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SUBGROUPS OF CONJUGATE CLASSES IN EXTENSIONS

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Often in various mathematical problems one encounters an extension B of the group G by the group π in which one wishes to extract certain information about B from information given in terms of G, π, the action of π on G, and the class of the extension in $H^2(\pi$, centre $G)$. An example of this type of problem is to determine some intrinsically defined subgroup of B, for instance the centre of B, given knowledge of the corresponding subgroup for G and π, and, of course, the usual information concerning the extension.

In this paper we shall use the fact that any extension is congruent to a crossed product extension [**2**] to investigate a class of subgroups which naturally generalizes the notion of the centre. The definition of this class appears in § 3.

1. Let

$$E: 0 \to G \xrightarrow{\kappa} B \xrightarrow{\sigma} \pi \to 1$$

by an extension of G by π. We write G and B additively and π multiplicatively although none of the groups are necessarily abelian. It was shown in [**2**] that any such extension gives rise in a canonical way to a homomorphism

$$\psi: \pi \to \operatorname{Aut} G/\operatorname{In} G.$$

Given any function

$$\phi: \pi \to \operatorname{Aut} G, \qquad \phi(1) = \mathrm{id},$$

such that ϕ composed with the quotient map τ from $\operatorname{Aut} G$ to $\operatorname{Aut} G/\operatorname{In} G$ is ψ, there exists a function

$$f: \pi \times \pi \to G, \qquad f(x, 1) = f(1, x) = 0 \qquad \forall\, x \in \pi,$$

such that the following identities are valid:

$$\phi(x)f(y, z) + f(x, yz) = f(x, y) + f(xy, z), \tag{1}$$

$$\phi(x)\phi(y) = \mu[f(x, y)]\phi(xy), \tag{2}$$

where $\mu: G \to \operatorname{In} G$ is the obvious homomorphism. These functions allow one to define a group operation on $G \times \pi$ which is given by the formula

$$(g, x) + (g_1, y) = (g + \phi(x)g_1 + f(x, y), xy).$$

This group is denoted by $[G, \phi, f, \pi]$ and is called a crossed product. The exact sequence

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$$0 \to G \xrightarrow{i_1} [G, \phi, f, \pi] \xrightarrow{p_2} \pi \to 1$$

is called a crossed product extension of G by π. This new extension is congruent to the original extension, i.e., there exists a homomorphism

$$\rho: B \to [G, \phi, f, \pi]$$

such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & G & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & \pi & \to & 1 \\ & & \| & & \downarrow \rho & & \| & & \\ 0 & \to & G & \xrightarrow{i_1} & [G, \phi, f, \pi] & \xrightarrow{p_2} & \pi & \to & 1 \end{array}$$

commutes. ρ is necessarily an isomorphism. However, ρ is not unique. We will call any such ρ a congruence homomorphism.

We recall briefly how one obtains ϕ and f and what degree of freedom one has in choosing them, given a fixed extension E. One method is to choose a normalized coset representative; i.e., a function

$$u: \pi \to B, \qquad u(1) = 0, \qquad \sigma \circ u = \mathrm{id}_\pi.$$

Then ϕ, f, and ρ are uniquely determined by the following formulae:

(i) $\phi(x)g = u(x) + g - u(x), \forall g \in G, x \in \pi,$

(ii) $f(x, y) = u(x) + u(y) - u(xy), \forall x, y \in \pi,$

(iii) $\rho(b) = (b - u \circ \sigma(b), \sigma b), \forall b \in B.$

The second method is to choose any ϕ in the automorphism class of ψ and then choose a normalized coset representative in such a way that (i) holds. This was done in detail in [**2**]. The third method is to fix ϕ in the automorphism class of ψ ($\phi \in \psi$); then since the extension exists, there is a function $f: \pi \times \pi \to G$ such that the identities (1) and (2) are valid and E is congruent to the crossed product extension $[G, \phi, f, \pi]$. The different choices for f are in a one-to-one correspondence with the factor sets

$$h: \pi \times \pi \to \text{centre } G$$

which are cohomologous to zero, as identities (1) and (2) indicate. Once the choice is made, then ρ must be chosen so that if

$$u = \rho^{-1} \circ p_2: \pi \to B,$$

then equations (i), (ii), and (iii) are satisfied.

2. Let $[G, \phi, f, \pi]$ be a crossed product group.

Definition. A sub-crossed product of $[G, \phi, f, \pi]$ is a subgroup H of $[G, \phi, f, \pi]$ of the form $[G', \phi', f', \pi']$, where

$$G' \subseteq G, \quad \pi' \subseteq \pi, \phi(\pi') \subseteq \operatorname{Aut} G', \quad f(\pi' \times \pi') \subseteq G', \quad \phi' = \phi|\pi',$$

and

$$f' = f\,|\pi' \times \pi'.$$

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Note that not all subgroups of $[G, \phi, f, \pi]$ have this form.

Definition. Let $A \subset B$ be a subgroup of B. We say that A *splits* with respect to $[G, \phi, f, \pi]$ if there exists a sub-crossed product of $[G, \phi, f, \pi]$, $[G', \phi', f', \pi']$, and a congruence homomorphism ρ such that $\rho|A$ is an isomorphism from A onto $[G', \phi', f', \pi']$, i.e. there is a congruence homomorphism ρ which makes the following diagram commute.

$$\begin{tikzcd}
 & G \arrow[rr, tail, "\kappa"] \arrow[dd, equal] \arrow[dl, equal] & & B \arrow[rr, two heads, "\sigma"] \arrow[dd, "\rho"] \arrow[dl, equal] & & \pi \arrow[dd, equal] \arrow[dl, equal] \\
G' \arrow[rr, tail] \arrow[dd, equal] & & A \arrow[rr, two heads] \arrow[dd, "\rho|A"'] & & \pi' \arrow[dd, equal] & \\
 & G \arrow[rr, tail] \arrow[dl, equal] & & {[G, \phi, f, \pi]} \arrow[rr, two heads] \arrow[dl, equal] & & \pi \arrow[dl, equal] \\
G' \arrow[rr, tail] & & {[G', \phi', f', \pi']} \arrow[rr, two heads] & & \pi' &
\end{tikzcd}$$

We say that A is *absolutely split* with respect to $[G, \phi, f, \pi]$ if A is split for any choice of congruence homomorphism ρ.

It is obvious that if one considers the extension

$$E': 0 \to \kappa^{-1}(A) \to A \xrightarrow{\sigma} \sigma A \to 1,$$

then E' is congruent to the extension

$$0 \to G' \to [G', \phi', f', \pi'] \to \pi' \to 1$$

under the isomorphism $\rho|A$, and therefore it follows that if A is split with respect to $[G, \phi, f, \pi]$ then $G' = \kappa^{-1}(A)$ and $\pi' = \sigma A$.

The following proposition shows that for any subgroup A of B, there exist ϕ and f such that A is split with respect to $[G, \phi, f, \pi]$ since it is obviously always possible to choose a normalized coset representative with $u(\sigma A) \subseteq A$.

PROPOSITION 1. *Let $u: \pi \to B$ be a normalized coset representative and let ϕ_u, f_u, and ρ_u be the uniquely determined functions as in* § 1. *Then A splits with respect to $[G, \phi_u, f_u, \pi]$ and ρ_u if and only if $u(\sigma A) \subseteq A$.*

Proof. Since $E: G \rightarrowtail B \twoheadrightarrow \pi$ is congruent to the crossed product extension via $\rho_u: B \to [G, \phi, f, \pi]$, we have that $\rho_u \circ u: \pi \to [G, \phi, f, \pi]$ is the map $x \to (0, x)$. Now if $x \in \sigma A$ and A is split with respect to $[G, \phi, f, \pi]$, then $(0, x) \in \rho_u(A)$ and so $u(x) = \rho_u^{-1}(0, x) \in A$.

Conversely, if $u(\sigma A) \subseteq A$, then since $\phi_u(x)$ is given by means of conjugation by $u(x)$ and $G \cap A$ is normal in A, we have $\phi_u|\sigma A: \sigma A \to \mathrm{Aut}(\kappa^{-1}A)$ and $f_u: \sigma A \times \sigma A \to \kappa^{-1}(A)$ since $f_u(x, y) = u(x) + u(y) - u(xy)$. Finally,

$$\rho_u A = [\kappa^{-1}A, \phi', f', \sigma A]$$

since if $(g, x) \in \rho_u A$, then $x \in p_2\rho_u(A) = \sigma A$ and so $\rho_u u(x) = (0, x) \in \rho_u A$. Therefore $(g, 1) \in \rho_u A$ and so $g \in \kappa^{-1}A$. It follows that

$$(g, x) \in [\kappa^{-1}A, \phi', f', \sigma A].$$

The opposite inclusion is clear.

Our aim in this paper is to investigate the following two problems with respect to the subgroups of B defined in the next section. One problem is to find criterion on ϕ and f so that one of the subgroups in question is split or absolutely split with respect to $[G, \phi, f, \pi]$. We are especially concerned with the problem of the existence of an appropriate f when ϕ is held fixed, and in particular, when G is abelian. The other problem is to find some more explicit description of $\kappa^{-1}A$ and σA in terms of ϕ, f, G, and π so that when A is split with respect to $[G, \phi, f, \pi]$ it is possible to compute the subcrossed product $[\kappa^{-1}A, \phi', f', \sigma A]$ and hence determine A.

We conclude this section with an example that shows that it is not always possible, given a fixed ϕ, to find f such that the centre of B is split with respect to $[G, \phi, f, \pi]$. Let

$$E: 0 \to \mathrm{SL}_2(Z_5) \to \mathrm{GL}_2(Z_5) \xrightarrow{\det} Z_5^* \to 1$$

and let $\phi: Z_5^* \to \mathrm{Aut}(\mathrm{SL}_2(Z_5))$ be the homomorphism which sends the generator α of Z_5^* into conjugation by $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)$. We know that one can choose a normalized coset representative $u: Z_5^* \to \mathrm{GL}_2(Z_5)$ which realizes the homomorphism ϕ. Since centre $\mathrm{SL}_2(Z_5) \approx Z_2$, there are only two possible choices for u, namely

$$u(\beta) = \begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} \quad \text{or} \quad u(\beta) = \begin{pmatrix}-\beta & 0\\ 0 & -1\end{pmatrix}, \quad \beta \in Z_5^*.$$

However, in neither case does u map $\det(\text{centre } \mathrm{GL}_2(Z_5))$ into centre $\mathrm{GL}_2(Z_5)$ and so the centre of $\mathrm{GL}_2(Z_5)$ cannot be written as a sub-crossed product when one makes this choice of ϕ beforehand.

3. Let α be any infinite cardinal number or 2. We will write $||X||$ for the cardinality of the set X.

Definition. Let G be any group; then

$$C(\alpha, G) = \{g \in G|\ ||hgh^{-1}: h \in G|| < \alpha\}.$$

For example, $C(2, G)$ is the centre of G and $C(\aleph_0, G)$ is the subgroup of G consisting of those elements having finitely many conjugates. This last group has been studied in [**1**; **3**].

PROPOSITION 2. *$C(\alpha, G)$ is a characteristic subgroup of G.*

Proof. Since any group element and its inverse have the same number of conjugates and since the conjugate of a product is the product of the conjugates, we see that $C(\alpha, G)$ is a subgroup of G. If $T: G \to G$ is any automorphism of G, then

$$T: \{\text{conjugates of } h\} \to \{\text{conjugates of } T(h)\}$$

is a bijection of sets and therefore $C(\alpha, G)$ is characteristic. Let

$$E: 0 \to G \to B \to \pi \to 1$$

be an extension of G by π and let $\psi: \pi \to \operatorname{Aut} G/\operatorname{In} G$ be the associated "action". Recall that τ is the natural homomorphism from $\operatorname{Aut} G$ to $\operatorname{Aut} G/\operatorname{In} G$. In our description of $C(\alpha, B)$ we will need the following subgroups of G and π.

Definition.

$$F_\alpha(G) = \{g \in G \mid \|w(g), \forall w \in \tau^{-1}\psi(\pi)\| < \alpha\},$$

$$\Gamma_\alpha(\pi) = \{x \in \pi \mid \text{there exists } w_x \in \tau^{-1}\psi(x) \text{ such that } \|(1 - w_x)G\| < \alpha\}.$$

Remarks. (i) If G is abelian and $\alpha = 2$, then $F_\alpha(G)$ consists of the fixed points of G relative to $\psi(\pi) \subseteq \operatorname{Aut} G$. If $\alpha = \aleph_0$, then similarly $F_\alpha(G)$ consists of the points of G having finite orbits.

(ii) If G is not abelian, then if we choose any $\phi \in \psi$, i.e. any $\phi: \pi \to \operatorname{Aut} G$ in the automorphism class of ψ, then $F_\alpha(G)$ consists of the points of $C(\alpha, G)$ whose orbits relative to $\phi(\pi)$ have at most cardinality α.

(iii) If $\alpha = 2$, then $\Gamma_\alpha(\pi) = \operatorname{kernel} \psi$. This can be seen as follows. Clearly, $\operatorname{kernel} \psi \subseteq \Gamma_2(\pi)$ for we can always choose $w = 1$. On the other hand, it is clear that $x \in \Gamma_\alpha(\pi)$ if and only if there exists an inner automorphism $\mu[g_x]$ such that $\|(1 - \mu[g_x]\phi(x))G\| < \alpha$ when $\phi \in \psi$. Therefore if $x \in \Gamma_\alpha(\pi)$, then there is an inner automorphism $\mu[g_x]$ such that $h = (\mu[g_x]\phi(x))h$ for all $h \in G$, i.e. $\phi(x)(h) = \mu[g_x^{-1}](h)$ so that $\phi(x)$ is inner and therefore

$$\psi(x) = \tau\phi(x) = 1.$$

(iv) If G is abelian, then $\operatorname{In} G = (1)$ and so $\psi: \pi \to \operatorname{Aut} G$. It follows that the definition of $\Gamma_\alpha(\pi)$ simplifies to

$$\Gamma_\alpha(\pi) = \{x \in \pi \mid \|(1 - \psi(x))G\| < \alpha\}.$$

PROPOSITION 3. (1) $F_\alpha(G)$ is a normal subgroup of $G \subseteq C(\alpha, G)$.

(2) $\Gamma_\alpha(\pi)$ is a normal subgroup of π.

Proof. (1) If $g, h \in F_\alpha(G)$, then

$$\{w(gh^{-1}), w \in \tau^{-1}\psi(\pi)\} = \{w(g)w(h)^{-1}, w \in \tau^{-1}\psi(\pi)\}$$
$$\subseteq \{w(g), w \in \tau^{-1}\psi(\pi)\} \times \{w(h)^{-1}, w \in \tau^{-1}\psi(\pi)\}$$

and therefore $||w(gh^{-1}), w \in \tau^{-1}\psi(\pi)|| < \alpha$ since $g, h \in F_\alpha(G)$ and inversion is a bijection of sets.

$F_\alpha(G) \subseteq C(\alpha, G)$ since $\tau^{-1}\phi(\pi) \supseteq \mathrm{In}\, G$, and finally, $F_\alpha(G)$ is normal in G since for fixed $h \in G$,

$$\{w(hgh^{-1}), w \in \tau^{-1}\psi(\pi)\} = \{w(g), w \in \tau^{-1}\psi(\pi)\}.$$

(2) That $\Gamma_\alpha(\pi)$ is a subgroup follows from the following two equations:

$$(1 - w_x)g + w_x(1 - w_y)g = (1 - w_x w_y)g;$$
$$(1 - w_x)g = -w_x(1 - w_x{}^{-1})g.$$

If $w_x \in \tau^{-1}\psi(x)$ and $w_y \in \tau^{-1}\psi(y)$, then $w_x w_y \in \tau^{-1}\psi(xy)$ and $w_x{}^{-1} \in \tau^{-1}\psi(x^{-1})$.

To show that $\Gamma_\alpha(\pi)$ is normal in π, choose ϕ and f so that $[G, \phi, f, \pi]$ exists and yields an extension congruent to E. Now if $y \in \pi$,

$$\phi(yxy^{-1}) = \mu[f(y, xy^{-1})^{-1}]\phi(y)\mu[f(x, y^{-1})^{-1}]\phi(x)\phi(y^{-1})$$

by (2). But

$$\phi(y)\mu[f(x, y^{-1})^{-1}] = \mu[\lambda]\phi(y) \quad \text{for some } \lambda \in G.$$

that is

$$\phi(yxy^{-1}) = \mu[\tilde{\lambda}]\phi(y)\phi(x)\phi(y^{-1}) \quad \text{for some } \tilde{\lambda} \in G.$$

Therefore

$$\phi(x) = \phi(y)^{-1}\mu[\tilde{\lambda}^{-1}]\phi(yxy^{-1})\phi(y^{-1})^{-1}.$$

It follows that

$$1 - \mu[g_x]\phi(x) = 1 - \mu[g_x]\phi(y)^{-1}\mu[\tilde{\lambda}^{-1}]\phi(yxy^{-1})\phi(y^{-1})^{-1}$$
$$= -\phi(y^{-1})(1 - \mu[\gamma]\phi(yxy^{-1}))\phi(y^{-1})^{-1}$$

for some $\gamma \in G$ since $\phi(y)^{-1}$ and $\phi(y^{-1})$ differ by an inner automorphism of G.

Therefore, if $x \in \Gamma_\alpha(\pi)$, i.e. there exists $\mu[g_x]$ such that

$$||(1 - \mu[g_x]\phi(x))G|| < \alpha,$$

and since y is fixed, we have $||(1 - \mu[\gamma]\phi(yxy^{-1}))G|| < \alpha$, that is $yxy^{-1} \in \Gamma_\alpha(\pi)$.

We now state a lemma giving the conjugation formula in a crossed product group. The proof is straightforward but tedious. We will write g^h for the conjugation hgh^{-1}.

LEMMA. *In the crossed product group*, $[G, \phi, f, \pi]$,

$$(g_0, x_0)^{(g,x)} = (g + \phi(x)g_0 + f(x, x_0) - f(x_0{}^x, x) - \phi(x_0{}^x)g, x_0{}^x).$$

THEOREM 1. (i) $\kappa^{-1}(C(\alpha, B)) = F_\alpha G$.

(ii) $\sigma(C(\alpha, B)) \subseteq \pi_\alpha = \Gamma_\alpha(\pi) \cap C(\alpha, \pi)$.

Proof. (i) By Proposition 1 we can assume that $C(\alpha, B)$ is split with respect to $[G, \phi, f, \pi]$ with ρ a congruence homomorphism. Now

$$\rho(C(\alpha, B)) = [G', \phi', f', \pi'],$$

where $G' = \kappa^{-1}(C(\alpha, B))$ and $\pi' = \sigma(C(\alpha, B))$, and so it is sufficient to determine $C(\alpha, [G, \phi, f, \pi])$. Now suppose that $\bar{g} \in G' = \kappa^{-1}C(\alpha, B)$ so that $(\bar{g}, 1) \in \rho(C(\alpha, B))$. We see from the lemma that

$$(\bar{g}, 1)^{(g,x)} = (g + \phi(x)\bar{g} - \phi(1)g, 1).$$

Thus $||\mu[g]\phi(x)\bar{g}$ for $g \in G$ and $x \in \pi|| < \alpha$ and therefore $\bar{g} \in F_\alpha(G)$. Conversely, if $\bar{g} \in F_\alpha(G)$, then the lemma shows that $(\bar{g}, 1) \in \rho(C(\alpha, B))$ and so $\bar{g} \in G'$.

(ii) Let $x_0 \in \sigma(C(\alpha, B)) = \pi'$, i.e., there exists $\bar{g} \in G$ such that

$$(\bar{g}, x_0) \in \rho[C(\alpha, B)].$$

Using the lemma again we see that

$$(\bar{g}, x_0)^{(g,1)} = (g + \bar{g} - \phi(x_0)g, x_0).$$

Since $(\bar{g}, x_0) \in \rho(C_x(\alpha, B))$, this last set has cardinality $< \alpha$ for $g \in G$. But $\bar{g} - \phi(x_0)g = \mu[\bar{g}]\phi(x_0)(-g) + \bar{g}$; therefore the set $\{(1 - \mu[\bar{g}]\phi(x_0))G\}$ has cardinality $<\alpha$ and $x_0 \in \Gamma_\alpha(\pi)$. On the other hand, if we conjugate $(\bar{g}, x_0)$ by any (g, x), we see that the set of conjugates of x_0 must have cardinality $<\alpha$ since $(\bar{g}, x_0) \in \rho(C(\alpha, B))$ and so $x_0 \in C(\alpha, \pi)$.

Let us now suppose that G is abelian and try to find a "lower bound" for $\sigma(C(\alpha, B))$. This lower bound will depend on the choice of factor set representing the extension.

Definition. Let $f: \pi \times \pi \to G$ be a factor set for the extension E associated with $\phi: \pi \to \operatorname{Aut} G$. By $B(\alpha, f)$ we will mean the set

$$\{x \in \pi|\ ||f(y, x) - f(x^y, y)|| < \alpha \quad \forall y \in \pi\}.$$

THEOREM 2. *Suppose that G is abelian, and let $[G, \phi, f, \pi]$ be any crossed product group. If $x_0 \in \pi_\alpha$, then $(0, x_0) \in C(\alpha, [G, \phi, f, \pi])$ if and only if $x_0 \in B(\alpha, f)$.*

COROLLARY 1. $B(\alpha, f) \cap \pi_\alpha \subseteq \sigma C(\alpha, B)$.

COROLLARY 2. *If E is any extension of the abelian group G by the group π and if $f: \pi \times \pi \to G$ is any factor set representing the extension, then for any choice of coset representative $u: \pi \to B$ such that $f_u = f$, we have that if $x_0 \in \pi_\alpha$, then $u(x_0) \in C(\alpha, B)$ if and only if $x_0 \in B(\alpha, f)$.*

Proof of Theorem 2. Suppose that $x_0 \in \pi_\alpha$. The lemma shows that

$$(0, x_0)^{(g,x)} = (g + \phi(x)(0) + f(x, x_0) - f(x_0^x x) - \phi(x_0^x)g, x_0^x).$$

Since G is abelian, the first entry is

$$g - \phi(x_0^x)g + f(x, x_0) - f(x_0^x, x).$$

Now since $x_0 \in C(\alpha, \pi)$, x_0 has $\beta < \alpha$ distinct conjugates, say $\{x_0, x_1, \ldots\}$. Since $x_0 \in \Gamma_\alpha(\pi)$ and $\Gamma_\alpha(\pi)$ is normal in π, each $x_\gamma, \gamma \leqq \beta$, is in $\Gamma_\alpha(\pi)$. Therefore

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for each $\gamma \leqq \beta$ the set $\{(1 - \phi(x_\gamma))G\}$ has cardinality $<\alpha$. Since there are $\beta < \alpha$ such conjugates, the set $\{g - \phi(x_0^x)g;\ g \in G, x \in \pi\}$ has cardinality $< \alpha$. It follows that $x_0 \in B(\alpha, f)$ if and only if $(0, x_0) \in C(\alpha, [G, \phi, f, \pi])$.

Unfortunately the above theorem is false without the hypothesis that G be abelian. Even the following more restricted question can be answered in the negative without the hypothesis of commutativity for G.

Let $\phi \in \psi$ and let f be any factor set associated with ϕ. Then if

$$x_0 \in \pi_\alpha \cap B(\alpha, f),$$

does there exist $g_0 \in G$ such that

$$(g_0, x_0) \in C(\alpha, [G, \phi, f, \pi])?$$

In terms of extensions and not crossed products, this can be stated as: If $x_0 \in B(\alpha, f) \cap \pi_\alpha$, does there exist $u\colon \pi \to B$ such that $f_u = f$ and

$$u(x_0) \in C(\alpha, B)?$$

That this question has a negative answer is seen by means of the following example in which $C(\alpha, B)$ is the centre of B.

Let $G = D_8$, the dihedral group of order 8. Let $\pi = Z_2 \times Z_2$ and $\psi = 0$. Choose $\phi \in \psi$ to be the following homomorphism.

$$\phi(-1, 1) = \mu[\alpha],$$

$$\phi(1, -1) = \mu[\beta],$$

$$\phi(-1, -1) = \mu[\alpha\beta],$$

where α and β generate D_8, $\alpha^4 = 1$, $\beta^2 = 1$, and $\beta\alpha = \alpha^3\beta$. Since ϕ is a homomorphism, the associated factor set f is zero. It follows that the obstruction to this abstract kernel vanishes, and therefore the extension exists. It is easily seen that $\pi_2 \cap B(2, f) = \pi$, and so we must show that if $1 \neq x_0 \in Z_2 \times Z_2$, there does not exist any $g_0 \in G$ such that $(g_0, x_0) \in \text{centre}[G, \phi, f, \pi]$. Suppose that we are given $x_0 \in \pi$ such that g_0 exists. Then from the conjugation lemma we have, taking $g = 1$, that $\phi(x)g_0 = g_0$ for all $x \in \pi$. This implies that $g_0 = 1$ or $g_0 = \alpha^2$. However,

$$(1, x_0) \in \text{Centre} \Leftrightarrow g\phi(x_0)g^{-1} = 1 \text{ for all } g \in G$$
$$\Leftrightarrow \phi(x_0) = \text{id} \Leftrightarrow x_0 = 1.$$

If $g_0 = \alpha^2$, then $(\alpha^2, x_0) \in \text{centre} \Leftrightarrow g\alpha^2\phi(x_0)g^{-1} = \alpha^2$ for all $g \in G$. But $\alpha^2 \in \text{centre } D_8$ and so this states that $g\phi(x_0)g^{-1} = 1$ and again we see that $x_0 = 1$. We conclude therefore that $\sigma(\text{centre}) = (1)$, and so

$$\pi_2 \cap B(2, f) \not\subseteq \sigma(\text{centre}).$$

This example shows again that in the non-abelian case the choice of $\phi \in \psi$ must be involved in the description of the centre of the extension. Also, this extension was split and so even in this simple case the choice of ϕ becomes crucial.

Returning now to the abelian case, we combine our previous results into the following theorem.

THEOREM 3. *Let $f: \pi \times \pi \to G$ be any factor set associated with the extension $G \rightarrowtail B \twoheadrightarrow \pi$, where G is abelian. Then the following are equivalent:*

(a) *f splits $C(\alpha, B)$ absolutely, i.e. $C(\alpha, B)$ is split absolutely with respect to $[G, \phi, f, \pi]$;*

(b) *f splits $C(\alpha, B)$;*

(c) *$\sigma(C(\alpha, B)) \subseteq B(\alpha, f)$.*

Proof. (a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c). If f splits $C(\alpha, B)$, then there exists a congruence homomorphism $\rho: B \to [G, \phi, f, \pi]$. Let $u: \pi \to B$ be the associated coset representative. By Proposition 1, $u(\sigma(C(\alpha, B))) \subseteq C(\alpha, B)$. Therefore, if $x_0 \in \sigma(C(\alpha, B))$, then $(0, x_0) \in \rho(C(\alpha, B))$. Conjugating by $(0, x)$ shows that $x_0 \in B(\alpha, f)$.

(c) $\Rightarrow$ (a). If $\sigma(C(\alpha, B)) \subseteq B(\alpha, f)$, then by Theorem 1 and Corollary 1 to Theorem 2 we see that $\sigma(C(\alpha, B)) = \pi_\alpha \cap B(\alpha, f)$. Therefore if u is any coset representative such that $f_u = f$, (note that this is equivalent to choosing a congruence homomorphism of B to $[G, \phi, f, \pi]$), then if $x_0 \in \sigma(C(\alpha, B))$ we have by Corollary 2 to Theorem 2 that $u(x_0) \in C(\alpha, B)$. By Proposition 1, $C(\alpha, B)$ is split by u, i.e., f splits $C(\alpha, B)$ absolutely.

Note that from Proposition 1 we know that in the abelian case, there always exists an f which splits $C(\alpha, B)$. Theorem 3 states a condition so that splitting fs may be recognized and also states that for this class of subgroups splitting and absolute splitting are equivalent.

Under the same hypothesis as in the preceding theorem, we have the following result.

COROLLARY. *If $\sigma(C(\alpha, B)) \subseteq B(\alpha, f)$, then $C(\alpha, B)$ corresponds under any congruence homomorphism $\rho: B \to [G, \phi, f, \pi]$ to $[F_\alpha G, \phi', f', \pi_\alpha \cap B(\alpha, f)]$.*

Note that if the extension is split ($f \equiv 0$), then $C(\alpha, B)$ corresponds to $[F_\alpha G, \phi', f', \pi_\alpha]$.

We conclude with two examples in the abelian case. The first shows that $B(\alpha, f) \not\supseteq \sigma(C(\alpha, B))$ and therefore it is not true that every f splits $C(\alpha, B)$. The second example shows that it is possible for the sub-crossed product $[F_2 G, \phi', f', \sigma(C(2, B))]$ to exist but have no congruence homomorphism ρ which splits $C(2, B)$ with respect to $[G, \phi, f, \pi]$.

(1) Let

$$B = \left\{ \begin{pmatrix} 1 & 2a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in Z \right\}$$

and let

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$$G = \left\{ \begin{pmatrix} 1 & 0 & 2x \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y \in Z \right\}.$$

Then $G \lhd B$ and a simple calculation shows that

$$\text{centre } B = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; b \in Z \right\}.$$

If $\pi = B/G$, then

$$\pi = \left\{ \begin{pmatrix} 1 & 2a & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a \in Z, \beta, \gamma \in Z_2 \right\};$$

define $u: \pi \to B$ by

$$u\begin{pmatrix} 1 & 2a & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a & r(\beta) \\ 0 & 1 & r(\gamma) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if} \begin{pmatrix} 1 & 2a & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $r: Z_2 \to Z$ by $r(0) = 0$ and $r(1) = 1$; and

$$u\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we denote

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \pi$$

by x_0, and f_u is the associated factor set to u, then an easy calculation shows that

$$f(x_0, y) \neq f(y, x_0) \quad \text{if } y = \begin{pmatrix} 1 & 2a & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \pi,$$

but $x_0 \in \sigma(\text{centre } B)$ and so $\sigma(\text{centre } B) \not\subseteq B(2, f)$ and centre B does not split with respect to f.

(ii) For this example, we let

$$B = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in Z_4 \right\}$$

and

$$G = \left\{ \begin{pmatrix} 1 & 0 & 2b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; b, c \in Z_4 \right\}.$$

Then it is easily seen that

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$$\text{centre } B = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; b \in Z_4 \right\}.$$

Choose u as in the previous example. An easy calculation shows that

$$f \mid \sigma(\text{centre } B)^2 \colon \sigma(\text{centre } B)^2 \to F_2(G)$$

and therefore the sub-crossed product $[F_2(G), \phi', f', \sigma(\text{centre } B)]$ exists. However, this is not the centre of B because if it were, this choice of f would split the centre of B, and therefore split it absolutely. It follows that u would have to map $\sigma(\text{centre } B)$ into centre B which it does not.

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