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A NOTE ON HOPF BIFURCATION WITH DIHEDRAL GROUP SYMMETRY

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Abstract. We consider the standard action of the dihedral group $\mathbf{D}_n$ of order $2n$ on $\mathbf{C}$. This representation is absolutely irreducible and so the corresponding Hopf bifurcation occurs on $\mathbf{C} \oplus \mathbf{C}$. Golubitsky and Stewart (Hopf bifurcation with dihedral group symmetry: Coupled nonlinear oscillators. In: *Multiparameter Bifurcation Series*, M. Golubitsky and J. Guckenheimer, eds., Contemporary Mathematics **46**, Am. Math. Soc., Providence, R.I. 1986, 131–173) and van Gils and Valkering (Hopf bifurcation and symmetry: standing and travelling waves in a circular chain. *Japan J. Appl. Math.* **3**, 207–222, 1986) prove the generic existence of three branches of periodic solutions, up to conjugacy, in systems of ordinary differential equations with $\mathbf{D}_n$-symmetry, depending on one real parameter, that present Hopf bifurcation. These solutions are found by using the Equivariant Hopf Theorem. We prove that generically, when $n \neq 4$ and assuming Birkhoff normal form, these are the *only* branches of periodic solutions that bifurcate from the trivial solution.

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1. Introduction. When $n \geq 3$ the dihedral group $\mathbf{D}_n$ of order $2n$ has one and two-dimensional irreducible representations. Thus, in systems with $\mathbf{D}_n$-symmetry, Hopf bifurcation from a $\mathbf{D}_n$-invariant steady-state may occur by eigenvalues of multiplicity one or two crossing the imaginary axis. In this note we consider generic $\mathbf{D}_n$-Hopf bifurcation in the double eigenvalue case. Specifically, we consider the standard action of $\mathbf{D}_n$ on $V = \mathbf{C} \oplus \mathbf{C}$ (see Section 3). That is, V is the sum of two (isomorphic) absolutely irreducible representations where $\mathbf{D}_n$ acts on $\mathbf{C} \equiv \mathbf{R}^2$ in the standard way as symmetries of the regular n-gon. Although $\mathbf{D}_n$ has many distinct two-dimensional irreducible representations there is no loss of generality in making this assumption. Essentially it is possible to arrange for a standard action by relabeling the group elements and dividing by the kernel of the action.

Suppose we have a system of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda) \tag{1.1}$$

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where $x \in V$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, and $f : V \times \mathbf{R} \to V$ is smooth and commutes with $\mathbf{D}_n$:

$$f(\sigma \cdot x, \lambda) = \sigma \cdot f(x, \lambda) \quad (\sigma \in \mathbf{D}_n,\ x \in V,\ \lambda \in \mathbf{R})$$

Note that with these conditions we have $f(0, \lambda) \equiv 0$. Assume that $(df)_{(0,0)}$ has an imaginary eigenvalue, say i, after rescaling time if necessary. Golubitsky and Stewart [**2**] and van Gils and Valkering [**9**]) (see also Golubitsky *et al.* [**4**]) prove the generic existence of three branches of periodic solutions, up to conjugacy, of (1.1) bifurcating from the trivial solution. These solutions are found by using the Equivariant Hopf Theorem (Golubitsky *et al.* [**4**] Theorem XVI 4.1). They thus correspond to three (conjugacy classes of) maximal isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ (acting on V), each having a two-dimensional fixed-point subspace. In this note we prove in Theorem 4.2 that if we assume (1.1) satisfying the conditions of the Equivariant Hopf Theorem and f is in Birkhoff normal form then, when $n \neq 4$ and $n \geq 3$, the *only* branches of small-amplitude periodic solutions of period near 2π of (1.1) that bifurcate generically from the trivial equilibrium are the branches of solutions guaranteed by the Equivariant Hopf Theorem.

The case when $n = 4$ differs markedly from those other n. Swift [**8**] studies the dynamics of all possible square-symmetric codimension one Hopf bifurcations. In particular, it is shown that periodic solutions with submaximal symmetry bifurcate from the origin for open regions of the parameter space of the cubic coefficients in the Birkhoff normal form.

This paper is organized in the following way. In Section 2 we start by reviewing a few concepts and results related with the general theory of Hopf bifurcation with symmetry – we follow the approach of Golubitsky *et al.* [**4**]. In Section 3 we recall the conjugacy classes of $\mathbf{D}_n \times \mathbf{S}^1$ (with action on V) obtained by Golubitsky *et al.* [**4**]. For each n, there are five conjugacy classes and three of them correspond to isotropy subgroups with two-dimensional fixed-point subspaces. The next step is to recall the general form of the vector field f of (1.1). We assume that f is in Birkhoff normal form to all orders and so f commutes also with $\mathbf{S}^1$. Specifically, we choose coordinates such that $\theta \cdot z = e^{i\theta} z$ for all $\theta \in \mathbf{S}^1$, $z \in V$. Finally in Section 4 we obtain our main result – Theorem 4.2. We prove that when $n \neq 4$ and $n \geq 3$ generically the only branches of small-amplitude periodic solutions of (1.1) that bifurcate from the trivial equilibrium are those guaranteed by the Equivariant Hopf Theorem. The proof of this theorem relies mostly in the general form of f and the use of Morse Lemma.

2. Background.

We say that a system of ODEs

$$\dot{x} = f(x, \lambda), \quad f(0, 0) = 0 \tag{2.2}$$

where $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is a smooth function undergoes a *Hopf bifurcation* at $\lambda = 0$ if $(df)_{0,0}$ has a pair of *simple* purely imaginary eigenvalues. Here $(df)_{0,0}$ denotes the $n \times n$ Jacobian matrix of derivatives of f with respect to the variables x_j, evaluated at $(x, \lambda) = (0, 0)$. Under additional hypotheses of nondegeneracy, the standard Hopf Theorem implies the occurrence of a branch of periodic solutions. See for example Golubitsky and Schaeffer [**1**] Theorem VIII 3.1. Suppose now that Γ is a compact Lie group with a linear action on $V = \mathbf{R}^n$ and f *commutes* with Γ (or it is Γ-*equivariant*):

$f(\gamma \cdot x, \lambda) = \gamma \cdot f(x, \lambda)$ for all $\gamma \in \Gamma,\ x \in V,\ \lambda \in \mathbf{R}$. This imposes restrictions on the corresponding imaginary eigenspace that may complicate the analysis, and in general the standard Hopf Theorem does not apply directly. We outline the concepts and results involved in the study of (2.2) in presence of symmetry. We follow Golubitsky *et al.* [**4**] Chapter XVI. See also Golubitsky and Stewart [**3**] Chapter 4.

We are interested in branches of periodic solutions of (2.2) occurring by Hopf bifurcation from the trivial solution $(x, \lambda) = (0, 0)$. Suppose then that $(df)_{(0,0)}$ has a pair of imaginary eigenvalues $\pm\omega i$. It follows that the corresponding imaginary eigenspace $E_{\omega i}$ contains a Γ-*simple* subspace W of V ([**4**] Lemma XVI 1.2). Thus $W \cong W_1 \oplus W_1$ where W_1 is absolutely irreducible for Γ, or W is irreducible but non-absolutely irreducible for Γ. Moreover, generically the imaginary eigenspace itself is Γ-simple and coincides with the corresponding real generalized eigenspace of $(df)_{(0,0)}$. By rescaling time and choosing appropriate coordinates we may assume that $\omega = 1$ and

$$(df)_{0,0}|_{E_i} = \begin{pmatrix} 0 & -\mathrm{Id}_{m\times m} \\ \mathrm{Id}_{m\times m} & 0 \end{pmatrix} \equiv J$$

where $2m = \dim E_i$. See [**4**, Proposition XVI 1.4 and Lemma XVI 1.5].

The *orbit* of the action of Γ on $x \in V$ is defined to be

$$\Gamma x = \{\gamma \cdot x : \gamma \in \Gamma\}$$

and the *isotropy subgroup* of $x \in V$ is the subgroup Σ_x of Γ defined by

$$\Sigma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}$$

Points on the same group orbit have isotropy subgroups that are conjugate.

Note that if f as above is Γ-equivariant and if $x(t)$ is a solution of (2.2), then $\gamma \cdot x(t)$ is also a solution of (2.2). In particular, if f vanishes on $x \in V$, then it vanishes on the orbit Γx. Further, if the *fixed-point subspace* of $\Sigma \in \Gamma$ is

$$\mathrm{Fix}(\Sigma) = \{x \in V : \gamma \cdot x = x,\ \forall \gamma \in \Sigma\}$$

then

$$f\,(\mathrm{Fix}\,(\Sigma) \times \mathbf{R}) \subseteq \mathrm{Fix}\,(\Sigma)$$

We describe now what we mean by a symmetry of a periodic solution $x(t)$ of (2.2). Suppose that $x(t)$ is 2π-periodic in t. (If not, we can rescale time to make the period 2π, and consequently to change the eigenvalues $\pm\omega i$ of $(df)_{(0,0)}$ to $\pm i$.) Let $\gamma \in \Gamma$. Then $\gamma \cdot x(t)$ is another 2π-periodic solution of (2.2). If $\gamma \cdot x(t)$ and $x(t)$ intersect then the uniqueness of solutions implies that the trajectories must be identical. So either the two trajectories are identical or they do not intersect. Suppose that the trajectories are identical. Then uniqueness of solutions implies that there exists $\theta \in \mathbf{S}^1$ (we identify the circle group $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$) such that

$$\gamma \cdot x(t) = x(t - \theta)$$

We call $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ a *spatio-temporal symmetry* of the solution $x(t)$. Denote the space of 2π-periodic mappings by $\mathcal{C}_{2\pi}$. Note that $\mathbf{S}^1$ acts on $\mathcal{C}_{2\pi}$. This action of $\mathbf{S}^1$ is

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usually called the *phase-shift* action. The collection of all symmetries of $x(t)$ forms a subgroup

$$\Sigma_{x(t)} = \{(\gamma, \theta) \in \Gamma \times \mathbf{S}^1 : \gamma \cdot x(t) = x(t - \theta)\}$$

Assume now the generic hypothesis that $L = (df)_{0,0}$ has only one pair of imaginary eigenvalues, say $\pm i$. The method for finding periodic solutions (with period approximately 2π) of (2.2) rests on prescribing in advance the symmetry of the solutions we seek. We can apply a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i} = J$ acting on the imaginary eigenspace E_i of L. Moreover the reduced equation of f commutes with $\Gamma \times \mathbf{S}^1$. See [**4**] Lemma XXVI 3.2. Now small-amplitude periodic solutions of (2.2) of period near 2π correspond to zeros of a reduced equation $\phi(x, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. To find periodic solutions of (2.2) with symmetries Σ is equivalent to find zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$. See [**4**] Chapter XVI Section 4.

Consider (2.2) where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is smooth and commutes with a compact Lie group Γ and make the generic hypothesis that $\mathbf{R}^n$ is Γ-simple. Choose coordinates so that $(df)_{(0,0)} = J$ where $m = n/2$. The eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\rho(\lambda)$ where $\sigma(0) = 0$ and $\rho(0) = 1$ ([**4**] Lemma XVI 1.5). Suppose that

$$\sigma'(0) \neq 0. \tag{2.3}$$

Consider the action of $\mathbf{S}^1$ on $\mathbf{R}^n$ defined by:

$$\theta \cdot x = e^{i\theta J} x \quad (\theta \in \mathbf{S}^1,\ x \in \mathbf{R}^n)$$

The Equivariant Hopf Theorem [**4**, Theorem XVI 4.1] states that for each isotropy subgroup of $\Gamma \times \mathbf{S}^1$ with two-dimensional fixed-point subspace there exists a unique branch of small-amplitude periodic solutions of (2.2) with period near 2π, having that symmetry.

A tool for seeking periodic solutions that are not guaranteed by the Equivariant Hopf Theorem and also for calculating the stabilities of the periodic solutions is to use a *Birkhoff normal form* of f: by a suitable coordinate change, up to any given order, the vector field f can be made to commute with Γ and $\mathbf{S}^1$ (in the Hopf case). This result is the equivariant version of the Poincaré-Birkhoff Normal Form Theorem [**4**, Theorem XVI 5.1]. If we assume that the original vector field is in Birkhoff normal form (it commutes also with $\mathbf{S}^1$) then it is possible to perform a Liapunov-Schmidt reduction on (2.2) such that the reduced equation ϕ has the form

$$\phi(v, \lambda, \tau) = f(v, \lambda) - (1 + \tau)Jv$$

where τ is the period-scaling parameter ([**4**] Theorem XVI 10.1).

We finish this section by recalling a few results about invariant theory of compact groups. As before Γ is a compact Lie group with a linear action on a finite-dimensional (real) vector space V. A smooth function $f : V \to \mathbf{R}$ is said to be Γ-*invariant* if $f(\gamma \cdot x) = f(x)$ for all $\gamma \in \Gamma,\ x \in V$. The set of all smooth Γ-invariant functions is a ring under the usual operations of sum and product. By the Hilbert-Weyl Theorem ([**4**] Theorem XII 4.2) and its generalization to smooth functions by Schwarz [**7**] this ring is finitely generated. The set of all Γ-equivariant smooth mappings on V is a module over the ring of the Γ-invariant smooth functions. The Hilbert-Weyl Theorem

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also implies that there exists a finite-set of Γ-equivariant polynomial mappings $\mathbf{X}_1, \ldots, \mathbf{X}_t$ that *generate* the module of the Γ-equivariant smooth mappings on V over the ring of the smooth Γ-invariants (see **[4]** Theorem XII 5.2 and Poénaru **[5]**).

3. The action of $\mathbf{D}_n \times \mathbf{S}^1$. In this section we review the standard action of $\mathbf{D}_n \times \mathbf{S}^1$ on $\mathbf{C}^2$, the corresponding isotropy lattice and the general form of a $\mathbf{D}_n \times \mathbf{S}^1$-equivariant bifurcation problem. We follow Golubitsky *et al.* **[4]**, Chapter XVIII.

Let us assume that $\Gamma = \mathbf{D}_n$ where $n \geq 3$ acts on $\mathbf{C} \equiv \mathbf{R}^2$ in the standard way as symmetries of the regular n-gon. This action is generated by

$$\begin{aligned} \zeta \cdot z &= e^{i\zeta} z \quad \text{ where } \zeta = 2\pi/n \\ \kappa \cdot z &= \bar{z} \end{aligned}$$

Thus the cyclic subgroup $\mathbf{Z}_n$ of $\mathbf{D}_n$ consists of rotations of the plane through the angles $0, \zeta, 2\zeta, \ldots, (n-1)\zeta$, the *flip* κ is reflection in the x-axis and $\mathbf{D}_n = \langle \zeta, \kappa \rangle$. Suppose now that Γ acts on $\mathbf{C}^2$ by the diagonal action

$$\gamma \cdot (z_1, z_2) = (\gamma \cdot z_1, \gamma \cdot z_2) \quad (\gamma \in \mathbf{D}_n)$$

Note that $\mathbf{C}$ is absolutely irreducible for $\mathbf{D}_n$ and so $V = \mathbf{C}^2$ is $\mathbf{D}_n$-simple. We choose coordinates on $V = \mathbf{C}^2$ such that the action of $\mathbf{D}_n$ is generated by

$$\begin{aligned} \zeta \cdot (z_1, z_2) &= (e^{i\zeta} z_1, e^{-i\zeta} z_2) \\ \kappa \cdot (z_1, z_2) &= (z_2, z_1) \end{aligned} \tag{3.4}$$

Suppose we have the system of ODEs (1.1) where $f : V \times \mathbf{R} \to V$ commutes with $\mathbf{D}_n$ and $(df)_{0,0}$ has eigenvalues $\pm i$. Note that since $\mathrm{Fix}_V(\mathbf{D}_n) = \{0\}$ then $f(0, \lambda) \equiv 0$. Our aim is to study the generic existence of branches of periodic solutions of (1.1) near the bifurcation point $(x, \lambda) = (0, 0)$. We assume that f is in Birkhoff normal form, that is, f also commutes with $\mathbf{S}^1$, where we may assume that $\mathbf{S}^1$ acts on V by

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) \quad (\theta \in \mathbf{S}^1) \tag{3.5}$$

The isotropy lattice. Consider the subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ defined by

$$\begin{aligned} \widetilde{\mathbf{Z}}_n = \{(\gamma, -\gamma) : \gamma \in \mathbf{Z}_n\}, \quad \mathbf{Z}_2(\kappa) = \{\mathbf{1}, \kappa\}, \\ \mathbf{Z}_2(\kappa, \pi) = \{\mathbf{1}, (\kappa, \pi)\}, \quad \mathbf{Z}_2(\kappa\zeta) = \{\mathbf{1}, \kappa\zeta\} \end{aligned} \tag{3.6}$$

where $\zeta = 2\pi/n$ and $\mathbf{Z}_n = \langle \zeta \rangle$. Given the action of $\mathbf{D}_n \times \mathbf{S}^1$ $(n \geq 3)$ on V by (3.4) and (3.5), for each n there are five conjugacy classes of isotropy subgroups. They are listed, together with their orbit representatives and fixed-point subspaces in Tables 1, 2 and 3. (See Golubitsky *et al.* **[4]**, pp. 368–371.) Note that, up to conjugacy, for each n, we have three isotropy subgroups with two-dimensional fixed-point subspaces. It follows from the Equivariant Hopf Theorem, that there are (*at least*) three branches of periodic solutions occurring generically in Hopf bifurcation with $\mathbf{D}_n$-symmetry.

Invariant theory for $\mathbf{D}_n \times \mathbf{S}^1$. We calculate now the general form of a $\mathbf{D}_n \times \mathbf{S}^1$-equivariant bifurcation problem for the action of $\mathbf{D}_n \times \mathbf{S}^1$ on $\mathbf{C}^2$ generated by (3.4)

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Table 1. Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when n is odd.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa)$	$\{(w,w) : w \in \mathbf{C}\}$
$(a,-a)$	$\mathbf{Z}_2(\kappa,\pi)$	$\{(w,-w) : w \in \mathbf{C}\}$
$(a,w),\ w \neq \pm a, 0$	$\mathbf{1}$	$\mathbf{C}^2$

Table 2. Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when $n \equiv 2 \pmod 4$. Here $\mathbf{Z}_2^c = \{(0,0),(\pi,\pi)\}$.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$	$\{(w,w) : w \in \mathbf{C}\}$
$(a,-a)$	$\mathbf{Z}_2(\kappa,\pi) \oplus \mathbf{Z}_2^c$	$\{(w,-w) : w \in \mathbf{C}\}$
$(a,w),\ w \neq \pm a, 0$	$\mathbf{Z}_2^c$	$\mathbf{C}^2$

Table 3. Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when $n \equiv 0 \pmod 4$. Here $\mathbf{Z}_2^c = \{(0,0),(\pi,\pi)\}$.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$	$\{(w,w) : w \in \mathbf{C}\}$
$\left(a, e^{2\pi i/n}a\right)$	$\mathbf{Z}_2(\kappa\zeta) \oplus \mathbf{Z}_2^c$	$\{\left(w, e^{2\pi i/n}w\right) : w \in \mathbf{C}\}$
$(a,w),\ w \neq \pm a, 0$	$\mathbf{Z}_2^c$	$\mathbf{C}^2$

and (3.5). Define

$$m = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even} \end{cases} \tag{3.7}$$

Golubitsky *et al.* [**4**] (Proposition XVIII 2.1) prove that if $n \geq 3$ and m is as in (3.7), then the ring of the smooth $\mathbf{D}_3 \times \mathbf{S}^1$-invariant functions $f : \mathbf{C}^2 \to \mathbf{R}$ is generated by the polynomials

$$\begin{aligned} N = |z_1|^2 + |z_2|^2, \quad P = |z_1|^2|z_2|^2,\ S = (z_1\bar{z}_2)^m + (\bar{z}_1 z_2)^m, \\ T = i(|z_1|^2 - |z_2|^2)((z_1\bar{z}_2)^m - (\bar{z}_1 z_2)^m) \end{aligned} \tag{3.8}$$

Moreover, every smooth $\mathbf{D}_n \times \mathbf{S}^1$-equivariant function $f : \mathbf{C}^2 \to \mathbf{C}^2$ has the form

$$f(z_1,z_2) = A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix} + C\begin{bmatrix} \bar{z}_1^{m-1} z_2^m \\ z_1^m \bar{z}_2^{m-1} \end{bmatrix} + D\begin{bmatrix} z_1^{m+1}\bar{z}_2^m \\ \bar{z}_1^m z_2^{m+1} \end{bmatrix} \tag{3.9}$$

where A, B, C, D are complex-valued $\mathbf{D}_n \times \mathbf{S}^1$-invariant smooth functions.

4. Generic Hopf bifurcation with $\mathbf{D}_n$-symmetry. In Section 3 we show the conjugacy classes of isotropy subgroups for the action of $\mathbf{D}_n \times \mathbf{S}^1$ on $V = \mathbf{C}^2$ (Tables 1, 2 and 3). Up to conjugacy, for each $n \geq 3$, we have three isotropy subgroups with two-dimensional fixed-point subspaces. It follows from the Equivariant Hopf Theorem, that there are (*at least*) three branches of periodic solutions corresponding to each one of these isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ occurring in generic Hopf bifurcation with $\mathbf{D}_n$-symmetry. We prove in Theorem 4.2 that, when $n \neq 4$, generically these are the only branches of periodic solutions obtained through bifurcation from the trivial equilibrium in bifurcation problems with $\mathbf{D}_n$-symmetry (assuming Birkhoff normal form).

Suppose that the function $f : V \times \mathbf{R} \to V$ is $\mathbf{D}_n \times \mathbf{S}^1$-equivariant and smooth, and satisfies the conditions of the Equivariant Hopf Theorem. Thus we assume that

$$(df)_{0,\lambda}(z) = \mu(\lambda)z \tag{4.10}$$

where μ is a smooth function from $\mathbf{R}$ to $\mathbf{C}$ such that

$$\mu(0) = i, \quad \mathrm{Re}(\mu'(0)) \neq 0 \tag{4.11}$$

From [**4**] Theorem XVI 10.1 the small-amplitude periodic solutions of the equation

$$\dot{z} = f(z, \lambda) \tag{4.12}$$

of period near 2π are in one-to-one correspondence with the zeros of the equation

$$g(z, \lambda, \tau) = 0 \tag{4.13}$$

where $g = f - (1 + \tau)iz$ and τ is the period-scaling parameter. From (3.9) the general form of $f = (f_1, f_2)$ is

$$\begin{aligned} f_1(z_1, z_2, \lambda) &= \mu(\lambda)z_1 + Az_1 + Bz_1^2\bar{z}_1 + C\bar{z}_1^{m-1}z_2^m + Dz_1^{m+1}\bar{z}_2^m \\ f_2(z_1, z_2, \lambda) &= \mu(\lambda)z_2 + Az_2 + Bz_2^2\bar{z}_2 + Cz_1^m\bar{z}_2^{m-1} + D\bar{z}_1^m z_2^{m+1} \end{aligned} \tag{4.14}$$

where A, B, C, D are smooth $\mathbf{D}_n \times \mathbf{S}^1$-invariant functions from $V \times \mathbf{R}$ to $\mathbf{C}$ (thus they may depend on λ) and m is defined by (3.7). Since we are assuming (4.10) it follows that $A(0, \lambda) \equiv 0$. Let us consider g as in (4.13). Thus g has form

$$\begin{aligned} g_1(z, \lambda, \tau) &= (\nu + A)\, z_1 + Bz_1^2\bar{z}_1 + C\bar{z}_1^{m-1}z_2^m + Dz_1^{m+1}\bar{z}_2^m \\ g_2(z, \lambda, \tau) &= (\nu + A)\, z_2 + Bz_2^2\bar{z}_2 + Cz_1^m\bar{z}_2^{m-1} + D\bar{z}_1^m z_2^{m+1} \end{aligned} \tag{4.15}$$

where $\nu = \mu(\lambda) - (1 + \tau)i$.

LEMMA 4.1. *Consider f as in (4.14). Let $(z_1, z_2) = (r_1e^{i\phi_1}, r_2e^{i\phi_2})$ with $r_1, r_2 \in \mathbf{R}$ and let $\phi = \phi_2 - \phi_1$. Then we can write $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} r_1e^{i\phi_1}h(r_1,r_2,\phi,\lambda) \\ r_2e^{i\phi_2}h(r_2,r_1,-\phi,\lambda) \end{smallmatrix}\right]$ where h is a smooth function from $\mathbf{R}^4$ to $\mathbf{C}$.*

Proof. Let N, P, S and T be as in (3.8). Taking $(z_1, z_2) = (r_1e^{i\phi_1}, r_2e^{i\phi_2})$ and $\phi = \phi_2 - \phi_1$ we can write each of the invariant polynomials in the form

$$N = r_1^2 + r_2^2, \quad P = r_1^2r_2^2, \quad S = 2r_1^mr_2^m\cos(m\phi), \quad T = 2r_1^mr_2^m\sin(m\phi)\big(r_1^2 - r_2^2\big)$$

Recall now (4.14) and denote by

$$X_2 = \begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} \bar{z}_1^{m-1}z_2^m \\ z_1^m\bar{z}_2^{m-1} \end{bmatrix}, \quad X_4 = \begin{bmatrix} z_1^{m+1}\bar{z}_2^m \\ \bar{z}_1^m z_2^{m+1} \end{bmatrix}$$

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Then

$$X_j = \begin{bmatrix} r_1 e^{i\phi_1} h_j(r_1, r_2, \phi) \\ r_2 e^{i\phi_2} h_j(r_2, r_1, -\phi) \end{bmatrix} \tag{4.16}$$

where

$$\begin{aligned} &h_2(r_1, r_2, \phi) = r_1^2, \quad h_3(r_1, r_2, \phi) = r_1^{m-2} r_2^m(\cos(m\phi) + i\sin(m\phi)), \\ &h_4(r_1, r_2, \lambda) = r_1^m r_2^m(\cos(m\phi) - i\sin(m\phi)) \end{aligned} \tag{4.17}$$

It follows the result if we consider (4.14). □

THEOREM 4.2. *Consider* (4.12) *with* f *as in* (4.14) *where* $A(0, \lambda) \equiv 0$ *and* $\mu : \mathbf{R} \to \mathbf{C}$ *is smooth and satisfies* (4.11). *Suppose that* $n \neq 4$ *and* $n \geq 3$. *Then, generically, the system* (4.12) *admits only branches of periodic solutions that bifurcate from* (0, 0) *corresponding to the isotropy subgroups of* $\mathbf{D}_n \times \mathbf{S}^1$ *with two-dimensional fixed-point subspaces.*

Proof. We have that $\mathrm{Fix}_V(\mathbf{D}_n) = \{0\}$, consequently $f(0, \lambda) \equiv 0$. Therefore $(0, \lambda)$ is an equilibrium point of (4.12) for all values of λ. Since we are assuming that $(df)_{0,\lambda}(z) = \mu(\lambda)z$, where $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$, the stability of this equilibrium changes when λ crosses zero.

The space V is $\mathbf{D}_n$-simple and we are assuming (4.10) and (4.11) and so the conditions of the Equivariant Hopf Theorem are satisfied. Therefore, for each isotropy subgroup Σ of $\mathbf{D}_n \times \mathbf{S}^1$ with a two-dimensional fixed-point subspace, the system (4.12) admits a unique branch of periodic solutions with symmetry Σ by bifurcation from $(z, \lambda) = (0, 0)$. Moreover, this corresponds to a branch of zeros of (4.13) with the corresponding symmetry. We study now the existence of branches of periodic solutions of (4.12) with submaximal symmetry that bifurcate from (0, 0). We begin by looking for branches of zeros (z_1, z_2) of (4.15) with $z_1 z_2 \neq 0$. These satisfy

$$\begin{cases} \dfrac{g_1(z, \lambda, \tau)}{z_1} = 0 \\ \dfrac{g_2(z, \lambda, \tau)}{z_2} = 0 \end{cases} \tag{4.18}$$

Taking $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ and $\phi = \phi_2 - \phi_1$, by Lemma 4.1 we can write f in the form

$$\begin{bmatrix} r_1 e^{i\phi_1} h(r_1, r_2, \phi, \lambda) \\ r_2 e^{i\phi_2} h(r_2, r_1, -\phi, \lambda) \end{bmatrix}$$

and so (4.18) can be written as

$$\begin{cases} \nu + A + Br_1^2 + Cr_1^{m-2} r_2^m(\cos(m\phi) + i\sin(m\phi)) + D(r_1 r_2)^m(\cos(m\phi) - i\sin(m\phi)) = 0 \\ \nu + A + Br_2^2 + Cr_1^m r_2^{m-2}(\cos(m\phi) - i\sin(m\phi)) + D(r_1 r_2)^m(\cos(m\phi) + i\sin(m\phi)) = 0 \end{cases} \tag{4.19}$$

Taking the difference of the equations of (4.19) we obtain

$$\begin{aligned} B\left(r_1^2 - r_2^2\right) + C\,(r_1 r_2)^{m-2}\left(\cos(m\phi)\left(r_2^2 - r_1^2\right) + i\sin(m\phi)\left(r_1^2 + r_2^2\right)\right) \\ - 2iD\,(r_1 r_2)^m \sin(m\phi) = 0 \end{aligned} \tag{4.20}$$

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and so the real and imaginary parts of (4.20) should verify

$$\begin{cases} (r_2^2 - r_1^2)(C_R(r_1r_2)^{m-2}\cos(m\phi) - B_R) + \sin(m\phi)(r_1r_2)^{m-2}\big(2D_Ir_1^2r_2^2 - C_I\big(r_1^2+r_2^2\big)\big) = 0 \\ (r_2^2 - r_1^2)(C_I(r_1r_2)^{m-2}\cos(m\phi) - B_I) + \sin(m\phi)(r_1r_2)^{m-2}\big(C_R\big(r_1^2+r_2^2\big) - 2D_Rr_1^2r_2^2\big) = 0 \end{cases} \tag{4.21}$$

Here we use the notation $B_R = \mathrm{Re}(B)$, $B_I = \mathrm{Im}(B), \ldots$. Thus $B = B_R + iB_I, \ldots$, where $B_R, B_I, \ldots$, are smooth $\mathbf{D}_n \times \mathbf{S}^1$-invariant functions from $V \times \mathbf{R}$ to $\mathbf{R}$.

Assume the generic hypothesis

$$B_R(0) \neq 0$$

where $B_R(0)$ denotes the function B_R evaluated at the origin. Recall that $n \geq 3$ and $n \neq 4$. By (3.7) it follows that $m - 2 \geq 1$ and so in a sufficiently small neighborhood of the origin the system (4.21) can be writen as

$$\begin{cases} r_2^2 - r_1^2 = \dfrac{\sin(m\phi)(r_1r_2)^{m-2}\big(C_I\big(r_1^2+r_2^2\big) - 2D_Ir_1^2r_2^2\big)}{C_R(r_1r_2)^{m-2}\cos(m\phi) - B_R} \\ \sin(m\phi)\big((B_IC_I + B_RC_R)\big(r_1^2+r_2^2\big) + P(r_1, r_2, \lambda, m)\big) = 0 \end{cases} \tag{4.22}$$

where

$$P(r_1, r_2, \lambda, m) = (r_1r_2)^{m-2}\cos(m\phi)\big(2(C_ID_I + C_RD_R)r_1^2r_2^2 - \big(C_R^2 + C_I^2\big)\big(r_1^2+r_2^2\big)\big)$$
$$-2r_1^2r_2^2(B_RD_R + B_ID_I)$$

Assume the generic hypothesis

$$(B_IC_I + B_RC_R)(0) \neq 0$$

where $(B_IC_I + B_RC_R)(0)$ denotes the function $B_IC_I + B_RC_R$ evaluated at the origin. By Morse Lemma (see for example Poston and Stewart [**6**] Theorem 4.2), the equation

$$(B_IC_I + B_RC_R)\big(r_1^2+r_2^2\big) + P(r_1, r_2, \lambda, m) = 0$$

in a sufficiently small neighborhood of the origin admits only the trivial solution $(r_1, r_2) = (0, 0)$. Recall (3.7) and note that we are assuming $n \geq 3$ and $n \neq 4$. Thus $m - 2 \geq 1$. It follows that the system (4.22) in a sufficiently small neighborhood of the origin admits only branches of solutions (containing $(r_1, r_2) = (0, 0)$ and) satisfying

$$\begin{cases} \sin(m\phi) = 0 \\ r_1^2 = r_2^2 \end{cases} \tag{4.23}$$

Thus $\phi = k\pi/m$ for some integer k. We show below that these solutions correspond to the branches of periodic solutions of (4.12) guaranteed by the Equivariant Hopf Theorem. Note that the case $n = 4$ and so $m - 2 = 0$ is special. The existence of branches of periodic solutions of (4.12) with submaximal symmetry that bifurcate from $(0, 0)$ in a generic Hopf bifurcation with $\mathbf{D}_4$-symmetry is proved by Swift [**8**].

We show now the correspondence between the solutions of (4.23) and the periodic solutions of (4.12).

(i) We begin with the case when n is odd. We recall that

$$\mathrm{Fix}\,(\mathbf{Z}_2(\kappa)) = \{(w, w) : w \in \mathbf{C}\}$$

(see Table 1). It follows that $n+1 \geq 4$ is even and so $k(1+n)\pi/n \in \mathbf{Z}_n$. Moreover,

$$e^{\frac{ik(1+n)\pi}{n}} \cdot \left(w, e^{\frac{2ik\pi}{n}} w\right) = \left(we^{\frac{ik(1+n)\pi}{n}}, we^{\frac{2ik\pi}{n} - \frac{ik(1+n)\pi}{n}}\right)$$

and

$$\frac{2k\pi}{n} - \frac{k(1+n)\pi}{n} - \frac{k(1+n)\pi}{n} = -2k\pi$$

So, periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa)$ correspond to zeros of (4.13) where $r_1 = r_2$ and $\phi = 2k\pi/n,\ k \in \mathbf{Z}$ (or $r_1 = -r_2$ and $\phi = 2k\pi/n - \pi,\ k \in \mathbf{Z}$). In the case of $\mathbf{Z}_2(\kappa, \pi)$, we have

$$\mathrm{Fix}(\mathbf{Z}_2(\kappa, \pi)) = \{(w, -w) : w \in \mathbf{C}\}$$

Observe that if $n \equiv 3 \pmod 4$ then

$$e^{\frac{(2k+1)(1+n)\pi i}{2n}} \cdot \left(w, e^{\frac{(2k+1)i\pi}{n}} w\right) = \left(we^{\frac{(2k+1)(1+n)\pi i}{2n}}, we^{\frac{(2k+1)\pi i}{n} - \frac{(2k+1)(1+n)\pi i}{2n}}\right)$$

where

$$\frac{(2k+1)\pi}{n} - \frac{(2k+1)(1+n)\pi}{2n} - \frac{(2k+1)(1+n)\pi}{2n} = -(2k+1)\pi$$

and if $n \equiv 1 \pmod 4$ then

$$e^{\frac{(2k+1)(1-n)\pi i}{2n}} \cdot \left(w, e^{\frac{(2k+1)i\pi}{n}} w\right) = \left(we^{\frac{(2k+1)(1-n)\pi i}{2n}}, we^{\frac{(2k+1)\pi i}{n} - \frac{(2k+1)(1-n)\pi i}{2n}}\right)$$

where

$$\frac{(2k+1)\pi}{n} - \frac{(2k+1)(1-n)\pi}{2n} - \frac{(2k+1)(1-n)\pi}{2n} = (2k+1)\pi$$

So, periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa, \pi)$ correspond to zeros of (4.13) where $r_1 = r_2$ and $\phi = (2k+1)\pi/n,\ k \in \mathbf{Z}$ (or $r_1 = -r_2$ and $\phi = (2k+1)\pi/n - \pi, k \in \mathbf{Z}$).
(ii) We consider now the case where $n \equiv 2 \pmod 4$. We recall that

$$\mathrm{Fix}\,(\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c) = \{(w, w) : w \in \mathbf{C}\}, \quad \mathrm{Fix}\,(\mathbf{Z}_2(\kappa, \pi) \oplus \mathbf{Z}_2^c) = \{(w, -w) : w \in \mathbf{C}\}$$

(see Table 2). We prove, by the same method used in (i), that periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.13) where $r_1 = r_2$ and $\phi = 2k\pi/m,\ k \in \mathbf{Z}$, and periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa, \pi) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.13) where $r_1 = r_2$ and $\phi = (2k+1)\pi/m, k \in \mathbf{Z}$.
(iii) Finally we study the case where $n \equiv 0 \pmod 4$ and $n \neq 4$. We recall that

$$\mathrm{Fix}\,(\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c) = \{(w, w) : w \in \mathbf{C}\}, \quad \mathrm{Fix}\,(\mathbf{Z}_2(\kappa\zeta) \oplus \mathbf{Z}_2^c) = \{(w, e^{2\pi i/n} w) : w \in \mathbf{C}\}$$

(see Table 3). We prove, by the same method used in (i), that periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.13) where $r_1 = r_2$

and $\phi = 2k\pi/m,\ k \in \mathbf{Z}$, and that periodic solutions of (4.12) with symmetry (conjugate to) $\mathbf{Z}_2(\kappa\zeta) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.13) where $r_1 = r_2$ and $\phi = (2k+1)\pi/m, k \in \mathbf{Z}$.

We finish the proof considering the cases where $z_1 = 0$ and $z_2 \neq 0$. Let N, P, S and T be as in (3.8). In that case $N = |z_2|^2$, $P = S = T = 0$ and (4.15) takes the form

$$\begin{cases} g_1(z, \lambda, \tau) = 0 \\ g_2(z, \lambda, \tau) = (\nu + A)\, z_2 + B z_2^2 \bar{z}_2 \end{cases}$$

In this case we obtain zeros corresponding to a branch of periodic solutions with symmetry conjugate to $\widetilde{\mathbf{Z}}_n$. If $z_2 = 0$ and $z_1 \neq 0$ the situation is similar to this one. $\square$

REMARK 4.3. From the above proof, the nondegeneracy conditions (referred to in the word "generically") in Theorem 4.2 that guarantee that the only branches of periodic solutions with symmetry corresponding to isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ with two-dimensional fixed-point subspaces can bifurcate at $\lambda = 0$ for the equations (4.12) with f as in (4.14) are

$$B_R(0) \neq 0, \quad (B_I C_I + B_R C_R)(0) \neq 0$$

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