commutes, it suffices to do so after post-composing with $\widetilde{\alpha}$. This follows from the above considerations by a diagram chase. Hence, the back face of Equation (5-3) commutes, which yields the commutativity of Equation (5-2) for this particular choice of étale map $\mathrm{Spec}\, A \to \mathcal{M}_{\mathrm{Ell}}$.

PROOF OF CLAIM 5.3. As $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^\infty) \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$ is an ind-étale cover (see [DFHH14, Section 12 and Lemma 5.1]), it is faithfully flat. By base-change, we see that $A^{\mathrm{ord}} \to W$ and $(A^{\mathrm{ord}})^{\mathrm{ss}} \to W^{\mathrm{ss}}$ are also faithfully flat, and hence $\widetilde{\alpha}$ is injective if we can show α is injective. To do this, we show $\alpha_*\colon A_*^{\mathrm{ord}} \to (A_*^{\mathrm{ss}})^{\mathrm{ord}}$ is injective. Using the notation above, we find ourselves with the following commutative diagram of graded rings, where all maps are the indicated localisations or completions:

$$
\begin{array}{ccccc}
A_* & \longrightarrow & A_*[v_1^{-1}] & \longrightarrow & A_*[v_1^{-1}]_p^\wedge = A_*^{\mathrm{ord}} \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha_*} \\
A_*^{\mathrm{ss}} & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}] & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}]_p^\wedge = (A_*^{\mathrm{ss}})^{\mathrm{ord}}
\end{array}
$$

Let us now make the following remarks from this diagram.

(1) From our choice of A, we have $A = A_1 \times A_2$, where A_1 represents $\mathcal{M}_1(4)^{\mathrm{sm}}$ and A_2 represents $\mathcal{M}_1(5)$. Moreover, both A_1 and A_2 are integral domains; see [Zhu14, Proposition 2.1] and [BO16, Theorem 1.1.1], respectively, where the following isomorphisms are constructed:

$$A_1 \simeq \mathbf{Z}[\tfrac{1}{4}, a, b, \Delta^{-1}] \quad A_2 \simeq \mathbf{Z}[\tfrac{1}{5}, b, \Delta^{-1}] \quad \Delta = a^2 b^4 (a^2 - 16b) = b^5(b^2 - 11b - 1).$$

It then follows that γ can be written in the following commutative diagram of graded rings:

$$
\begin{array}{ccc}
A_* & \xrightarrow{\ \gamma\ } & A_*^{\mathrm{ss}} = (A_*)^{\wedge}_{(v_1)} \\
\downarrow & & \downarrow \\
A_{*,1} \times A_{*,2} & \xrightarrow{\ \gamma_1 \times \gamma_2\ } & A_{*,1}^{\mathrm{ss}} \times A_{*,2}^{\mathrm{ss}}
\end{array}
$$

The ring A is Noetherian as it is finitely presented over $\mathrm{Spec}\,\mathbf{Z}$, so both A_1 and A_2 are Noetherian integral domains. In particular, the completion maps γ_i are flat for $i = 1, 2$. If we know these maps γ_i are nonzero, then it immediately follows that they are injective. To see that they are nonzero, it suffices to show that v_1 is not a unit inside both $A_{*,1}$ and $A_{*,2}$. This is where our choice of A comes in. If our fixed prime $p \neq 2, 5$, then for both $i = 1, 2$, the image of the map

$$\mathrm{Spf}\,\widehat{A_i} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}^{\mathrm{sm}} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$$

contains a supersingular elliptic curve, as all supersingular elliptic curves are contained in the smooth locus of $\widehat{\mathcal{M}}_{\mathrm{Ell}}$. This implies that v_1 cannot be a unit, else $\mathrm{Spf}\,\widehat{A_i} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$ would define only ordinary elliptic curves of height one. Similarly, if $p = 2$, then the p-completion of A is $\widehat{A_2}$, and we again see v_1 is not a unit so $\gamma_2 = \gamma$ is injective. The same holds for $\widehat{A_1}$ when $p = 5$. This implies that $\gamma_1 \times \gamma_2$ is always injective, and hence γ is injective.

(2) As β is the v_1-localisation of γ, and localisation is exact, we see that β is also injective.

(3) Standard arguments show that the p-completion of β, also known as α_*, is also injective. Indeed, limits are left exact, so it suffices to show each α_*^k in the following commutative diagram of rings is injective, for every $k \geq 1$:

$$
\begin{array}{ccc}
A_*[v_1^{-1}] & \longrightarrow & A_*[v_1^{-1}]/p^k \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_*^k} \\
A_*^{\mathrm{ss}}[v_1^{-1}] & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}]/p^k
\end{array}
$$

Given an element $\bar{x}$ such that $\alpha_*^k(\bar{x}) = 0$, then we first note that any lift x over $\bar{x}$ is sent to a $\beta(x)$ such that $p^k \beta(x) = 0$. However, $A_*^{\mathrm{ss}}[v_1^{-1}]$ is flat over $\mathbf{Z}$, as we have the following

composite of flat maps:

$$\mathbf{Z} \to \mathbf{Z}_p \to \widehat{A} \to A_* \xrightarrow{\gamma} A_*^{\mathrm{ss}} \to A_*^{\mathrm{ss}}[v_1^{-1}];$$

the second map is flat as $\mathrm{Spec}\, A \to \mathcal{M}_{\mathrm{Ell}}$ is étale and $\mathcal{M}_{\mathrm{Ell}}$ is smooth over $\mathbf{Z}$, and the third map is flat as each $\omega_E^{\otimes k}(\mathrm{Spf}\, A)$ is a line bundle and hence projective of rank 1. This implies that $A_*^{\mathrm{ss}}[v_1^{-1}]$ is torsion-free, and hence $\beta(x) = 0$. As β is injective, this implies $x = 0$ and $\bar{x} = 0$, and hence α_*^k is injective.

It follows that $\widetilde{\alpha}$ is injective. $\square$

Proof for a general étale morphism. Let $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$ be an arbitrary étale morphism and consider the Cartesian diagram of stacks

$$
\begin{array}{ccc}
\mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, A \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, R & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}
\end{array}
$$

where $\mathrm{Spec}\, A = \mathcal{M}_1^{\mathrm{sm}}(4) \sqcup \mathcal{M}_1^{\mathrm{sm}}(5)$ is the stack of the previous paragraph and note that $\mathrm{Spec}\, B$ is affine as $\mathcal{M}_{\mathrm{Ell}}$ is separated; see part (2) of [Sta, Tag 01SG]. All of the morphisms above are étale by base-change, so we can consider the morphism of $\mathbf{E}_\infty$-rings $\mathscr{O}(A) \to \mathscr{O}(B)$.

CLAIM 5.4. *The morphism of $\mathbf{E}_\infty$-rings $\mathscr{O}(A) \to \mathscr{O}(B)$ is étale.*

PROOF. Recall from [Lur17, Section 7.5] that a morphism $\mathcal{A} \to \mathcal{B}$ of $\mathbf{E}_\infty$-rings is *étale* if the morphism $\pi_0\mathcal{A} \to \pi_0\mathcal{B}$ of discrete commutative rings is étale and the natural map of $\pi_0\mathcal{B}$-modules

$$\pi_0\mathcal{B} \underset{\pi_0\mathcal{A}}{\otimes} \pi_*\mathcal{A} \to \pi_*\mathcal{B}$$

is an isomorphism. The fact that $\pi_0\mathscr{O}(A) \to \pi_0\mathscr{O}(B)$ is étale follows from the facts that $A \to B$ is étale and $\mathscr{O}$ defines natural elliptic cohomology theories. The condition on the higher homotopy groups also follows as $\mathscr{O}$ defines natural elliptic cohomology theories; see Remark 2.4. $\square$

By [Lur17, Theorem 7.5.0.6], the π_0-functor induces an equivalence of ∞-categories

$$\mathrm{CAlg}_{\mathscr{O}(A)}^{\text{ét}} \xrightarrow{\pi_0} \mathrm{CAlg}_A^{\text{ét}},$$

where the superscript indicates subcategories of étale algebras. By Claim 5.4, for any étale $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebra $\mathcal{B}$ such that $\pi_0\mathcal{B}$ is isomorphic to B as an A-algebra, there is an equivalence of $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebras $\mathscr{O}(B) \simeq \mathcal{B}$, which is unique up to contractible choice. As we have proven Lemma 5.1 for $\mathrm{Spec}\, A$, it follows from the proof of Theorem 2.1 above that the equivalence of homotopy commutative ring spectra $\mathrm{h}\phi(A)\colon \mathscr{O}^{\mathrm{top}}(A) \simeq \mathscr{O}(A)$ can be lifted to a morphism of $\mathbf{E}_\infty$-rings. The composite $\mathscr{O}(A) \simeq \mathscr{O}^{\mathrm{top}}(A) \to \mathscr{O}^{\mathrm{top}}(B)$ is also an étale $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebra recognising B, and hence we obtain a natural equivalence of $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebras $\mathscr{O}^{\mathrm{top}}(B) \simeq \mathscr{O}(B)$. As $\mathscr{O}^{\mathrm{top}}(B)$

is θ-compatible (see [DFHH14, Section 12 and Remark 6.3]), we see $\mathscr{O}(B)$ is also θ-compatible, meaning that Equation (5-2) commutes for $R = B$. Finally, let us turn our attention to $\mathscr{O}(R) \to \mathscr{O}(B)$.

CLAIM 5.5. *The morphism induced by $\mathscr{O}(R) \to \mathscr{O}(B)$ on zeroth p-adic K-theory is injective.*

Assuming the above claim, it immediately follows that Equation (5-2) commutes for our arbitrary R. Indeed, Claim 5.5 provides us with an injection of θ-algebras induced by $\mathscr{O}(R) \to \mathscr{O}(B)$, which allows us to check the commutativity of Equation (5-2) in the same diagram for $R = B$, which we know commutes by the above paragraph.

PROOF OF CLAIM 5.5. The morphism $\operatorname{Spec} B \to \operatorname{Spec} R$ can be factored into the following diagram of formal stacks:

$$
\begin{array}{ccccccc}
\operatorname{Spf} W_B & \longrightarrow & \operatorname{Spf} W_R^{\mathrm{sm}} & \longrightarrow & \operatorname{Spf} W_R & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^\infty) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spf} \widehat{B}^{\mathrm{ord}} & \longrightarrow & \operatorname{Spf} \widehat{R}^{\mathrm{ord,sm}} & \longrightarrow & \widehat{R}^{\mathrm{ord}} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spf} \widehat{B} & \longrightarrow & \operatorname{Spf} \widehat{R}^{\mathrm{sm}} & \longrightarrow & \operatorname{Spf} \widehat{R} & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{Ell}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R^{\mathrm{sm}} & \longrightarrow & \operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Spec} A & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} & &
\end{array}
\tag{5-4}
$$

Every square above is Cartesian, and the $\widehat{(-)}$ indicates base-change over $\operatorname{Spf} \mathbf{Z}_p$. By [DFHH14, Section 12 and Proposition 6.1], the morphism $W_R \to W_B$ above is isomorphic to the morphism induced by $\mathscr{O}(R) \to \mathscr{O}(B)$ on p-adic K-theory, and hence it suffices to see the composite map

$$
W_R \to W_R^{\mathrm{sm}} \to W_B,
\tag{5-5}
$$

featured in the upper-left corner of Equation (5-4), is injective. As $\operatorname{Spec} A \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$ is an étale cover, then by base-change, we see $W_R^{\mathrm{sm}} \to W_B$ is also faithfully flat and hence injective. Observe that $W_R \to W_R^{\mathrm{sm}}$ is an open immersion of formal affine schemes by base-change as $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}} \to \mathcal{M}_{\mathrm{Ell}}$ is an open immersion of stacks. Moreover, we claim the open immersion $R \to R^{\mathrm{sm}}$ has scheme theoretically dense image as Δ is a nonzero divisor in R; see [Sta, Tag 01RE]. Indeed, to see Δ is not a zero divisor, it suffices to show that the image of $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ has nontrivial intersection with the image of $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$. This is clear on the level of underlying topological spaces, as the inclusion $|\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}| \to |\mathcal{M}_{\mathrm{Ell}}|$ is equivalent to open immersion of coarse moduli spaces $|\mathbf{A}_{\mathbf{Z}}^1| \to |\mathbf{P}_{\mathbf{Z}}^1|$

which adds the point at ∞, and the map $|\operatorname{Spec} R| \to |\mathcal{M}_{\mathrm{Ell}}|$ is open as étale morphisms are in particular flat and locally of finite presentation; see [Sta, Tag 06R7]. As all the right vertical maps in Equation (5-4) are flat, and $R \to R^{\mathrm{sm}}$ is quasi-compact (as a map of affine schemes), then [Sta, Tag 0CMK] tells us that $W_R \to W_R^{\mathrm{sm}}$ also has scheme theoretically dense image. Another application of [Sta, Tag 01RE] shows this open immersion $W_R \to W_R^{\mathrm{sm}}$ must be injective. Therefore, the composite in Equation (5-5) is injective.   □

This finishes our proof of Lemma 5.1.

REMARK 5.6. There is a potential improvement that could be made to this article. If the reader can find an affine étale **cover** $\operatorname{Spec} A \to \mathcal{M}_{\mathrm{Ell}}$ such that the analogous Claim 5.3 holds, which is a purely algebro-geometric pursuit, then the rest of Theorem 2.1 follows formally. Indeed, in this case, one can prove that $\mathcal{O}^{\mathrm{top}}$ is uniquely defined on the Čech nerve of such a cover by copying the proofs of Theorem 2.1 and Lemma 5.1 seen above. One can then use spectral deformation theory and descent to show that Theorem 2.1 follows from this particular case. With this in mind, the reader might also notice that restricting Theorem 1.3 to the moduli stack of smooth elliptic curves vastly simplifies the above proof.

Acknowledgements

References

[Beh20]   M. Behrens, 'Topological modular and automorphic forms', in: *Handbook of Homotopy Theory* (ed. H. Miller) (CRC Press, Boca Raton, FL, 2020), 221–261.

[BO16]   M. Behrens and K. M. Ormsby, 'On the homotopy of $Q(3)$ and $Q(5)$ at the prime 2', *Algebr. Geom. Topol.* **16**(5) (2016), 2459–2534.

[Dav20]   J. M. Davies, 'On Lurie's theorem and applications', Preprint, 2022, arXiv:2007.00482.

[Dav21]   J. M. Davies, 'Constructing and calculating Adams operations on topological modular forms', Preprint, 2021, arXiv:2104.13407.

[Dav22a]   J. M. Davies, 'Stable operations and topological modular forms', Dissertation, Utrecht University, 2022.

[Dav22b]   J. M. Davies, 'Stable operations on periodic topological modular forms', in preparation, 2022.

[DFHH14]   C. L. Douglas, J. Francis, A. G. Henriques and M. A. Hill (eds.), *Topological Modular Forms, Based on the Talbot Workshop, North Conway, NH, USA, March 25–31, 2007*, vol. 201 (American Mathematical Society, Providence, RI, 2014).

[DKS89]   W. G. Dwyer, D. M. Kan and J. H. Smith, 'Homotopy commutative diagrams and their realizations', *J. Pure Appl. Algebra* **57**(1) (1989), 5–24.

[DR73]   P. Deligne and M. Rapoport, 'Les schemas de modules de courbes elliptiques', in: *Modular Functions of One Variable II, Proceedings International Summer School, University of Antwerp 1972* (eds. P. Deligne and W. Kuyk), Lecture Notes in Mathematics, 349 (Springer, Berlin, 1973), 143–316.

[GH04]   P. G. Goerss and M. J. Hopkins, 'Moduli spaces of commutative ring spectra', in: *Structured Ring Spectra* (ed. A. Baker) (Cambridge University Press, Cambridge, 2004), 151–200.

[GHMR05] P. G. Goerss, H.-W. Henn, M. Mahowald and C. Rezk, 'A resolution of the $K(2)$-local sphere at the prime 3', *Ann. of Math. (2)* **162**(2) (2005), 777–822.

[GHN17]   D. Gepner, R. Haugseng and T. Nikolaus, 'Lax colimits and free fibrations in ∞-categories', *Doc. Math.* **22** (2017), 1225–1266.

[GM20]   D. Gepner and L. Meier, 'On equivariant topological modular forms', Preprint, 2020, arXiv:2004.10254.

[Goe10]   P. G. Goerss, 'Topological modular forms [after Hopkins, Miller and Lurie]', in: *Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011*, Société Mathématique de France (SMF), Paris, 2010, 221–255.

[HL16]   M. Hill and T. Lawson, 'Topological modular forms with level structure', *Invent. Math.* **203**(2) (2016), 359–416.

[Hoy14]   M. Hoyois, 'A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula', *Algebr. Geom. Topol.* **14**(6) (2014), 3603–3658.

[KM85]   N. M. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, 108 (Princeton University Press, Princeton, NJ, 1985).

[Law]   T. Lawson, 'Why does elliptic cohomology fail to be unique up to contractible choice?' MathOverflow, version 2021.06.07. https://mathoverflow.net/q/394745.

[Lur]   J. Lurie, Kerodon. https://kerodon.net.

[Lur09a] J. Lurie, 'A survey of elliptic cohomology', in: *Algebraic Topology: The Abel Symposium 2007, Proceedings of the Fourth Abel Symposium, Oslo, Norway, August 5–10, 2007* (ed. N. Baas) (Springer, Berlin, 2009), 219–277.

[Lur09b] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, 170 (Princeton University Press, Princeton, NJ, 2009).

[Lur17]   J. Lurie, 'Higher algebra', 2017. Available at https://www.math.ias.edu/lurie/.

[Lur18]   J. Lurie, 'Elliptic cohomology II: orientations', 2018. Available at https://www.math.ias.edu/lurie/.

[Lur19]   J. Lurie, 'Elliptic cohomology III: tempered cohomology', July 2019, version 2019. Available at https://www.math.ias.edu/lurie/.

[Mei21]   L. Meier, 'Connective models for topological modular forms of level n', *Algebr. Geom. Topol.*, to appear.

[Mei22]   L. Meier, 'Topological modular forms with level structure: decompositions and duality', *Trans. Amer. Math. Soc.* **375**(2) (2022), 1305–1355.

[PV19]   P. Pstrkagowski and P. VanKoughnett, 'Abstract Goerss–Hopkins obstruction theory', *Adv. Math.* **395** (2022), 108098.

[ST11]   S. Stolz and P. Teichner, 'Supersymmetric field theories and generalized cohomology', in: *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory* (ed. H. Sati) (American Mathematical Society (AMS), Providence, RI, 2011), 279–340.

[Sta]   The Stacks Project Authors, *Stacks Project*. https://stacks.math.columbia.edu.

[WX17]   G. Wang and Z. Xu, 'The triviality of the 61-stem in the stable homotopy groups of spheres', *Ann. of Math. (2)* **186**(2) (2017), 501–580.

[Zhu14]   Y. Zhu, 'The power operation structure on Morava E-theory of height 2 at the prime 3', *Algebr. Geom. Topol.* **14**(2) (2014), 953–977.

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