Canad. J. Math. Vol. **61** (3), 2009 pp. 641–655

Characterization of Parallel Isometric Immersions of Space Forms into Space Forms in the Class of Isotropic Immersions

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Abstract. For an isotropic submanifold M^n ($n \geqq 3$) of a space form $\widetilde{M}^{n+p}(c)$ of constant sectional curvature c, we show that if the mean curvature vector of M^n is parallel and the sectional curvature K of M^n satisfies some inequality, then the second fundamental form of M^n in $\widetilde{M}^{n+p}$ is parallel and our manifold M^n is a space form.

1 Introduction

An n-dimensional space form $M^n(c)$ is a complete connected Riemannian manifold of constant sectional curvature c. Locally it is congruent to a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$, or a hyperbolic space $H^n(c)$, if c is positive, zero, or negative, respectively.

We review the notion of isotropic immersions which plays a key role in this paper. An isometric immersion $f\colon M \to \widetilde{M}$ of an n-dimensional Riemannian manifold into an $(n+p)$-dimensional Riemannian manifold is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2 (= \lambda(x))$ does not depend on the choice of $X(\neq 0) \in T_xM$, where σ is the second fundamental form of the immersion f. If the immersion is isotropic at every point, then the immersion is said to be isotropic (see [7]). When the function $\lambda = \lambda(x)$ is constant on M, we call M a constant (λ-)isotropic submanifold. Note that a totally umbilic immersion is isotropic, but not *vice versa*.

Here we recall examples of isotropic immersions which are not totally umbilic. Let $f\colon M \to S^4(1)$ be a superminimal immersion of a Riemann surface into a sphere in the sense of Bryant [1]. It is known that in general this immersion is non-constant isotropic. Next, let $f\colon M(= G/K) \to \widetilde{M}$ be a G-equivariant isometric immersion of a rank one symmetric space $M(= G/K)$ into an arbitrary Riemannian manifold $\widetilde{M}$. Then we easily see that this submanifold (M, f) is a constant isotropic submanifold of $\widetilde{M}$, so that in particular, every standard minimal immersion (in the sense of do Carmo and Wallach [2]) $f\colon M \to S^N(c)$ of a compact rank one symmetric space M into a sphere is constant isotropic. Moreover, there exist many constant isotropic minimal immersions which are not standard minimal immersions into a sphere (see [10]). These examples tell us that the class of isotropic immersions into a sphere is an abundant class in submanifold theory.

Received by the editors May 12, 2006; revised October 18, 2006.
AMS subject classification: Primary 53C40; secondary 53C42.
Keywords: space forms, parallel isometric immersions, isotropic immersions, totally umbilic, Veronese manifolds, sectional curvatures, parallel mean curvature vector.

https://doi.org/10.4153/CJM-2009-034-4 Published online by Cambridge University Press

On the other hand, it is natural to give geometric characterizations of parallel isometric immersions of rank one symmetric spaces M^n into a space form $\widetilde{M}^{n+p}(c)$ of constant sectional curvature c. It is well known that this submanifold is either totally umbilic in $\widetilde{M}^{n+p}(c)$, one of compact rank one symmetric spaces embedded into some totally umbilic submanifold in $\widetilde{M}^{n+p}(c)$ through the first standard minimal embedding, or a sphere immersed into some totally umbilic submanifold in $\widetilde{M}^{n+p}(c)$ through the second standard minimal immersion (see [3, 9]). This fact implies that every parallel isometric immersion of a rank one symmetric space into a space form is isotropic. Hence it is interesting to consider the problem of how to characterize parallel isometric immersions of rank one symmetric spaces into a space form in the class of isotropic immersions. In this paper, we pay particular attention to parallel isometric immersions of compact space forms into space forms.

The main purpose of this paper is to characterize all parallel isometric immersions of compact space forms into space forms $\widetilde{M}^{n+p}(c)$ in the class of isotropic submanifolds M^n under conditions that the mean curvature vector of M^n in $\widetilde{M}^{n+p}(c)$ is parallel with respect to the normal connection and the sectional curvature K of M^n satisfies $K \geqq n/2(n+1) \cdot (c + H^2)$, where H is the length of the mean curvature vector of M^n in $\widetilde{M}^{n+p}(c)$ (Theorem 2.1). None of our results (Theorems 2.1, 3.1 and Corollaries 3.2, 3.3) hold if we replace the condition that "the mean curvature vector is parallel with respect to the connection" for a weaker condition that "the length of the mean curvature vector is constant".

2 Main Result

Theorem 2.1 *Let M be an $n\,(\geqq 3)$-dimensional connected compact oriented isotropic submanifold whose mean curvature vector is parallel with respect to the normal connection in an $(n+p)$-dimensional space form $\widetilde{M}^{n+p}(c)$ of constant sectional curvature c through an isometric immersion f. Suppose that every sectional curvature K of M^n satisfies $K \geqq (n/2(n+1))(c + H^2)$, where H is the length of the mean curvature vector of M^n in $\widetilde{M}^{n+p}(c)$. Then the immersion f has parallel second fundamental form and the submanifold (M, f) is congruent to one of the following.*

(i) *M^n is a compact space form $M^n(K)$ of constant sectional curvature $K = c + H^2$ and f is a totally umbilic embedding.*

(ii) *M^n is a compact space form $M^n(K)$ of constant sectional curvature*

$$K = \frac{n}{2(n+1)}(c + H^2),$$

and f is given by

$$f = f_2 \circ f_1 \colon\ M^n(K) \xrightarrow{f_1} S^{n(n+3)/2-1}(2(n+1)K/n) \xrightarrow{f_2} \widetilde{M}^{n+p}(c),$$

where f_1 is a minimal (parallel) immersion and f_2 is a totally umbilic embedding.

In order to prove Theorem 2.1 we prepare four lemmas.

Lemma 2.2 *The second fundamental form σ of an isotropic submanifold M in a Riemannian manifold $\widetilde{M}$ with Riemannian metric $\langle\ ,\ \rangle$ satisfies the following at each point $x \in M$.*

(i) $\langle \sigma(X,X), \sigma(X,Y)\rangle = 0$ *for each pair of orthonormal vectors X, Y in T_xM.*

(ii) $\|\sigma(X,X)\|^2 = \langle \sigma(X,X), \sigma(Y,Y)\rangle + 2\|\sigma(X,Y)\|^2$ *for each pair of orthonormal vectors X, Y in T_xM.*

(iii) $\langle \sigma(X,X), \sigma(U,V)\rangle = -2\langle \sigma(X,U), \sigma(X,V)\rangle$ *for each triplet of orthonormal vectors X, U, V in T_xM.*

Proof It is known that the second fundamental form σ is λ-isotropic if and only if σ satisfies the following equation:

$$\langle \sigma(X,Y), \sigma(Z,W)\rangle + \langle \sigma(X,Z), \sigma(Y,W)\rangle + \langle \sigma(X,W), \sigma(Y,Z)\rangle = \lambda^2(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,Z\rangle)$$

for arbitrary vectors $X, Y, Z, W \in TM$. Thus we get equations in (i), (ii), and (iii). ■

Lemma 2.3 *Let M^n be an n-dimensional isotropic submanifold of $\widetilde{M}^{n+p}$. We take $v \in U_xM = \{v \in T_xM : \|v\| = 1\}$ and an orthonormal basis $\{v = E_1, E_2, \dots, E_n\}$ of T_xM. Then these vectors satisfy the following.*

(i)
$$n(\|\sigma(v,v)\|^2 - \langle \mathfrak{h}, \sigma(v,v)\rangle) = 2\sum_{j=2}^{n} \|\sigma(v,E_j)\|^2,$$

where $\mathfrak{h}$ is the mean curvature vector of M^n in $\widetilde{M}^{n+p}$.

(ii)
$$\sum_{2\leqq \ell, j\leqq n} \langle \sigma(v,E_j), \sigma(v,E_\ell)\rangle\langle \sigma(v,v), \sigma(E_j,E_\ell)\rangle = \|\sigma(v,v)\|^2 \sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 - 2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^4 - 2\sum_{2\leqq \ell\neq j\leqq n}\langle \sigma(v,E_j), \sigma(v,E_\ell)\rangle^2.$$

(iii)
$$\sum_{2\leqq \ell, j\leqq n} \langle \sigma(v,v), \sigma(E_j,E_\ell)\rangle^2 = (n-1)\|\sigma(v,v)\|^4 - 4\|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 + 4\sum_{j=2}^{n}\|\sigma(v,E_j)\|^4 + 4\sum_{2\leqq \ell\neq j\leqq n}\langle \sigma(v,E_j), \sigma(v,E_\ell)\rangle^2.$$

Proof For (i), by Lemma 2.2(ii) and $\mathfrak{h} = (1/n)\sum_{j=1}^{n}\sigma(E_j,E_j)$ we get

$$2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 = \sum_{j=2}^{n}\big(\|\sigma(v,v)\|^2 - \langle \sigma(v,v), \sigma(E_j,E_j)\rangle\big) = n\|\sigma(v,v)\|^2 - n\langle \sigma(v,v), \mathfrak{h}\rangle.$$

For (ii) from Lemma 2.2(ii) and (iii) we have

$$
\begin{aligned}
&\sum_{2\leqq \ell, j\leqq n} \langle \sigma(v,E_j),\sigma(v,E_\ell)\rangle\langle \sigma(v,v),\sigma(E_j,E_\ell)\rangle \\
&\quad = \sum_{j=2}^{n}\langle \sigma(v,E_j),\sigma(v,E_j)\rangle\langle \sigma(v,v),\sigma(E_j,E_j)\rangle \\
&\qquad + \sum_{2\leqq \ell\neq j\leqq n} \langle \sigma(v,E_j),\sigma(v,E_\ell)\rangle\langle \sigma(v,v),\sigma(E_j,E_\ell)\rangle \\
&\quad = \sum_{j=2}^{n}\langle \sigma(v,E_j),\sigma(v,E_j)\rangle\big(\|\sigma(v,v)\|^2 - 2\|\sigma(v,E_j)\|^2\big) \\
&\qquad + \sum_{2\leqq \ell\neq j\leqq n} \langle \sigma(v,E_j),\sigma(v,E_\ell)\rangle\big(-2\langle \sigma(v,E_j),\sigma(v,E_\ell)\rangle\big) \\
&\quad = \|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 - 2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^4 \\
&\qquad - 2\sum_{2\leqq \ell\neq j\leqq n}\langle \sigma(v,E_j),\sigma(v,E_\ell)\rangle^2.
\end{aligned}
$$

The same computation as that in (ii) yields the equation in (iii). ■

We recall fundamental equations for a Riemannian submanifold M^n of a space form $\widetilde{M}^{n+p}(c)$. We denote by R (resp. $R^\perp$) the curvature tensor (resp. the normal curvature tensor) of M and A the shape operator of M^n in $\widetilde{M}^{n+p}(c)$. Then, for any $X,Y,Z,W\in TM$ and any $\xi\in T^\perp M$ we have

$$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y, \tag{2.1}$$

$$R^\perp(X,Y)\xi = \sigma(X,A_\xi Y) - \sigma(Y,A_\xi X), \tag{2.2}$$

$$
\begin{aligned}
(\nabla_X\nabla_Y\sigma)(Z,W) - (\nabla_Y\nabla_X\sigma)(Z,W) &= R^\perp(X,Y)\sigma(Z,W) \\
&\quad - \sigma(R(X,Y)Z,W) - \sigma(Z,R(X,Y)W),
\end{aligned} \tag{2.3}
$$

$$(\nabla_X\sigma)(Y,W) = (\nabla_Y\sigma)(X,W). \tag{2.4}$$

In the following, we regard U_xM as an $(n-1)$-dimensional unit sphere $S^{n-1}(1)$ in $T_xM(\cong\mathbb{R}^n)$, and denote by Δ the Laplacian on $S^{n-1}(1)$.

Lemma 2.4 *For an orthonormal basis $\{v = E_1, E_2,\dots,E_n\}$ of T_xM we consider smooth curves $v_j(t) = (\cos t)v + (\sin t)E_j$ $(j = 2,3,\dots,n)$ on $S^{n-1}(1)$. Let*

$$\Psi(v) = \langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(v,v)\rangle \text{ for } v\in S^{n-1}(1)$$

for a Riemannian submanifold M^n (which is not necessarily isotropic) in a general Riemannian manifold $\widetilde{M}^{n+p}$. Then

(2.5)
$$\begin{aligned}(\Delta\Psi)_v = &-6(n+4)\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(v,v)\rangle\\ &+6\sum_{j=1}^{n}\langle(\nabla_{E_j}\nabla_{E_j}\sigma)(v,v),\sigma(v,v)\rangle+4\sum_{j=1}^{n}\langle(\nabla_{E_j}\nabla_v\sigma)(v,v),\sigma(E_j,v)\rangle\\ &+6\sum_{j=1}^{n}\langle(\nabla_v\nabla_{E_j}\sigma)(E_j,v),\sigma(v,v)\rangle+12\sum_{j=1}^{n}\langle(\nabla_v\nabla_{E_j}\sigma)(v,v),\sigma(E_j,v)\rangle\\ &+2\sum_{j=1}^{n}\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(E_j,E_j)\rangle.\end{aligned}$$

Proof We first have

$$\begin{aligned}\frac{d}{dt}\Psi(v_j(t)) = &\langle(\nabla_{\dot v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(v_j,v_j)\rangle+\langle(\nabla_{v_j}\nabla_{\dot v_j}\sigma)(v_j,v_j),\sigma(v_j,v_j)\rangle\\ &+2\langle(\nabla_{v_j}\nabla_{v_j}\sigma)(\dot v_j,v_j),\sigma(v_j,v_j)\rangle+2\langle(\nabla_{v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(\dot v_j,v_j)\rangle,\end{aligned}$$

where $v_j=v_j(t)$, $\dot v_j(t)=\frac{d}{dt}v_j(t)=(-\sin t)v+(\cos t)E_j$. We similarly obtain

$$\begin{aligned}\frac{d^2}{dt^2}\Psi(v_j(t)) = &\langle(\nabla_{\ddot v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(v_j,v_j)\rangle+3\langle(\nabla_{v_j}\nabla_{\ddot v_j}\sigma)(v_j,v_j),\sigma(v_j,v_j)\rangle\\ &+2\langle(\nabla_{v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(\ddot v_j,v_j)\rangle+6\langle(\nabla_{\dot v_j}\nabla_{\dot v_j}\sigma)(v_j,v_j),\sigma(v_j,v_j)\rangle\\ &+4\langle(\nabla_{\dot v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(\dot v_j,v_j)\rangle+6\langle(\nabla_{v_j}\nabla_{\dot v_j}\sigma)(\dot v_j,v_j),\sigma(v_j,v_j)\rangle\\ &+12\langle(\nabla_{v_j}\nabla_{\dot v_j}\sigma)(v_j,v_j),\sigma(\dot v_j,v_j)\rangle+2\langle(\nabla_{v_j}\nabla_{v_j}\sigma)(v_j,v_j),\sigma(\dot v_j,\dot v_j)\rangle,\end{aligned}$$

where $\ddot v_j=\frac{d^2}{dt^2}v_j(t)=(-\cos t)v+(-\sin t)E_j$. Hence, from the above equation we can verify that $(\Delta\Psi)_v=\left(\sum_{j=2}^{n}\frac{d^2}{dt^2}\Psi(v_j(t))\right)_{t=0}$ is given by (2.5). ■

Lemma 2.5 *The second fundamental form σ of a Riemannian submanifold M^n with parallel mean curvature vector in a space form $\widetilde{M}^{n+p}(c)$ satisfies the following at each point $x\in M$.*

(i) $$\int_{U_xM\ni v}\sum_{j=1}^{n}\langle(\nabla_{E_j}\nabla_v\sigma)(v,v),\sigma(E_j,v)\rangle=0.$$

(ii) $$\int_{U_xM\ni v}\sum_{j=1}^{n}\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(E_j,E_j)\rangle=0.$$

https://doi.org/10.4153/CJM-2009-034-4 Published online by Cambridge University Press

Proof Let $\Phi(v) = \sum_{j=1}^{n} \langle (\nabla_{E_j} \nabla_v \sigma)(v, v), \sigma(E_j, v)\rangle$ for $\forall v \in S^{n-1}(1)$. Then the computation in the proof of Lemma 2.4, together with the assumption that the mean curvature vector is parallel with respect to the normal connection, tells us the following.

$$(\Delta\Phi)_v = -(4n+8)\Phi(v) + 6 \sum_{1\leqq j,k\leqq n} \langle (\nabla_{E_j} \nabla_{E_k} \sigma)(v, v), \sigma(E_j, E_k)\rangle.$$

Here, let $\widetilde{\Phi}(v) = \sum_{1\leqq j,k\leqq n} \langle (\nabla_{E_j} \nabla_{E_k} \sigma)(v, v), \sigma(E_j, E_k)\rangle$. Then, again by using the assumption that the mean curvature vector is parallel, the same computation as above yields $(\Delta\widetilde{\Phi})_v = -2n\widetilde{\Phi}(v)$. Therefore, by Green's theorem we see that $\int_{U_xM\ni v} \Phi(v) = 0$, so that we get (i) in our lemma. Similarly we have (ii) in Lemma 2.5. ■

Proof of Theorem 2.1 We are now in a position to prove Theorem 2.1. Suppose that our manifold M satisfies the hypothesis of Theorem 2.1. We note that the fourth term in the right hand side of (2.5) vanishes because of (2.4) and the assumption that the mean curvature vector is parallel. So, it follows from Lemmas 2.4 and 2.5 that

(2.6)
$$\begin{aligned}(n+4)\int_{U_xM} \langle (\nabla_v \nabla_v \sigma)(v, v), \sigma(v, v)\rangle = \int_{U_xM\ni v} \Big(&\sum_{j=1}^{n} \langle (\nabla_{E_j} \nabla_{E_j} \sigma)(v, v), \sigma(v, v)\rangle \\ &+ 2\sum_{j=1}^{n} \langle (\nabla_v \nabla_{E_j} \sigma)(v, v), \sigma(E_j, v)\rangle \Big).\end{aligned}$$

This, together with (2.1), (2.2), Lemma 2.2, and the hypothesis that the mean curvature vector is parallel, shows

$$\begin{aligned}\sum_{j=1}^{n} \langle (\nabla_{E_j} \nabla_{E_j} \sigma)(v, v), \sigma(v, v)\rangle = {}&2c \sum_{j=2}^{n} \|\sigma(v, E_j)\|^2 \\ &+ 2\|\sigma(v, v)\|^2 \sum_{j=2}^{n} \|\sigma(v, E_j)\|^2 - 8 \sum_{2\leqq \ell,j\leqq n} \langle \sigma(v, E_j), \sigma(v, E_\ell)\rangle^2.\end{aligned} \tag{2.7}$$

Also, we have

$$\begin{aligned}\sum_{j=1}^{n} \langle (\nabla_v \nabla_{E_j} \sigma)(v, v), \sigma(E_j, v)\rangle = {}&\sum_{j=1}^{n} \langle (\nabla_{E_j} \nabla_v \sigma)(v, v), \sigma(E_j, v)\rangle \\ &+ 2c \sum_{j=2}^{n} \|\sigma(v, E_j)\|^2 + 2\|\sigma(v, v)\|^2 \sum_{j=2}^{n} \|\sigma(v, E_j)\|^2 \\ &- 8 \sum_{2\leqq j,k\leqq n} \langle \sigma(v, E_j), \sigma(v, E_k)\rangle^2.\end{aligned} \tag{2.8}$$

In consideration of (2.6), (2.7), (2.8), and Lemma 2.5(i) we find that

$$(2.9)\quad (n+4)\int_{U_xM}\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(v,v)\rangle=\int_{U_xM\ni v}\Big(6c\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 + 6\|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2-24\sum_{2\leqq j,k\leqq n}\langle\sigma(v,E_j),\sigma(v,E_k)\rangle^2\Big).$$

On the other hand, by the result of Ros [8] for $T(v)=\langle\sigma(v,v),\sigma(v,v)\rangle$, we know that

$$\begin{aligned}0&=\int_{UM}(\nabla^2T)(v,v,v,v,v,v)\\&=\int_{UM}(2\|(\nabla_v\sigma)(v,v)\|^2+2\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(v,v)\rangle),\end{aligned}$$

so that

$$(2.10)\qquad \int_{UM}\langle(\nabla_v\nabla_v\sigma)(v,v),\sigma(v,v)\rangle=-\int_{UM}\|(\nabla_v\sigma)(v,v)\|^2\leqq 0.$$

Our aim here is to show that the submanifold M has parallel second fundamental form. Hence, by virtue of (2.4), (2.9), and (2.10) we have only to prove the following inequality:

$$c\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2+\|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2-4\sum_{2\leqq j,k\leqq n}\langle\sigma(v,E_j),\sigma(v,E_k)\rangle^2\geqq 0.$$

To do this, we set

$$A=\sum_{j=2}^{n}K(v,E_j)=\sum_{j=2}^{n}\langle R(v,E_j)E_j,v\rangle\quad\text{and}\quad B=\sum_{2\leqq j\neq k\leqq n}K(E_j,E_k).$$

It follows from (2.1) and Lemma 2.2(ii) that

$$(2.11)\qquad A=(n-1)c+(n-1)\|\sigma(v,v)\|^2-3\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2.$$

By the definition of the mean curvature vector $\mathfrak{h}$ we see that

$$(2.12)\quad n^2\|\mathfrak{h}\|^2=\|\sigma(v,v)\|^2+2\sum_{j=2}^{n}\langle\sigma(v,v),\sigma(E_j,E_j)\rangle + \sum_{2\leqq j,k\leqq n}\langle\sigma(E_j,E_j),\sigma(E_k,E_k)\rangle.$$

In view of (2.1), (2.12), and Lemma 2.3(i) we find that

$$n^2\|\mathfrak{h}\|^2 = 2n\|\sigma(v,v)\|^2 - n\langle\sigma(v,v),\mathfrak{h}\rangle - 2(n-1)c + 2A + B - (n-1)(n-2)c + \sum_{2\leqq j\neq k\leqq n}\|\sigma(E_j,E_k)\|^2. \tag{2.13}$$

On the other hand, we have

$$\sum_{j=2}^{n}\langle\sigma(v,v),\sigma(E_j,E_j)\rangle = n\langle\sigma(v,v),\mathfrak{h}\rangle - \|\sigma(v,v)\|^2,$$

and

$$\sum_{1\leqq j,k\leqq n}\langle\sigma(E_j,E_j),\sigma(E_k,E_k)\rangle = (n-1)\|\sigma(v,v)\|^2 + (n-1)(n-2)\|\sigma(v,v)\|^2 - 2\sum_{2\leqq j\neq k\leqq n}\|\sigma(E_j,E_k)\|^2$$

because of Lemma 2.2(ii). Substituting these equations into the right hand side of (2.12), we get

$$\sum_{2\leqq j\neq k\leqq n}\|\sigma(E_j,E_k)\|^2 = \frac{n(n-2)}{2}\|\sigma(v,v)\|^2 + n\langle\sigma(v,v),\mathfrak{h}\rangle - \frac{n^2}{2}\|\mathfrak{h}\|^2,$$

which, together with (2.13), implies

$$3n^2\|\mathfrak{h}\|^2 = n(n+2)\|\sigma(v,v)\|^2 - 2n(n-1)c + 4A + 2B. \tag{2.14}$$

It follows from (2.11) and (2.14) that

$$\|\sigma(v,v)\|^2 = \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}\|\mathfrak{h}\|^2 - \frac{4}{n(n+2)}A - \frac{2}{n(n+2)}B, \tag{2.15a}$$

$$\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 = \frac{n(n-1)}{n+2}c + \frac{n(n-1)}{n+2}\|\mathfrak{h}\|^2 - \frac{n^2+6n-4}{3n(n+2)}A - \frac{2(n-1)}{3n(n+2)}B. \tag{2.15b}$$

Moreover, from (2.11), (2.15a), and Lemma 2.3(i) we obtain

$$B = (n-2)A + \frac{3}{2}n^2(\|\mathfrak{h}\|^2 - \langle\sigma(v,v),\mathfrak{h}\rangle). \tag{2.16}$$

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Substituting (2.16) into the right-hand sides of (2.15a)and (2.15b), we can see that

$$\|\sigma(v,v)\|^2 = \frac{2(n-1)}{n+2}c - \frac{2}{n+2}A + \frac{3n}{n+2}\langle\sigma(v,v),\mathfrak{h}\rangle, \tag{2.17a}$$

$$\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 = \frac{n(n-1)}{n+2}c - \frac{n}{n+2}A + \frac{n(n-1)}{n+2}\langle\sigma(v,v),\mathfrak{h}\rangle. \tag{2.17b}$$

We next estimate $\sum_{2\leqq j,k\leqq n}\langle\sigma(v,E_j),\sigma(v,E_k)\rangle^2$. By (2.1), for $j=2,\dots,n$ we have

$$\|\sigma(v,E_j)\|^4 = \big(c - K(v,E_j) + \langle\sigma(v,v),\sigma(E_j,E_j)\rangle\big)\,\|\sigma(v,E_j)\|^2,$$

which, combined with Lemma 2.2(ii), yields

$$\begin{aligned}4\sum_{j=2}^{n}\|\sigma(v,E_j)\|^4 = \frac{4}{3}c\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 - \frac{4}{3}\sum_{j=2}^{n}K(v,E_j)\|\sigma(v,E_j)\|^2 \\ + \frac{4}{3}\|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2.\end{aligned} \tag{2.18}$$

It follows from (2.18) and the assumption that every sectional curvature K of M satisfies $K \geqq \frac{n}{2(n+1)}(c+\|\mathfrak{h}\|^2)$ that

$$\begin{aligned}4\sum_{j=2}^{n}\|\sigma(v,E_j)\|^4 \leqq \Big(\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2\Big) \\ \times\Big(\frac{4}{3}c - \frac{4n}{6(n+1)}(c+\|\mathfrak{h}\|^2) + \frac{4}{3}\|\sigma(v,v)\|^2\Big).\end{aligned} \tag{2.19}$$

For simplicity we take an orthonormal basis $\{E_2,\dots,E_n\}$ of U_xM satisfying

$$\langle\sigma(v,E_j),\sigma(v,E_k)\rangle = \delta_{jk}\langle\sigma(v,E_j),\sigma(v,E_j)\rangle \text{ for } j,k=2,\dots,n,$$

so that

$$\sum_{2\leqq j,k\leqq n}\langle\sigma(v,E_j),\sigma(v,E_k)\rangle^2 = \sum_{j=2}^{n}\|\sigma(v,E_j)\|^4. \tag{2.20}$$

On the other hand, by the assumption on the sectional curvature K of M we know that

$$A \geqq \frac{n(n-1)}{2(n+1)}(c+\|\mathfrak{h}\|^2), \qquad B \geqq \frac{n(n-1)(n-2)}{2(n+1)}(c+\|\mathfrak{h}\|^2). \tag{2.21}$$

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It follows from the second inequality in (2.21) and (2.16) that

$$\langle \sigma(v,v), \mathfrak{h}\rangle \leqq \|\mathfrak{h}\|^2 + \frac{2(n-2)}{3n^2}A - \frac{(n-1)(n-2)}{3n(n+1)}(c+\|\mathfrak{h}\|^2). \tag{2.22}$$

Hence, from (2.19), (2.20), (2.17a), (2.22) and the first inequality in (2.21) we can see that

$$\begin{aligned}
&c\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 + \|\sigma(v,v)\|^2\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2 - 4\sum_{2\leqq j,k\leqq n}\langle\sigma(v,E_j),\sigma(v,E_k)\rangle^2\\
&\quad\geqq \Big(\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2\Big)\Big(c+\|\sigma(v,v)\|^2 - \frac{4}{3}c + \frac{4n}{6(n+1)}(c+\|\mathfrak{h}\|^2) - \frac{4}{3}\|\sigma(v,v)\|^2\Big)\\
&\quad= \Big(\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2\Big)\Big(-\frac{n}{n+2}c + \frac{2n}{3(n+1)}(c+\|\mathfrak{h}\|^2) + \frac{2}{3(n+2)}A\\
&\qquad\qquad - \frac{n}{n+2}\langle\sigma(v,v),\mathfrak{h}\rangle\Big)\\
&\quad\geqq \Big(\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2\Big)\Big(-\frac{n}{n+2}c + \frac{2n}{3(n+1)}(c+\|\mathfrak{h}\|^2) + \frac{2}{3(n+2)}A\\
&\qquad\qquad - \frac{n}{n+2}\|\mathfrak{h}\|^2 - \frac{2(n-2)}{3n(n+2)}A + \frac{(n-1)(n-2)}{3(n+1)(n+2)}(c+\|\mathfrak{h}\|^2)\Big)\\
&\quad= \Big(\sum_{j=2}^{n}\|\sigma(v,E_j)\|^2\Big)\Big(\frac{4}{3n(n+2)}A - \frac{2(n-1)}{3(n+1)(n+2)}(c+\|\mathfrak{h}\|^2)\Big) \geqq 0,
\end{aligned}$$

so that M has parallel second fundamental form. This, together with the assumption that M is connected, shows that the length $\|\mathfrak{h}\|$ is a constant function on the submanifold M. Moreover, the last inequality shows that our discussion is divided into the following two cases.

One is to investigate the case of $\sigma(v,E_j) = 0$ for $j = 2,\dots,n$. This, combined with Lemmas 2.3(i) and 2.2(ii), implies $\|\sigma(v,v)\|^2 = \langle\sigma(v,v),\mathfrak{h}\rangle = \|\mathfrak{h}\|^2$. So equation (2.1) yields $K(v,E_j) = c + \|\mathfrak{h}\|^2$, which is a constant for $j = 2,\dots,n$. As v is an arbitrary (unit) vector of T_xM, our discussion shows that the manifold M^n has constant sectional curvature $K \equiv c + \|\mathfrak{h}\|^2$ at the point x and x is an umbilic point on M^n.

The other is to consider the case of $\sigma(v,E_j) \neq 0$ for some j. In this case, the last inequality tells us that M^n has constant sectional curvature $K \equiv \frac{n}{2(n+1)}(c+\|\mathfrak{h}\|^2)$ at the point x. Then by the connectivity of M we see that our manifold M^n is a space form either $M^n(c+\|\mathfrak{h}\|^2)$ or $M^n\big(n(c+\|\mathfrak{h}\|^2)/2(n+1)\big)$. Therefore, by the classification theorems of parallel submanifolds in a space form (see [3,9]) we get the conclusion. ∎

3 Concluding Remarks

Our aim here is to establish Theorem 3.1. Note that in Theorem 3.1 we do not suppose that the immersion f is isotropic.

Theorem 3.1 *Let M be a 2-dimensional, connected, compact, oriented submanifold whose mean curvature vector is parallel with respect to the normal connection in a $(2+p)$-dimensional space form $\widetilde{M}^{2+p}(c)$ of constant sectional curvature c through an isometric immersion f. Suppose that every sectional curvature K of M^2 satisfies $K \geqq (1/3)(c+H^2)$, where H is the length of the mean curvature vector of M^2 in $\widetilde{M}^{2+p}(c)$. Then the immersion f has parallel second fundamental form and the submanifold (M, f) is congruent to one of the following.*

(i) *M^2 is a compact space form $M^2(K)$ of constant sectional curvature $K = c + H^2$ and f is a totally umbilic embedding.*

(ii) *M^2 is a compact space form $M^2(K)$ of constant sectional curvature $K = (1/3)(c + H^2)$ and f is given by $f = f_2 \circ f_1 \colon M^2(K) \xrightarrow{f_1} S^4(3K) \xrightarrow{f_2} \widetilde{M}^{2+p}(c)$, where f_1 is a minimal (parallel) immersion and f_2 is a totally umbilic embedding.*

Proof Let $\{X, Y\}$ be a local field of orthonormal frames on M^2. We set $Z = (1/\sqrt{2})(X - \sqrt{-1}Y)$. In the following, our computation is expressed in terms of $Z, \overline{Z}$. We can set

$$\nabla_{\overline{Z}} Z = aZ, \nabla_{\overline{Z}}\overline{Z} = -a\overline{Z}, \nabla_Z \overline{Z} = \overline{a}\overline{Z}, \nabla_Z Z = -\overline{a}Z, \tag{3.1}$$

where a is a local smooth function on M^2. We here review some fundamental facts. M^2 is totally umbilic in an ambient Riemannian manifold $\widetilde{M}$ if and only if $\sigma(Z, Z) = 0$ on M^2. The mean curvature vector $\mathfrak{h}$ of M^2 in $\widetilde{M}$ is parallel if and only if $\nabla^{\perp}_{\widetilde{X}}\sigma(Z, \overline{Z}) = 0$ for $\forall \widetilde{X} \in TM$, which is equivalent to saying that $(\nabla_{\widetilde{X}}\sigma)(Z, \overline{Z}) = 0$ for $\forall \widetilde{X} \in TM$. Hence, from (2.4) and the parallelism of the mean curvature vector we have

$$(\nabla_{\overline{Z}}\sigma)(Z, Z) = (\nabla_Z \sigma)(Z, \overline{Z}) = 0. \tag{3.2}$$

Then from (3.1) and (3.2) we see that

$$\begin{aligned}
\nabla_{\overline{Z}}\|\sigma(Z,Z)\|^2 &= \langle \nabla^{\perp}_{\overline{Z}}\sigma(Z,Z), \sigma(\overline{Z},\overline{Z})\rangle + \langle \sigma(Z,Z), \nabla^{\perp}_{\overline{Z}}\sigma(\overline{Z},\overline{Z})\rangle \\
&= \langle 2\sigma(\nabla_{\overline{Z}}Z, Z), \sigma(\overline{Z},\overline{Z})\rangle + \langle \sigma(Z,Z), (\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z}) + 2\sigma(\nabla_{\overline{Z}}\overline{Z},\overline{Z})\rangle \\
&= \langle \sigma(Z,Z), (\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z})\rangle.
\end{aligned}$$

This, together with (3.1), implies

$$\begin{aligned}
\Delta\|\sigma(Z,Z)\|^2 &= \nabla_Z\nabla_{\overline{Z}}\|\sigma(Z,Z)\|^2 - \nabla_{\nabla_Z\overline{Z}}\|\sigma(Z,Z)\|^2 \\
&= \langle \nabla^{\perp}_Z\sigma(Z,Z), (\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z})\rangle + \langle \sigma(Z,Z), \nabla^{\perp}_Z(\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z})\rangle \\
&\qquad - \overline{a}\langle \sigma(Z,Z), (\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z})\rangle \\
&= \|(\nabla_Z\sigma)(Z,Z)\|^2 + \langle \sigma(Z,Z), (\nabla_Z\nabla_{\overline{Z}}\sigma)(\overline{Z},\overline{Z})\rangle.
\end{aligned}$$

Here, it follows from (2.1), (2.2), (2.3), and (3.2) that

$$(\nabla_Z \nabla_{\overline{Z}} \sigma)(\overline{Z}, \overline{Z}) = R^{\perp}(Z, \overline{Z})\sigma(\overline{Z}, \overline{Z}) - 2\sigma(R(Z, \overline{Z})\overline{Z}, \overline{Z})$$
$$= \langle \sigma(\overline{Z}, \overline{Z}), \sigma(\overline{Z}, \overline{Z})\rangle \sigma(Z, Z) + (2c + 2\|\mathfrak{h}\|^2 - 3\|\sigma(Z, Z)\|^2)\sigma(\overline{Z}, \overline{Z}).$$

Then the above computation yields

$$\Delta\|\sigma(Z, Z)\|^2 = \|(\nabla_Z \sigma)(Z, Z)\|^2 + |\langle \sigma(Z, Z), \sigma(Z, Z)\rangle|^2 + (2c + 2\|\mathfrak{h}\|^2 - 3\|\sigma(Z, Z)\|^2)\|\sigma(Z, Z)\|^2. \tag{3.3}$$

On the other hand, from (2.1) we have

$$K = \langle R(X, Y)Y, X\rangle = \langle R(\overline{Z}, Z)\overline{Z}, Z\rangle = c + \|\mathfrak{h}\|^2 - \|\sigma(Z, Z)\|^2, \tag{3.4}$$

which, combined with the assumption $K \geqq (c + \|\mathfrak{h}\|^2)/3$, shows that

$$2c + 2\|\mathfrak{h}\|^2 - 3\|\sigma(Z, Z)\|^2 = 3K - (c + \|\mathfrak{h}\|^2) \geqq 0.$$

Thus, from (3.3) we can see that

$$0 = \int_M \Delta\|\sigma(Z, Z)\|^2$$
$$= \int_M \big(\|(\nabla_Z \sigma)(Z, Z)\|^2 + |\langle \sigma(Z, Z), \sigma(Z, Z)\rangle|^2 + (3K - c - \|\mathfrak{h}\|^2)\|\sigma(Z, Z)\|^2 \big)$$
$$\geqq 0,$$

so that M^n has parallel second fundamental form, and moreover $\|\sigma(Z, Z)\|^2$ is constant on M^n because of Hopf's Lemma. Hence equation (3.4) shows that the manifold M^n is a space form $M^n(K)$. When $K > (c + \|\mathfrak{h}\|^2)/3$, $\sigma(Z, Z) = 0$ on M^n, which implies that the immersion f is totally umbilic and $K = c + \|\mathfrak{h}\|^2$. When $K = (c + \|\mathfrak{h}\|^2)/3$, the immersion f is given by Theorem 3.1(ii) (see [3, 9]). ■

Theorem 3.1 is a generalization of Itoh's result for the case of $n = 2$ (see [4]). As immediate consequences of Theorems 2.1 and 3.1 we obtain the following corollaries.

Corollary 3.2 *Let M be an $n\,(\geqq 3)$-dimensional, connected, compact, oriented, isotropic submanifold whose mean curvature vector is parallel with respect to the normal connection in an $(n+p)$-dimensional space form $\widetilde{M}^{n+p}(c)$ of constant sectional curvature c through an isometric immersion f. Suppose that every sectional curvature K of M^n satisfies $K \geqq (c + H^2)/2$, where H is the length of the mean curvature vector of M^n in $\widetilde{M}^{n+p}(c)$. Then M^n is a compact space form $M^n(K)$ of constant sectional curvature $K = c + H^2$ and f is a totally umbilic embedding.*

https://doi.org/10.4153/CJM-2009-034-4 Published online by Cambridge University Press

Corollary 3.3 *Let M be a 2-dimensional, connected, compact, oriented submanifold whose mean curvature vector is parallel with respect to the normal connection in a $(2+p)$-dimensional space form $\widetilde{M}^{2+p}(c)$ of constant sectional curvature c through an isometric immersion f. Suppose that every sectional curvature K of M^n satisfies $K \geqq (c+H^2)/2$, where H is the length of the mean curvature vector of M^2 in $\widetilde{M}^{2+p}(c)$. Then M^2 is a compact space form $M^2(K)$ of constant sectional curvature $K = c + H^2$ and f is a totally umbilic embedding.*

Here we recall the following characterization of totally umbilic submanifolds in a space form, which is a local statement [6].

Proposition 3.4 *Let M^n be a Riemannian submanifold in a space form $\widetilde{M}^{n+p}(c)$. Then the following are equivalent:*

(i) *M^n is totally umbilic in $\widetilde{M}^{n+p}(c)$,*
(ii) *M^n is an isotropic submanifold with flat normal connection of $\widetilde{M}^{n+p}(c)$.*

At the end of this paper we shall show that in the assumptions of Theorems 2.1, 3.1 and Corollaries 3.2, 3.3, if we replace the condition that "the mean curvature is parallel with respect to the connection" for a weaker condition that "the length of the mean curvature vector is constant", these results are no longer true.

Remark. There exist many connected, compact, oriented, $n(\geqq 2)$-dimensional submanifolds M^n's of an $(n+p)$-dimensional sphere $S^{n+p}(c)$ of constant sectional curvature c satisfying the following four conditions.

(1) M^n is an isotropic submanifold of $S^{n+p}(c)$.
(2) The length H of the mean curvature vector $\mathfrak{h}$ of M^n in $S^{n+p}(c)$ is constant on M^n.
(3) Every sectional curvature K of M^n satisfies $K \geqq (n/2(n+1))(c+H^2)$.
(4) The mean curvature vector $\mathfrak{h}$ is not parallel with respect to the normal connection,(so that the second fundamental form of M^n in $S^{n+p}(c)$ is not parallel).

Proof We shall construct an example of a submanifold satisfying the above four conditions in $S^{n+p}(c)$, which is due to the first author [5]. Let $\chi_1\colon S^n(n/2(n+1)) \to S^{n+(n(n+1)/2)-1}(1)$ be the second standard minimal immersion and $\chi_2\colon S^n(n/2(n+1)) \to S^n(n/2(n+1))$ the identity mapping. Using these minimal immersions, we define the following minimal immersion:

(3.5)
$$\chi_t(=(\chi_1,\chi_2))\colon S^n\Big(\frac{n}{2(n+1)}\Big) \to S^{n+(n(n+1)/2)-1}\Big(\frac{1}{\cos^2 t}\Big) \times S^n\Big(\frac{n}{2(n+1)\sin^2 t}\Big).$$

for $t \in (0,\pi/2)$. Here the differential $(\chi_t)_*$ of the mapping χ_t is given by $(\chi_t)_*X = (\cos t\cdot(\chi_1)_*X, \sin t\cdot(\chi_2)_*X)$ for each $X \in TS^n(n/2(n+1))$. The product space of spheres in (3.5) can be embedded into a sphere as a Clifford hypersurface:

(3.6)
$$S^{n+(n(n+1)/2)-1}\Big(\frac{1}{\cos^2 t}\Big) \times S^n\Big(\frac{n}{2(n+1)\sin^2 t}\Big) \to S^{n+(n(n+3)/2)}\Big(\frac{n}{n+(n+2)\sin^2 t}\Big).$$

Combining (3.5) and (3.6), we obtain the following isometric immersion f_t:

$$f_t\colon S^n\Big(\frac{n}{2(n+1)}\Big) \to S^{n+(n(n+3)/2)}\Big(\frac{n}{n+(n+2)\sin^2 t}\Big). \tag{3.7}$$

By virtue of the result in [5], we find the following properties of f_t for each $t \in (0, \pi/2)$.

(a) The length H_t of the mean curvature vector $\mathfrak{h}_t$ for f_t is given by

$$H_t = \frac{(n+2)\sin t \cos t}{\sqrt{2(n+1)\bigl(n+(n+2)\sin^2 t\bigr)}} \neq 0. \tag{3.8}$$

(b) The mean curvature vector $\mathfrak{h}_t$ is not parallel with respect to the normal connection $\nabla^{\perp}$. The length of the derivative of $\mathfrak{h}_t$ is given by

$$\|\nabla^{\perp}\mathfrak{h}_t\|^2 = \frac{n(n+2)^2}{4(n+1)^2}\sin^2 t\cos^2 t \neq 0.$$

(c) f_t is constant λ_t-isotropic. λ_t is given by

$$\lambda_t = \sqrt{\frac{n-1}{n+1}\cos^4 t + \frac{(c_1\cos^2 t - c_2\sin^2 t)^2}{c_1+c_2}} \neq 0,$$

where $c_1 = 1/\cos^2 t$ and $c_2 = n/\bigl(2(n+1)\sin^2 t\bigr)$.

Now, in particular we set $\cos t = 1/\sqrt{n+1}$ and $\sin t = \sqrt{n/(n+1)}$ in (3.7). Then we have the following isometric immersion f:

$$f\colon S^n\Bigl(\frac{n}{2(n+1)}\Bigr) \to S^{n+(n(n+3)/2)}\Bigl(\frac{n+1}{2n+3}\Bigr). \tag{3.9}$$

The rest of the proof is to show that the length H of the mean curvature vector for the isometric immersion f given by (3.9) satisfies the following inequalty:

$$K > \frac{c+H^2}{2}\Bigl(> \frac{n}{2(n+1)}(c+H^2)\Bigr).$$

It follows from (3.8) that

$$H = \frac{n+2}{(n+1)\sqrt{2(2n+3)}}.$$

This, together with $K = n/\bigl(2(n+1)\bigr)$ and $c = (n+1)/(2n+3)$, yields that

$$K - \frac{c+H^2}{2} = \frac{n^2-2}{4(n+1)^2} > 0.$$

Hence, by the continuity of H_t we have many examples satisfying the conditions (1), (2), (3), and (4) of our Remark. ∎

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