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LETTER TO THE EDITOR

Dear Editor,

A formula for tail probabilities of Cox distributions

1. Introduction and results

A Cox distribution with $n > 0$ phases can be defined as the time until absorption into state 0, starting from state n, of the Markov process depicted in Figure 1. The process remains in state k, $1 \leq k \leq n$, an exponentially distributed amount of time with parameter μ_k. Upon departure from state k the process moves to state 0 with probability α_k and moves to state $k-1$ with probability $\bar{\alpha}_k = 1 - \alpha_k$. To avoid trivial situations we assume that $\alpha_k < 1$ for all k.

Cox distributions are useful when approximating general nonnegative distributions using exponential phases. It has been shown that the class of Cox distributions is dense in the class of all nonnegative distributions [3].

Define Y_k as the time until absorption in 0 starting from state k, and let $F_k = 1 - \bar{F}_k$ be the distribution function of Y_k.

Theorem 1. *Let all μ_k, $1 \leq k \leq n$, be different. Then $\bar{F}_k$ for $k = 1, \ldots, n$ is given by*

$$\bar{F}_k(t) = \sum_{i=1}^{k} c_{i,k} \mathrm{e}^{-\mu_i t}$$

for all $t \geq 0$, where

$$c_{i,k} = \begin{cases} 1 & \text{if } i = k = 1, \\ \dfrac{\mu_k c_{i,k-1} \bar{\alpha}_k}{\mu_k - \mu_i} & \text{if } k > 1, i < k, \\ 1 - \displaystyle\sum_{j=1}^{k-1} c_{j,k} & \text{otherwise, i.e. if } i = k > 1. \end{cases}$$

Theorem 1 is a special case of Theorem 2 below, which deals with the case of general parameter values. We formulate Theorem 1 because of its simplicity and relevance for applications. Before continuing with Theorem 2, we introduce some additional notation.

Define:

$$m(j) = \#\{i \mid \mu_i = \mu_j, 1 \leq i < j\},$$

that is, the number of times that μ_j occurs in $\mu_1, \ldots, \mu_{j-1}$;

$$h(j,k) = \begin{cases} \min\limits_i \{\mu_i = \mu_j,\ j < i \leq k\} & \text{if such an } i \text{ exists}, \\ 0 & \text{otherwise}, \end{cases}$$

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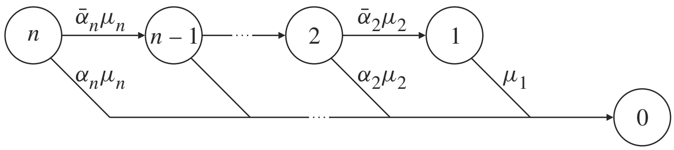

FIGURE 1: A Cox distribution with n phases.

that is, the lowest higher-numbered phase with the same parameter in the Cox distribution with k phases;

$$n(j) = \begin{cases} \max_i\{\mu_i = \mu_j, 1 \le i < j\} & \text{if } m(j) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

that is, the highest lower-numbered phase in the Cox distribution with the same parameter;

$$l(k) = \min_i\{\mu_i = \mu_k, 1 \le i \le k\},$$

that is, the lowest numbered phase with parameter μ_k.

For convenience we also take $c_{0,k} = 0$ for all k.

Theorem 2. *For arbitrary $\mu_k > 0$, $\bar{F}_k$ for $k = 1, \ldots, n$ is given by*

$$\bar{F}_k(t) = \sum_{i=1}^{k} c_{i,k} t^{m(i)} \mathrm{e}^{-\mu_i t} \tag{1}$$

for all $t \ge 0$, where

$$c_{i,k} = \begin{cases} 1 & \text{if } i = k = 1, \\ \dfrac{\mu_k c_{i,k-1}\bar{\alpha}_k - c_{h(i,k),k}(m(i)+1)}{\mu_k - \mu_i} & \text{if } \mu_i \ne \mu_k, \\ \dfrac{\mu_k c_{n(i),k-1}\bar{\alpha}_k}{m(i)} & \text{if } \mu_i = \mu_k, m(i) > 0, k > 1, \\ 1 - \displaystyle\sum_{1 \le j < k : m(j) = 0, j \ne i} c_{j,k} & \text{otherwise, i.e. if } \mu_i = \mu_k, m(i) = 0, k > 1. \end{cases}$$

Proof. We extend the proof of Riordan [2, pp. 110–111], who treated a special case of Theorem 1 (see Section 3 below). From properties of the exponential distribution, we find that, for small $h > 0$,

$$F_k(t+h) = \mu_k h(\alpha_k + \bar{\alpha}_k F_{k-1}(t)) + (1 - \mu_k h)F_k(t) + o(h),$$

where $o(h)$ has the usual meaning that $\lim_{h \to 0} o(h)h^{-1} = 0$. Rewriting and taking the limit as $h \to 0$ gives

$$\bar{F}_k'(t) = \mu_k(\bar{\alpha}_k \bar{F}_{k-1}(t) - \bar{F}_k(t))$$

for $k > 0$. Substituting in (1) leads to

$$\sum_{i=1}^{k} c_{i,k}\big[m(i)t^{m(i)-1}\mathrm{e}^{-\mu_i t} - \mu_i t^{m(i)}\mathrm{e}^{-\mu_i t}\big] = \sum_{i=1}^{k-1} \mu_k \bar{\alpha}_k c_{i,k-1} t^{m(i)} \mathrm{e}^{-\mu_i t} - \sum_{i=1}^{k} \mu_k c_{i,k} t^{m(i)} \mathrm{e}^{-\mu_i t}.$$

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Equating coefficients of $t^m \mathrm{e}^{-\mu t}$ for equal m and μ leads to the given expressions for $c_{i,k}$ when $\mu_i \neq \mu_k$ and when $\mu_i = \mu_k$, $m(i) > 0$. The expression when $\mu_i = \mu_k$, $m(i) = 0$ follows since $\bar{F}_k(0) = 1$.

2. Numerical considerations

Calculating the coefficients directly using (1) can lead to numerical problems. For example, if $t > 1$ and many phases have the same parameter, then, for large i, $c_{i,k}$ will approach 0 and $t^{m(i)}$ will get very big, leading to numerical instabilities. In this case it is better to scale the parameters such that t can be omitted, i.e. μ_i should be replaced by $t\mu_i$. This has been done in the algorithm below. Likewise we should be careful when $\mu_i \approx \mu_j$ for certain i, j; taking them equal might give a very good approximation while avoiding numerical difficulties. Finally, when $\bar{F}_k(t)$ gets close to 1, numerical problems can occur when computing $\bar{F}_{k+n}(t)$. If $\alpha_k = 0$ for all k, then $\bar{F}_k(t)$ is increasing in k. For this reason $\bar{F}_{k+n}(t)$ should be set equal to 1 if $\bar{F}_k(t) \approx 1$ in this case.

Algorithm to compute $F_K(t)$:

for $j, k = 1$ **to** K **do** calculate $m(j)$, $h(j,k)$, $n(j)$

$c_{1,1} = 1$

for $k = 2$ **to** K **do**

  $c_{0,k} = 0$

  for $i = k$ **downto** 1 **do**

    if $\mu_i \neq \mu_k$, **then** $c_{i,k} = \dfrac{t\mu_k c_{i,k-1}\bar{\alpha}_k - c_{h(i,k),k}(m(i)+1)}{t(\mu_k - \mu_i)}$

    elseif $m(i) > 0$, **then** $c_{i,k} = \dfrac{t\mu_k c_{n(i),k-1}\bar{\alpha}_k}{m(i)}$

    endif

  endfor

  $c_{l(k),k} = 1 - \displaystyle\sum_{1 \leq j < k : m(j) = 0, j \neq l(k)} c_{j,k}$

endfor

$$F_K(t) = 1 - \sum_{i=1}^{K} c_{i,k} \mathrm{e}^{-\mu_i t}$$

3. Special cases

If $\mu_i = \mu$ for all i then we obviously get a gamma distribution; in this case $c_{1,k} = 1$ for all k and $c_{i,k} = \mu c_{i-1,k-1}/(i-1)$, implying that $c_{i,k} = \mu^{i-1}/(i-1)!$ in case $1 < i \leq k$. These are indeed the coefficients of the gamma distribution.

Riordan [2, pp. 110–111] (see [1] for a discussion and other references) derived a closed-form expression for $\sum_{k=0}^{\infty} p_k \bar{F}_k(t)$ for the special case $\mu_i = C + i$. However, it is computationally more efficient to compute the coefficients recursively using Theorem 1, instead of using Riordan's closed-form solution.

The current result can be very useful for the calculation of waiting-time distributions in nonstandard queueing systems, such as queues with abandonments where the abandonment rate depends on the position of the customer.

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