LATTICE COVERINGS OF n-SPACE BY SPHERES

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1. Introduction. A *lattice* Λ in euclidean n-space, E_n is a group of vectors under vector addition generated by n independent vectors, $X_1, X_2, \ldots, X_n$, called a *basis* for the lattice. The absolute value of the $n \times n$ determinant the rows of which are the co-ordinates of a basis is called the *determinant of the lattice* and is denoted by $d(\Lambda)$. For any lattice Λ there is a unique minimal positive number r such that, if spheres of radius r are placed with centres at all points of Λ, the entire space is covered. The *density* of this covering may be defined as $(J_n r^n)/(d(\Lambda))$ where J_n is the volume of the unit sphere in n-dimensional euclidean space. This density will be denoted by $\theta_n(\Lambda)$. The *density of the most efficient lattice covering* of n-space by spheres, θ_n, is the absolute minimum of $\theta_n(\Lambda)$ considered as a function from the space of all lattices to the real numbers. It is known that the minimum always exists (viz. **10**). A lattice is called *extreme* if it yields a relative minimum of $\theta_n(\Lambda)$.

In 1952 Davenport (**8**) showed that for large n

$$\theta_n < \left(\frac{11\pi e}{54\cdot\sqrt{3}} + \epsilon\right)^{n/2} < (1.15)^n.$$

In the same year Bambah and Davenport (**3**) demonstrated that

$$4/3 - \epsilon_n < \theta_n$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Later, in 1959, Rogers (**13**) proved that

$$\theta_n = 0(n(\log_e n)^{(\frac{1}{2}\log_2 2\pi e)})$$

and in a joint paper Coxeter, Few, and Rogers (**6**) constructed a sequence of real numbers $\{\tau_n\}$ such that $\tau_n < \theta_n$ and $\tau_n \sim n/(e\sqrt{e})$.

For low dimensions it is a classical result that $\theta_2 = ((2\sqrt{3})/3)\pi$ (viz. **11**). In 1954, Bambah (**1**) proved that $\theta_3 = ((5\sqrt{5})/4)\pi$. A much simpler geometric proof of the three-dimensional result was given by Few (**9**). Barnes (**4**) further clarified the situation by showing that $\theta_2(\Lambda)$ and $\theta_3(\Lambda)$ have only one minimum, that is, the only extreme lattices are best possible.

In four dimensions Bambah (**2**) obtained the following:

$$1.5194 < \frac{4}{15\sqrt{3}}\pi^2 < \theta_4 \leqslant \frac{2}{5\sqrt{3}}\pi^2 < 1.7656.$$

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The upper bound was obtained by constructing a lattice which yielded a cover with that density. Bambah conjectured that this lattice is best possible.

By evaluation of τ_4 in the above-mentioned paper of Coxeter, Few, and Rogers, they obtained

$$1.658 < \tau_4 = \frac{192\pi}{5\sqrt{5}} \left(\tfrac{1}{2} \sec^{-1} 4 - \pi/5\right) < \theta_4.$$

In this work for each n a lattice Λ_n is constructed such that

$$\theta_n \leqslant \theta_n(\Lambda_n) = \frac{1}{\Gamma(\frac{n+2}{2})} \left[\frac{\pi n(n+2)}{12(n+1)^{1-1/n}}\right]^{n/2}.$$

It is further proved that each of these lattices is extreme, but that for sufficiently large n these lattices are not absolutely extreme. Thus for large enough n there are classes of lattices which yield locally best possible coverings, but not best possible coverings. For $n = 1, 2, 3$, these lattices are the best possible covering lattices. The lattice Λ_4 is the lattice which Bambah conjectured to be best possible; thus Bambah's conjecture will be strengthened to the extent that it will have been shown that his conjecture for the best possible covering is at least a locally best possible covering.

2. Fundamental notions. If f is any positive definite n-ary quadratic form with determinant $D(f)$, then the *inhomogeneous minimum* of f, $\mathfrak{m}(f)$, is defined by

$$\mathfrak{m}(f) = \max_{\alpha \in E_n} \min_{L \in \Lambda_0} \{f(\alpha + L)\}$$

where Λ_0 is the integral lattice. If Λ and f are an associated lattice and form (cf. **5** §I.4) then

$$\theta_n(\Lambda) = J_n \phi_n(f)$$

where $\phi_n(f) = (\mathfrak{m}(f)^{n/2})/(D(f)^{1/2})$. To study the extrema of $\theta_n(\Lambda)$ we shall find it more convenient to study the extrema of $\phi_n(f)$.

In this work it will also be necessary to make use of some results of Voronoi **(14)** on the theory of parallelohedra. Given a positive definite n-ary quadratic form, f and a lattice Λ, the *parallelohedron associated with f and* Λ, $P(f, \Lambda)$, is defined to be the set of all points X of E_n such that

$$f(X) \leqslant f(X - L) \tag{1}$$

for all points L of Λ. The bodies so defined have many interesting and useful properties.

First, they are closed, bounded, convex, symmetric about 0, with plane faces and a non-empty interior. Second, if translations of $P(f, \Lambda)$ are placed with centres at each point of Λ, they form a cover of E_n in which no two translations of the body have any interior points in common. Third, the only points of Λ which are necessary in the defining inequalities (1) are those finite numbers of points L of Λ for which

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$$f(L) < f(L''),\ L \neq \pm L'',\ \text{and}\ L \equiv L''\ (\text{mod } 2); \tag{2}$$

that is, for each congruence class of Λ(mod 2) that point, if it exists, at which f assumes its minimum over that congruence class. Last, Voronoi has shown that the number of vertices does not exceed $(n + 1)!$

Some of the above properties become more intuitive if one takes the following approach: Let

$$||X|| = f(X)^{1/2}$$

for any point X of E_n. Then, since f is a positive definite quadratic form, $||X||$ is a Hilbert norm for E_n and $P(f, \Lambda)$ is nothing more than the collection of points nearer the origin than any other point of Λ. Since the property of convexity is preserved under such changes of norm, it follows that a point Y of $P(f, \Lambda)$, for which $||Y||$ is maximal, must be a vertex, and since $P(f, \Lambda)$ is closed and bounded and since this norm is continuous, it follows that at least one vertex has this property.

It may further be shown that, if V is a vertex of $P(f, \Lambda)$ which is determined by the system of n equations

$$f(X) = f(X - L_i),\ L_i \text{ in } \Lambda, \qquad i = 1, 2, \ldots, n, \tag{3}$$

then there is a vertex $V'' = V - L_j$ of $P(f, \Lambda)$ which is determined by the system of n equations

$$f(X) = f(X - L_i''), \qquad i = 1, 2, \ldots, n, \tag{4}$$

where $L_i'' = L_i - L_j$, $i = 1, 2, \ldots, n$, $i \neq j$

$$L_j'' = -L_j \qquad \text{and also} \qquad f(V) = f(V'').$$

When the lattice concerned is the integral lattice Λ_0, one says simply the parallelohedron associated with f, $P(f)$. Also, if f is of the form $f(X) = x_1^2 + x_2^2 + \ldots + x_n^2$, so that the norm induced by f is the usual euclidean norm, $P(f, \Lambda)$ is denoted by $P(\Lambda)$.

The above properties of parallelohedra, while by no means inclusive, should suffice for the purpose of this work. It should also be mentioned that in his work Voronoi considered a linear transformation of the bodies obtained from Λ_0 rather than working with all lattices Λ.

3. Miscellaneous lemmas. In this section several lemmas will be proved which will be necessary for later arguments. As the proofs of these lemmas are quite apart from the general arguments which they will be used to support, they are collected here.

If B is any matrix B' will denote its transpose and, when applicable, $|B|$ its determinant.

LEMMA I. *Let $f(X) = XAX'$ be a positive definite quadratic form where $X = (x_1, x_2, \ldots, x_n)$ and A is an $n \times n$ positive definite symmetric matrix.*

Let $S_1, S_2, \ldots, S_n$ be n independent points of E_n. Then the simultaneous solution X_0 of the system of equations

$$f(X) = f(X - S_i) \qquad i = 1, 2, \ldots, n \tag{1}$$

is given by $2X_0 = (f(S_1), f(S_2), \ldots, f(S_n))(AS')^{-1}$ where S is the $n \times n$ matrix whose jth *row is S_j, $j = 1, 2, \ldots, n$.*

Proof. (1) may be reduced to $2XAS_i' = f(S_i)$ $i = 1, 2, \ldots, n$. It is now clear that this system may be written as one matrix equation

$$2XAS' = (f(S_1), f(S_2), \ldots, f(S_n)).$$

By post multiplying by $(AS')^{-1}$ we obtain the desired results.

LEMMA II. *If A is any non-singular $n \times n$ matrix and $Y = (y_1, y_2, \ldots, y_n)$ is any $1 \times n$ row vector, then*

$$YA^{-1}Y' = \frac{-1}{|A|}\begin{vmatrix} 0 & y_1 \quad y_2 \quad \cdots \quad y_n \\ y_1 & \\ y_2 & \\ \cdot & A \\ \cdot & \\ \cdot & \\ y_n & \end{vmatrix}.$$

Proof. Let $V = A^{-1}Y'$ where $V' = (v_1, v_2, \ldots, v_n)$, then $AV = Y'$ may be solved by Cramer's rule yielding

$$v_i = \frac{|A_i|}{|A|} \tag{2}$$

where A_i is the augmented matrix obtained from A by replacing the ith column by Y'. From the definition of V it follows that

$$YA^{-1}Y' = YV = y_1v_1 + y_2v_2 + \ldots + y_nv_n$$

using (2)

$$YA^{-1}Y' = \frac{1}{|A|}(y_1|A_1| + y_2|A_2| + \ldots + y_n|A_n|)$$

which is precisely what is obtained by expansion of the $(n + 1) \times (n + 1)$ determinant of the lemma along the 1st row.

LEMMA III. *Let $F(y_1, y_2, \ldots, y_m)$ be defined on an open region R. Suppose further that $F(Y)$ is homogeneous of degree zero and also has continuous first partial derivatives. Then*

$$\sum_{k=1}^{m} y_k \frac{\partial F}{\partial y_k} = 0$$

in R.

Proof. It is an immediate corollary of Euler's theorem that any function $G(y_1, y_2, \ldots, y_m)$ having continuous first partials which is homogeneous of degree q satisfies

$$\sum_{k=1}^{m} y_k \frac{\partial G}{\partial y_k} = qG.$$

LEMMA IV. *Let $F(Y)$ be as in Lemma III. Further assume that $F(Y)$ has continuous second partial derivatives in R. Then*

$$\sum_{i=1}^{m} y_i \frac{\partial^2 F}{\partial y_i \partial y_j} = -\frac{\partial F}{\partial y_j} \qquad j = 1, 2, \ldots, n.$$

Proof. One may either differentiate the conclusion of Lemma III, which is applicable by hypothesis, or one may observe that $(\partial F)/(\partial y_j)$ is homogeneous of degree -1 and proceed as in the proof of Lemma III.

LEMMA V. *Let $F(Y)$ be as in Lemma IV. Further assume R contains the point Y_0 all co-ordinates of which are* 1. *Then*

$$\sum_{i,j=1}^{m} \frac{\partial^2 F(Y_0)}{\partial y_i \partial y_j} = 0.$$

Proof. Applying Lemma IV and summing over j one obtains

$$\sum_{j,i=1}^{m} y_i \frac{\partial^2 F}{\partial y_i \partial y_j} = -\sum_{j=1}^{m} \frac{\partial F}{\partial y_j}. \tag{3}$$

From Lemma III applied to the function $F(Y)$ at the point Y_0 it follows that

$$\sum_{j=1}^{m} \frac{\partial F(Y_0)}{\partial y_j} = 0. \tag{4}$$

The conclusion of the lemma now follows by evaluating (3) at Y_0 and applying (4).

4. Parallelohedra of principal forms. In this section several properties of the parallelohedra associated with principal positive definite quadratic forms in the sense of Voronoi (**14**, especially §§ 102–105) will be discussed.

A form is principal in the sense of Voronoi if it can be written as

$$f(X) = \sum_{j=1}^{n} \rho_{0j} x_j^2 + \sum_{1 \leqslant i < j \leqslant n} \rho_{ij}(x_i - x_j)^2$$

where $0 < \rho_{ij}$ for all i, j. Putting $x_0 = 0, f(X)$ may be written

$$f(X) = \sum_{0 \leqslant i < j \leqslant n} \rho_{ij}(x_i - x_j)^2.$$

Adopting the convention that $\rho_{ij} = \rho_{ji}$ if $i < j$, we may prove a lemma which is the basis of arguments of symmetry.

LEMMA VI. *Let π be any permutation of* 0, 1, 2, . . . , n. *Then $f(X)$ is equivalent to*

$$g(X) = \sum_{0 \leqslant i < j \leqslant n} \rho_{\pi(i,j)} (x_i - x_j)^2.$$

Proof. Let π^* denote the inverse permutation of π. The transformation $x_i \to x_{\pi^*(i)} - x_{\pi^*(0)}$ is a unimodular transformation taking f into g.

There is no difficulty in verifying this for the two permutations π_1 which interchange 0 and 1 and π_2 which interchange 1 and 2. The lemma then follows by symmetry.

Let L_i, $i = 1, 2, \ldots, n$ denote the point of Λ_0 with ith co-ordinate 1 and the others 0 and L_0 denote the point with all co-ordinates -1. Let $L_{ij\ldots t} = L_i + L_j + \ldots + L_t$ where $i, j, \ldots, t$ are distinct integers between 0 and n inclusive.

The points $L_1, L_{12}, \ldots, L_{12\ldots n}$ determine, through equations (1) §II, a vertex denoted by $V_{1,2\ldots n,0}$. For any permutation π the points

(1) $$L_{\pi(1)}, L_{\pi(1,2)}, \ldots L_{\pi(1,2,\ldots n)}$$

determine the vertex denoted by $V_{\pi(1,2,\ldots n,0)}$. Distinct permutations yield distinct vertices and thus all $(n+1)!$ vertices are obtained in this manner.

By application of Lemma 1 we obtain

(2) $$V = V_{1,2,\ldots,n,0} = (1/2)(f(L_1), f(L_{12}), \ldots, f(L_{12\ldots n}))(AS')^{-1}$$

where S is the matrix with 0's above and 1's on and below the main diagonal and A given by

(3) $$A = \begin{bmatrix} \rho_{01}+\rho_{12}+\ldots+\rho_{1n} & -\rho_{12} & -\rho_{13} & \ldots & -\rho_{1n} \\ -\rho_{12} & \rho_{02}+\rho_{12}+\rho_{23}+\ldots+\rho_{2n} & -\rho_{23} & \ldots & -\rho_{2n} \\ -\rho_{13} & -\rho_{23} & \rho_{03}+\rho_{13}+\rho_{23}+\ldots+\rho_{3n} & \ldots & -\rho_{3n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ -\rho_{1n} & -\rho_{2n} & -\rho_{3n} & \ldots & \rho_{0n}+\rho_{1n}+\rho_{2n}+\ldots+\rho_{n-1,n} \end{bmatrix}$$

is the positive definite symmetric matrix of $f(X)$. Direct computation yields

$$f(L_{12\ldots k}) = \sum_{i=k+1}^{n+1} \rho_{1i} + \rho_{2i} + \ldots + \rho_{ki}.$$

It should be noted that subscripts should be taken mod $(n+1)$, for example, $\rho_{1,n+1} = \rho_{10} = \rho_{01}$. And from the symmetry lemma

(4) $$f(L_{\pi(12\ldots k)}) = \sum_{i=k+1}^{n+1} \rho_{\pi(1i)} + \rho_{\pi(2i)} + \ldots + \rho_{\pi(ki)}.$$

It is easy to verify that

$$(S')^{-1} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & & -1 & 0 \\ 0 & 0 & 0 & & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Thus from (2) we get

(5) $$Y = 2VA = (f(L_1), f(L_{12}) - f(L_1), \dots, f(L_{12\dots n}) - f(L_{12\dots(n-1)})).$$

More explicitly,

(6)
$$\begin{aligned} y_1 &= \rho_{01} + \rho_{12} + \rho_{13} + \dots + \rho_{1n} \\ y_2 &= \rho_{02} - \rho_{12} + \rho_{23} + \dots + \rho_{2n} \\ &\cdot \\ &\cdot \\ &\cdot \\ y_k &= \rho_{0k} - \rho_{1k} - \dots - \rho_{k-1,k} + \rho_{k,k+1} + \dots + \rho_{kn} \\ &\cdot \\ &\cdot \\ &\cdot \\ y_n &= \rho_{0n} - \rho_{1n} - \dots - \rho_{n-1,n} \end{aligned}$$

Hence $f(V) = VAV' = (1/4)(2VA)A^{-1}(2VA)' = YA^{-1}Y'/4$. Applying Lemma II

(7) $$f(V) = \frac{-1}{4|A|}\begin{vmatrix} 0 & Y \\ Y' & A \end{vmatrix}$$

where Y is given by (5) and (6). To find $f(V^*)$ where V^* is any other vertex it is only necessary to perform the appropriate permutation on the subscripts of the ρ_{ij}.

$L_i, L_{ij}, \dots, L_{ij\dots t}$ determine the vertex $V = V_{ijk,\dots,tu}$ and $-L_i, L_{ij} - L_i, \dots, L_{ijk\dots t} - L_i$ determine the vertex $V^* = V - L_i$ and $f(V^*) = f(V)$, as pointed out in §2 particularly in (4) and (5) of that section.

Further since

$$\begin{aligned} -L_i &= L_{j,k,\dots,t,u} \\ L_{ij} - L_i &= L_j \\ L_{ijk} - L_i &= L_{jk} \\ &\cdot \\ &\cdot \\ &\cdot \\ L_{ijk\dots t} - L_i &= L_{jk\dots t} \end{aligned}$$

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it follows that $V^* = V_{jkl\ldots tui}$; hence $f(V^*)$ is obtained from $f(V)$ by a cyclic permutation of the subscripts of the ρ_{ij}. Thus any cyclic permutation of the subscripts of the ρ_{ij} in $f(V)$ leave the function invariant as a function of the ρ_{ij}.

5. The lattices Λ_n. The lattice Λ_n is the lattice obtained from the integral lattice by multiplying Λ_0 by the following matrix

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} & \cdots & \frac{-1}{\sqrt{k(k+1)}} & \cdots & \frac{-1}{\sqrt{n(n+1)}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{12}} & \cdots & \frac{-1}{\sqrt{k(k+1)}} & \cdots & \frac{-1}{\sqrt{n(n+1)}} \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & \frac{\sqrt{k}}{\sqrt{k+1}} & \cdots & \frac{-1}{\sqrt{n(n+1)}} \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \cdots & \frac{\sqrt{n}}{\sqrt{n+1}} \end{bmatrix} \frac{\sqrt{n+1}\,2\sqrt{3}}{\sqrt{n(n+2)}}.$$

By well-known methods (cf. **5**) this matrix yields a positive definite quadratic form which is equivalent, through multiplication by the constant $(n(n+2))/(2^2 3)$ to the form $f(X)$ with matrix

$$A = \begin{bmatrix} n & -1 & \ldots -1 & \ldots -1 \\ -1 & n & -1 & -1 \\ \vdots & & & \\ -1 & -1 & n & -1 \\ \vdots & & & \\ -1 & -1 & -1 & n \end{bmatrix}.$$

This form may be written more explicitly

$$f(X) = n\sum_{i=1}^{n} x_i^2 - 2\Big(\sum_{1\leqslant i<j\leqslant n} x_i x_j\Big).$$

This form is the principal form in the sense of Voronoi with all the ρ_{ij} equal to unity, that is,

$$f(X) = \sum_{0\leqslant i<j\leqslant n} (x_i - x_j)^2. \tag{1}$$

In this section the following theorem will be proved:

THEOREM 1. *For every natural number n,*

$$\theta_n \leqslant \theta_n(\Lambda_n) = \frac{1}{\Gamma(\frac{n+2}{2})}\left[\frac{\pi n(n+2)}{12(n+1)^{1-1/n}}\right]^{n/2}$$

where Λ_n is the above-constructed lattice. Further for $n = 1, 2, 3$, equality holds.

It is clear from the definition of θ_n that $\theta_n \leqslant \theta_n(\Lambda_n)$; and it is also readily verified by comparison with the known values of θ_n for $n = 2, 3$, stated in the introduction that equality holds in those cases.

In the case $n = 1$ the sphere degenerates to the interval and the best density is clearly 1.

As pointed out in the discussion of the Voronoi Body, $P(f)$, in §2,

$$\mathfrak{m}(f) = \max_{V \in \mathfrak{V}} \{f(V)\} \tag{2}$$

where $\mathfrak{V}$ is the set of the $(n+1)!$ vertices of $P(f)$.

Since all the ρ_{ij} have the same value for this form

$$f(V_{123\ldots n0}) = f(V_{\pi(123\ldots n0)}) = \mathfrak{m}(f)$$

From formula (6) §4 it is easy to see that, evaluated at $\rho_{ij} = 1$, $y_k = n - 2(k-1)$.

From §4, formula (7), evaluating at $\rho_{ij} = 1$ it follows that by adding the last n rows to the bottom row, and then by adding this new bottom row to each of the 2nd through nth rows, we obtain

$$\mathfrak{m}(f) = \frac{-1}{4|A|}\cdot\begin{vmatrix} 0 & n & n-2 & \ldots & n-2(k-1) & \ldots & 2-n \\ 2n & n+1 & 0 & & 0 & & 0 \\ 2(n-1) & 0 & n+1 & & 0 & & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 2(n-k) & 0 & 0 & & n+1 & & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 2(2) & 0 & 0 & & 0 & & 0 \\ n & 1 & 1 & \ldots & 1 & \ldots & 1 \end{vmatrix}$$

By next subtracting the last column from the middle $(n-1)$ columns the determinant is reduced to a form in which it may be easily evaluated. Also the lower right $n \times n$ minor is $|A|$.

Thus $|A| = (n+1)^{n-1}$ and the large determinant, $\bar{D}$, is given by $\bar{D} = -(n/3)(n+2)(n+1)^{(n-1)}$. Combining these results

$$\mathfrak{m}(f) = \frac{n(n+2)}{12}.$$

It follows that

$$\phi_n(f) = \left(\frac{n(n+2)}{12}\right)^{n/2}\left(\frac{1}{(n+1)^{(n-1)}}\right)^{\frac{1}{2}}$$

From the formula

$$J_n = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}$$

and the relationship between $\phi_n(f)$ and $\theta_n(\Lambda)$ the conclusion of the theorem follows.

6. The extremity of the lattices Λ_n. In this section the following theorem will be proved:

THEOREM 2. *For every natural number n the lattice Λ_n is extreme; that is, it yields a relative minimum of the function $\theta_n(\Lambda)$.*

In this section $f(X)$ will always denote the form constructed in the last section for which all $\rho_{ij} = 1$. In order to show that $f(X)$ is extreme it is sufficient to show that for all sufficiently small ϵ_{ij} $i, j = 0, 1, 2, \ldots, n$ the form

$$g(X) = \sum_{0 \leqslant i < j \leqslant n} \rho_{ij}(x_i - x_j)^2$$

where $\rho_{ij} = 1 + \epsilon_{ij}$ satisfies $\phi_n(f) \leqslant \phi_n(g)$.

Since $\phi_n(g)$ is always positive it will suffice to consider the function $F(\rho)$ where $\rho = (\rho_{01}, \rho_{02}, \ldots, \rho_{0n}, \rho_{12}, \ldots, \rho_{n-1,n})$ defined by

$$F(\rho) = \phi_n(g)^{2/n} = \frac{\mathfrak{m}(g)}{D(g)^{1/n}}.$$

The notation $F(\rho)$ is adopted in order to emphasize that $F(\rho)$ is a function from the positive generalized octant of euclidean n^*-space where

$$n^* = \frac{n(n+1)}{2}.$$

$\mathfrak{m}(g)$, according to §5, formula (2), is the maximum of $g(V)$ over all vertices of $P(g)$, and as demonstrated in §4 this is the same as the maximum of $g(V_{\pi(1,2,\ldots,n,0)})$ taken over all permutations π_i, $i = 1, 2, \ldots, (n+1)!$ of the symbols $0, 1, \ldots, n$. Thus by defining

$$F_i(\rho) = g(V_{\pi_i(1,2,\ldots,n,0)})/D(g)^{1/n} \tag{1}$$

where it may be assumed that π_1 is the identity permutation, it follows that $F(\rho) = \max\{F_i(\rho): i = 1, 2, \ldots, (n+1)!\}$. In particular using (1) of the present section and formula (7) of §4,

$$F_1(\rho) = -\frac{\begin{vmatrix} 0 & Y \\ Y' & A \end{vmatrix}}{4|A|^{1+1/n}}.$$

From the above representation of $F_1(\rho)$ it is clear that $F_1(\rho)$ is an algebraic function of the ρ_{ij} which is homogeneous of degree zero. Further in the region $\rho_{ij} > 0$ the numerator and denominator are positive. By the symmetry lemma these remarks apply equally well to the $F_i(\rho)$. To prove the theorem it thus suffices to show that the function $F(\rho)$ has a minimum at the point ρ_0 all the co-ordinates of which are one. Since any permutation of the co-ordinates of ρ_0 leaves this point invariant and since $F_i(\rho)$ is obtained from $F_1(\rho)$ by a permutation of co-ordinates, $F_i(\rho_0) = F_1(\rho_0)$ for all $i = 1, 2, \ldots, (n+1)!$ Thus ρ_0 will yield a minimum of $F(\rho)$ if for every sufficiently short vector

$$\epsilon = (\epsilon_{01}, \epsilon_{02}, \ldots, \epsilon_{0n}, \ldots, \epsilon_{n-1,n})$$

there is some $i \leqslant (n+1)!$ such that

$$F_1(\rho_0) = F_i(\rho_0) \leqslant F_i(\rho_0 + \epsilon). \tag{2}$$

The demonstration of the inequality (2) will be accomplished by the following steps.

STEP 1. *It will be shown that the sum over* $i = 1, 2, \ldots, (n+1)!$ *of the directional derivatives of* $F_i(\rho)$ *evaluated at* ρ_0 *is zero for any fixed direction.*

STEP 2. *It will be shown that the sum over* $i = 1, 2, \ldots, (n+1)!$ *of the second-order terms in the Taylor approximation series for* $F_i(\rho)$ *expanded about* ρ_0 *is a positive semi-definite quadratic form in the* ϵ_{ij}.

STEP 3. *It will be shown that the above semi-definite form can be zero only when all* ϵ_{ij} *are equal.*

This will be sufficient for the demonstration of the theorem; for, let ϵ be any vector in E_{n^*} where

$$n^* = \frac{n(n+1)}{2}.$$

If ϵ is in such a direction that for some i the directional derivative of $F_i(\rho_0)$ is positive in that direction, then it may be supposed that ϵ is sufficiently small that the first order terms dominate the approximation of $F_i(\rho_0 + \epsilon)$ in that direction, and since the first order approximation is positive it follows that (2) holds with strict inequality.

In the case when no directional derivative is positive they must all be zero since their sum is zero.

In this case, if any $F_i(\rho)$ have non-zero sums of second order terms (2) will hold with strict inequality for sufficiently small ϵ. If no $F_i(\rho)$ has positive second order terms then all the ϵ_{ij} are equal, say $\epsilon_{ij} = \epsilon_0$. Since the $F_i(\rho)$ are homogeneous of degree zero

$$F_i(\rho + \epsilon) = F_i((1 + \epsilon_0)\rho) = (1 + \epsilon_0)^0 F_i(\rho_0) = F_i(\rho_0).$$

A compactness argument proves that (2) holds for all sufficiently small ϵ in the hyperplane $\epsilon\rho_0' = 0$. The homogeneity implies that for all sufficiently small ϵ, (2) holds with strict inequality unless g is a multiple of f.

Proof of Step 1. For any permutation π_k, π^*_k denotes the inverse permutation and

$$\pi_k(\rho) = (\rho_{\pi_k(01)}, \rho_{\pi_k(02)}, \ldots, \rho_{\pi_k(12)}, \ldots, \rho_{\pi_k(n-1,n)}).$$

Thus

$$F_k(\rho) = F_1(\pi_k(\rho)).$$

Thus

$$\frac{\partial F_k(\rho)}{\partial \rho_{ij}} = \left(\frac{\partial F_1(\rho)}{\partial \rho_{\pi^*_k(ij)}}\right)_{\rho=\pi_k(\rho)}.$$

But for ρ_0, $\rho_0 = \pi_k(\rho_0)$. Thus

$$\frac{\partial F_k(\rho_0)}{\partial \rho_{ij}} = \frac{\partial F_1(\rho_0)}{\partial \rho_{\pi^*_k(ij)}}. \tag{3}$$

Step 1 claims that

$$\sum_{k=1}^{(n+1)!} \sum_{i<j} \epsilon_{ij} \frac{\partial F_k(\rho_0)}{\partial \rho_{ij}} = 0.$$

The coefficient of ϵ_{ij} in this expression is

$$\sum_{k=1}^{(n+1)!} \frac{\partial F_k(\rho_0)}{\partial \rho_{ij}}.$$

It follows from (3) that the coefficient is

$$\sum_{k=1}^{(n+1)!} \frac{\partial F_1(\rho_0)}{\partial \rho_{\pi_k(ij)}}.$$

As k ranges from 1 to $(n+1)!$ the pair ij with $i < j$ is transformed into every other such pair precisely $2(n-1)!$ times: thus the coefficient of ϵ_{ij} is

$$2(n-1)! \sum_{0\leqslant i<j\leqslant n} \frac{\partial F_1(\rho_0)}{\partial \rho_{ij}}.$$

By application of Lemma III the validity of Step 1 is thus established.

Proof of Step 2. The second order term in the approximation of $F_t(\rho_0 + \epsilon)$ about ρ_0 is

$$\frac{1}{2} \sum_{0\leqslant k<l\leqslant n} \sum_{0\leqslant i<j\leqslant n} \epsilon_{kl}\, \epsilon_{ij} \frac{\partial^2 F_t(\rho_0)}{\partial \rho_{kl} \partial \rho_{ij}}. \tag{4}$$

In the sequel, without harm to the argument, the factor 1/2 will be omitted. The expression (4) may be written in matrix notation $\epsilon H_t \epsilon'$ where

$$H_t = \begin{bmatrix} \dfrac{\partial^2 F_t(\rho_0)}{(\partial \rho_{01})^2} & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{01}\partial \rho_{02}} & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{01}\partial \rho_{03}} & \cdots & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{01}\partial \rho_{n-1,n}} \\ \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{01}\partial \rho_{02}} & \dfrac{\partial^2 F_t(\rho_0)}{(\partial \rho_{02})^2} & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{02}\partial \rho_{03}} & \cdots & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{02}\partial \rho_{n-1,n}} \\ \vdots & & & & \\ \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{01}\partial \rho_{n-1,n}} & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{02}\partial \rho_{n-1,n}} & \dfrac{\partial^2 F_t(\rho_0)}{\partial \rho_{03}\partial \rho_{n-1,n}} & \cdots & \dfrac{\partial^2 F_t(\rho_0)}{(\partial \rho_{n-1,n})^2} \end{bmatrix}. \tag{5}$$

It has already been observed, Lemma V, that the sum of all the terms in this matrix is zero.

We shall need the analogue of (3), namely,

$$\frac{\partial^2 F_t(\rho_0)}{\partial\rho_{kl}\partial\rho_{ij}} = \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{\pi^*_t(kl)}\partial\rho_{\pi^*_t(ij)}}. \tag{6}$$

The sum over all the functions $F_t(\rho)$ of the second order approximation terms is

$$h(\epsilon) = \sum_{t=1}^{(n+1)!} \sum_{k<l} \sum_{i<j} \epsilon_{kl}\epsilon_{ij} \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{\pi^*_t(kl)}\partial\rho_{\pi^*_t(ij)}}.$$

In order to calculate $a_{kl,ij}$, the coefficient of $\epsilon_{kl}\epsilon_{ij}$ in this sum, three cases must be distinguished:

CASE 1. $(k, l) = (i, j)$. In this case the permutations take (k, l) with $k < l$, onto every other such pair precisely $2(n-1)!$ times. Thus the coefficient of $\epsilon_{kl}{}^2$ is given by

$$a_{kl,kl} = 2(n-1)! \sum_{0 \leqslant i<j \leqslant n} \frac{\partial^2 F_1(\rho_0)}{(\partial\rho_{ij})^2}. \tag{7}$$

It will be noticed that this expression is independent of k and l and hence will be denoted by a.

CASE 2. The quadruple (k, l, i, j) consists of precisely three distinct numbers. Since the order of differentiation is not important we assume temporarily that $k \leqslant i$ in

$$\frac{\partial^2 F_t(\rho_0)}{\partial\rho_{kl}\partial\rho_{ij}}.$$

The quadruple (k, l, i, j) can be permuted into any quadruple of the form (r, s, r, t) (r, s, s, t) (r, t, s, t), $r < s < t$ and only such quadruples, because of the temporary convention. Further it will be permuted into each such quadruple precisely $2(n-2)!$ times. Thus

$$a_{kl,ij} = 2(n-2)! \sum_{0 \leqslant r<s<t \leqslant n} \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{rs}\partial\rho_{rt}} + \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{rs}\partial\rho_{st}} + \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{rt}\partial\rho_{st}}. \tag{8}$$

This expression is independent of (k, l, i, j) and will be denoted by b.

CASE 3. $k, l, i,$ and j are all distinct. In this case let C be the set of the

$$\frac{(n+1)n(n-1)(n-2)}{4}$$

quadruples (r, s, t, u) for which $r < s$ and $t < u$. (k, l, i, j) will be permuted onto each of these quadruples precisely $4(n-3)!$ times, thus

$$a_{kl,ij} = 4(n-3)! \sum_{(r,s,t,u) \text{ in } C} \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{rs}\partial\rho_{tu}}. \tag{9}$$

This expression is independent of (k, l, i, j) and will be denoted by c.

We will now derive an elementary identity between a, b, and c which will give us a more calculable method to obtain c.

We observe first that in matrix notation $h(\epsilon) = \epsilon H \epsilon$ where the matrix

$$H = \sum_{1}^{(n+1)!} H_t$$

and H_t is given by (5). Since H_t is obtained from H by appropriately permuting the subscripts, each row of H_1, possibly rearranged, is permuted into another row. Considering all permutations, each row is permuted into every row precisely $2(n-1)!$ times. Thus the sum of the terms in any row of H is $2(n-1)!$ times the sum of all the terms in H_1. By Lemma V, this sum is zero.

In each row of H, a will occur precisely once, on the diagonal; b will occur precisely $2(n-1)$ times, and c the remaining

$$\frac{(n-1)(n-2)}{2}$$

times.

Since the sum of the row is zero we get

$$a + 2(n-1)b + \frac{(n-1)(n-2)}{2} c = 0. \tag{10}$$

Below is an outline of the major steps in computing a and b.

$\bar{D}$ denotes the $(n+1) \times (n+1)$ determinant of the numerator of $F_1(\rho)$ and $|A|$ denotes the determinant of the quadratic form associated with the point ρ of E_n^*. *All calculations are evaluated at the point* ρ_0.

As has already been computed

$$\bar{D} = \frac{n}{3}(n+2)(n+1)^{n-1}$$

and

$$|A| = (n+1)^{n-1}.$$

An easy computation from §4, formula (3) yields

$$\frac{\partial |A|}{\partial \rho_{01}} = 2(n+1)^{n-2}.$$

By Lemma VI $|A|$ is invariant under any permutation of the subscripts of the ρ_{ij}. It follows that

$$\frac{\partial |A|}{\partial \rho_{ij}} = \frac{\partial |A|}{\partial \rho_{01}} = 2(n+1)^{n-2}.$$

As indicated at the end of §4 $F_1(\rho)$ is invariant under cyclic permutations. Since $|A|$ is also invariant under cyclic permutations it follows that $\bar{D}$ is invariant under cyclic permutation. Thus

$$\frac{\partial F_1(\rho_0)}{\partial \rho_{0k}} = \frac{\partial F_1(\rho_0)}{\partial \rho_{i,i+k}} \qquad i = 1, 2, \ldots, n$$

and

$$\frac{\partial \bar{D}(\rho_0)}{\partial \rho_{0k}} = \frac{\partial \bar{D}(\rho_0)}{\partial \rho_{i,i+k}} \qquad i = 1, 2, \ldots, n$$

where $i + k$ is reduced $\bmod (n + 1)$ if necessary.

It may thus be computed that

$$\frac{\partial \bar{D}}{\partial \rho_{i,i+k}} = \frac{\partial \bar{D}}{\partial \rho_{0k}} = \frac{2}{3}(n + 1)^{n-3}[n(n + 1)(n + 2) + 6k(n + 1 - k)].$$

It follows from the above that

$$\frac{\partial F_1}{\partial \rho_{i,i+k}} = \frac{\partial F_1}{\partial \rho_{0k}} = \frac{(n + 1)^{1/n}}{6(n + 1)^3}[6k(n + 1 - k) - (n + 1)(n + 2)].$$

It will be noticed in formula (3) §4, that $|A|$ is linear in ρ_{01}; and thus by the invariance under permutation of the subscripts this is true of every ρ_{ij}. It results that

$$\frac{\partial^2 |A|}{(\partial \rho_{ij})^2} = 0.$$

Further computation and application of permutational invariance yield

$$\frac{\partial^2 |A|}{\partial \rho_{i,i+k} \partial \rho_{i(i+l)}} = \frac{\partial^2 |A|}{\partial \rho_{0k} \partial \rho_{0l}} = 3(n + 1)^{n-3} \qquad \begin{matrix} k \neq l \\ i, k, l = 1, 2, \ldots, n. \end{matrix}$$

Since the above is independent of i, k, and l

$$\frac{\partial^2 |A|}{\partial \rho_{ik} \partial \rho_{il}} = 3(n + 1)^{n-3} \qquad \begin{matrix} i \neq k \neq l \neq i \\ i, k, l = 0, 1, 2, \ldots, n. \end{matrix}$$

For $\bar{D}$ one may compute that

$$\frac{\partial^2 \bar{D}}{(\partial \rho_{ij})^2} = \frac{\partial^2 \bar{D}}{(\partial \rho_{0k})^2} = 4(n + 1)^{n-2} \qquad \begin{matrix} i \neq j \quad k = j - i \\ j, i = 0, 1, 2, \ldots, n. \end{matrix}$$

and

$$\frac{\partial^2 \bar{D}}{\partial \rho_{i,i+k} \partial \rho_{i,i+l}} = \frac{\partial^2 \bar{D}}{\partial \rho_{0k} \partial \rho_{0l}} = (n + 1)^{n-4}[(n + 1)(n^2 + 4n + 2) + 4(n + 1)(l + k) - 8(l^2 - lk + k^2)].$$

Since the evaluation at ρ_0 of the various partial derivatives of the functions of which $F_1(\rho)$ is composed are now known, it may readily be verified that

$$\frac{\partial^2 F_1(\rho_0)}{(\partial \rho_{0k})^2} = \frac{\partial^2 F_1(\rho_0)}{(\partial \rho_{i,i+k})^2} = \frac{2(n + 1)^{1/n}}{3n(n + 1)^2}\left[2n + 1 - \frac{6k(n + 1 - k)}{n + 1}\right] \quad i,k = 1,2,\ldots,n$$

and

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$$\frac{\partial^2 F_1(\rho_0)}{\partial\rho_{0k}\,\partial\rho_{0t}} = \frac{\partial^2 F_1(\rho_0)}{\partial\rho_{i,i+k}\partial\rho_{i,i+t}}$$
$$= \frac{(n+1)^{1/n}}{n(n+1)^3}\Bigg[\frac{7n^2+12n+8}{12} - \frac{n(t-k)^2}{n+1} - \frac{n+2}{n+1}$$
$$(k(n+1-k) + t(n+1-t))\Bigg] \qquad k \neq t \quad i, k, t = 1, 2, \ldots, n.$$

The common factor of

$$\frac{(n+1)^{1/n}}{n(n+1)^3}$$

shall be ignored in future argument. It thus results that

$$a = \frac{2}{3}(n+1)!(n^2-1) \qquad \text{and} \qquad b = \frac{1}{12}(n+1)!(n^2-4n-8).$$

The common positive factor of

$$\frac{(n+1)!}{3}$$

can safely be ignored.

Applying formula (10) we obtain $c = -(n+2)$.

This completes the description of the matrix H; it remains to show that the form $h(\epsilon)$ is a positive semi-definite form.

$h(\epsilon)$ may be expressed by

$$(11) \qquad h(\epsilon) = -\tfrac{1}{4}c\sum_{i<j}(\epsilon_{0i}+\epsilon_{1i}+\ldots+\epsilon_{in}-\epsilon_{0j}-\epsilon_{ij}-\ldots-\epsilon_{jn})^2$$
$$-\left(\frac{(n-3)c}{4}+b\right)\sum_{i=0}^{n-2}\sum_{i<j<k\leqslant n}(\epsilon_{ij}-\epsilon_{ik})^2.$$

The coefficient of the first sum is $(n+2)/4$ and that of the second is $(3n+2)/4$; since both of these are positive, $h(\epsilon)$ is positive semi-definite.

Proof of Step 3. It is clear from (11) that if all the ϵ_{ij} are equal, $h(\epsilon) = 0$. If $h(\epsilon) = 0$ it follows from the second sum of (11) that all the ϵ_{ij} are equal. Steps 1, 2, and 3 and hence the theorem are thus established.

7. The local character of Λ_n for large values of n. It was pointed out in the introduction that Hlawka has proved that there always exists a best possible covering of euclidean space by spheres. The question of whether or not there are solutions which are locally best possible, but not best possible has not been answered. In this section that question is answered in the affirmative for large n. To do this it is sufficient to show that the function $\theta_n(\Lambda)$, considered as a function from $E_n{}^*$ to the real numbers has a relative minimum which is not an absolute minimum.

It has already been shown, Theorem 2, that $\theta_n(\Lambda)$ has a relative minimum at Λ_n. The question is answered by the following theorem.

THEOREM 3. *For all sufficiently large values of n the following inequality holds:*

$$\theta_n < \theta_n(\Lambda_n).$$

Proof. Theorem 1 states

$$\theta_n(\Lambda_n) = \frac{1}{\Gamma(\frac{n+2}{2})}\left[\frac{n(n+2)\pi}{12(n+1)^{1-1/n}}\right]^{n/2}.$$

In view of Rogers' result that

$$\theta_n = 0(n(\log_e n)^{((1/2)\log_2 2\pi e)})$$

and that

$$0(n(\log_e n)^{(1/2)\log_2 2\pi e}) < 0\left(\left(\frac{e\pi}{6}\right)^{n/2}\right)$$

it follows that

$$\theta_n < \theta_n(\Lambda_n)$$

for all sufficiently large values of n.

8. Conclusion. Let us denote by $f_n(X)$ the form with which we have been working, §5, formula (1).

It is interesting to note that f_n is adjoint to the form U_n of Korkine and Zolotareff **(12)** (which is equivalent to the A_n of Coxeter **(6)**) and hence f_n is equivalent to Coxeter's A_n^{n+1}. The forms U_n and hence the A_n, are perfect and eutactic and therefore extreme in the sense of the packing problem. It is natural to ask if perhaps the forms adjoint to the other perfect eutactic forms are also extreme in the covering sense. Unfortunately, the answer is negative.

There are two perfect eutactic quaternary forms, the U_4 and V_4 of Korkine and Zolotareff (Coxeter: A_4, D_4).

We now consider the form $h(X) = 2x_1^2 + 2x_2^2 + (x_1 + x_2 - 2x_3)^2 + (x_1 + x_2 - 2x_4)^2$ which is adjoint to, and hence also equivalent to, V_4. $h(X)$ is simply 2ω where ω is the form considered by Voronoi **(14)** §115–116. Thus $h(X)$ may be considered as a reduced degenerate form of the IIIrd type in the Voronoi reduction.

For this form three of the congruence classes of Λ_0 (mod 2) do not have unique minimal representatives; namely, (0, 0, 1, 1), (0, 0, 1, −1), (0, 2, 1, 1), (2, 0, 1, 1); (1, 1, 1, 0), (1, −1, 1, 0), (1, −1, −1, 0), (1, 1, 1, 2); and (1, 1, 0, 1), (1, −1, 0, 1) (1, −1, 0, −1) (1, 1, 2, 1) are the sets of four points at which each of their congruence classes assumes minimal values. This collapsing of the unique representation is accompanied by a corresponding collapsing of some of the faces of the parallelohedron $P(h)$. In fact, the 12 distinct classes

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of vertices of the primitive parallelohedron of type III collapse into precisely 3 distinct classes, each representing four of the normal classes, as follows:

$V_1 = \frac{1}{2}(0, 0, 1, 1)$ represents the Voronoi vertex types I, IV, V, VIII.
$V_2 = \frac{1}{2}(1, 1, 1, 0)$ " " " II, III, X, XI.
$V_3 = \frac{1}{2}(1, 1, 0, 1)$ " " " VI, VII, IX, XII.

For each of these vertices $h(V) = 2$ and since $D(h) = 64$ we see that

$$\phi_4(h) = \frac{1}{2} > \frac{4}{5\sqrt{5}} = \phi_4(f_4).$$

Speaking intuitively, a lattice Λ for which $P(\Lambda)$ is a primitive parallelohedron with the vertices lying on a sphere is a likely candidate to be an extreme lattice; for, in order to obtain a better covering, one would have to vary the lattice in such a manner that the farthest vertex from the origin remains on the sphere, and yet the volume of the body increases. In the case of imprimitive bodies there is the possibility of pulling one or more of the vertices apart into a face and in this manner increase the volume of the body without sending any of the vertices outside the sphere.

With the above in mind we now consider the form $g(X) = h(X) + 2\epsilon x_3 x_4$, $2 > \epsilon > 0$. This is the degenerate reduced form of the third Voronoi type for which $\mu_1 = 2 - \epsilon$ and $\mu_6 = \epsilon$ and all the other coefficients are zero in the standard representation.

In the transition from $P(h)$ to $P(g)$ the vertex V_1 becomes a parallelogram with the following vertices:

$$V_1' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(4\epsilon^2 - \epsilon^3, 4\epsilon^2 - \epsilon^3, 32 + 8\epsilon - 4\epsilon^2, 32 + 8\epsilon - 4\epsilon^2),$$

a type I vertex;

$$V_1'' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(8\epsilon^2 - 2\epsilon^3, -16\epsilon + \epsilon^3, 32 - 2\epsilon^2, 32 - 2\epsilon^2),$$

a type IV vertex;

$$V_1''' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(-16\epsilon + 4\epsilon^2, -16\epsilon + 4\epsilon^2, 32 - 8\epsilon, 32 - 8\epsilon)$$

a type V vertex;

$$V_1'''' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(-16\epsilon + \epsilon^3, 8\epsilon^2 - 2\epsilon^3, 32 - 2\epsilon^2, 32 - 2\epsilon^2),$$

a type VIII vertex.

The vertex V_2 splits into the line segment bounded by the following vertices:

$$V_2' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(32 + 16\epsilon - 6\epsilon^2, 32 + 16\epsilon - 6\epsilon^2, 32 + 16\epsilon - 6\epsilon^2, 0),$$

of types II and XI;

$$V_2'' = \frac{1}{4(16 + 8\epsilon - 3\epsilon^2)}(32 + 8\epsilon - 4\epsilon^2, 32 + 8\epsilon - 4\epsilon^2, 32 + 8\epsilon, -16\epsilon),$$

of types III and X.

The situation regarding V_3 is identical to that of V_2 except that the x_3 and the x_4 co-ordinates are interchanged. V_3' is of types VI and XII; V_3'' is of types VII and IX.

By examination of the above vertices it is immediate that

$$m(g) = 2\frac{(16+8\epsilon-2\epsilon^2)}{(16+8\epsilon-3\epsilon^2)} = g(V_2'') = g(V_3'').$$

An easy computation yields $D(g) = 4(16+8\epsilon-3\epsilon^2)$. It therefore follows that

$$\phi_4(g) = \frac{2(16+8\epsilon-2\epsilon^2)^2}{(16+8\epsilon-3\epsilon^2)^{5/2}} < \frac{1}{2} = \phi_4(h),$$

for all sufficiently small values of ϵ.

We thus see that the covering yielded by h is neither better than that yielded by f_4 nor locally optimal.

It is interesting to note that the parallelohedron $P(h)$ has all vertices equidistant from the origin under the norm of h. It would be interesting to know if this is perhaps true of the inverses of all perfect eutactic forms.

The questions concerning how many local solutions there are as a function of n, what is the least dimension that has a local solution which is not an absolute solution, and for what values of n do the lattices Λ_n provide the best possible covers, remain open.

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