SOME DEFINITIONS OF KLEIN'S SIMPLE GROUP OF ORDER 168 AND OTHER GROUPS

by JOHN LEECH

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1. Introduction. In a previous note [**3**], Mennicke and I showed that the relations

$$(8, 7 \mid 2, 3): \quad A^8 = B^7 = (AB)^2 = (A^{-1}B)^3 = E$$

define a group of order 10752. As we remarked, the results of §§ 2, 3 of that note are not restricted in their application to this group; they apply to the group

$$[3, 7]^+: \quad B^7 = (AB)^2 = (A^{-1}B)^3 = E$$

and to any factor group of this group which in turn has Klein's simple group of order 168, defined by

$$(4, 7 \mid 2, 3): \quad A^4 = B^7 = (AB)^2 = (A^{-1}B)^3 = E,$$

as a factor group. In this note I use these results to establish alternative " weaker " definitions for Klein's group and for two groups discussed by Sinkov [**4**], namely (8, 7 | 2, 3) defined above and a factor group of this group of order 1344. These latter groups are eloquently discussed by Coxeter [**1**].

A mechanical technique for proving relations between generating elements of a group is discussed in an appendix. This was developed to prove certain relations stated in § 2 of this note, but it is of independent value and is described with reference to a simpler example in the appendix. It enables one to state without proof relations which can be proved thereby, since the details of proof can be supplied mechanically by following the precise rules given.

2. The kernel of the homomorphism between [3, 7]⁺ and (4, 7 | 2, 3). We showed in [**3**, § 2] that the seven elements

$$a = A^4, \quad c = B^{-1}A^4B, \quad e = B^{-2}A^4B^2, \quad g = B^{-3}A^4B^3,$$
$$b = B^{-4}A^4B^4, \quad d = B^{-5}A^4B^5, \quad f = B^{-6}A^4B^6,$$

generate a normal subgroup of $[3, 7]^+$ of index 168 whose quotient group is Klein's group (4, 7 | 2, 3). (In fact the subgroup is generated by any six of these elements, but it is convenient to keep all seven for reasons of symmetry.) It follows that any product of A's and B's which reduces to E in Klein's group is expressible as a product of these elements† $a, \ldots$. The mechanical technique described in the appendix was devised to facilitate the derivation of these expressions.

By this means we find that

$$A^{-1}aA = a, \quad A^{-1}bA = a^{-1}g^{-1}f^{-1}, \quad A^{-1}cA = f,$$
$$A^{-1}dA = b^{-1}, \quad A^{-1}eA = d^{-1}f^{-1}, \quad A^{-1}fA = a^{-1}c^{-1}, \quad A^{-1}gA = e^{-1},$$

† Throughout this note, sequences of dots designate the results of cyclic permutation of a, b, c, d, e, f, g.

as stated in [3]. Since conjugation by B effects only a cyclic permutation of $a, \ldots$, these expressions enable us to express any conjugate of an element of the subgroup by an element of the whole group in terms of the generators of the subgroup, which shows again that the subgroup is normal. In the same way we find that

$$(A^2B^4)^3 = afdb,$$
$$(A^2B^2)^4 = afge,$$
$$(A^3B^4)^3 = ae^{-1}b.$$

The products chosen are all those of the form A^mB^n with $m \leqq 4$ which have twice the period in $(8, 7 \mid 2, 3)$ that they have in Klein's group, except for certain conjugates, namely

$$AB^2 \sim A^2B^5 \sim A, \quad A^3B^6 \sim A^2, \quad A^3B^5 \sim A^2B^2.$$

Reference to the hyperbolic tessellation $\{3, 7\}$ (e.g. [3], figure 1) shows that these products are simple translations.

We also recall from [3] the relations

$$abcdefg = (A^4B^3)^7 = E,$$
$$afdbgec = (A^4B)^7 = E,$$

which follow since $(A^4B^3)^{-1}$ and A^4B are conjugates of B^2.

3. Klein's simple group of order 168. In this and the next section I give alternative definitions of Klein's group and of two groups discussed by Sinkov [4]. These groups are discussed as factor groups of

$$[3, 7]^+: \quad B^7 = (AB)^2 = (A^{-1}B)^3 = E,$$

and these relations are assumed hereafter without further reference in addition to those stated explicitly. In this section Klein's group is shown to be determined when the relation $A^4 = E$ is replaced by certain " weaker " relations 3.1, 3.2, 3.3 and 3.4.

3.1 A^4 *commutes with* B, i.e. $a = b = \ldots$. We have

$$a = A^{-1}aA = A^{-1}gA = e^{-1} = a^{-1},$$

so that $$a^2 = E.$$
Also $$abcdefg = a^7 = E.$$
Thus $$a = E.$$

3.2 $(A^2B^4)^3$ *commutes with* B, i.e. $afdb = fdbg = \ldots$.
Since $$afdbgec = E,$$
this implies $$afd = fdb = \ldots,$$
so that $$b = g = \ldots$$
and $$a = E,$$
by 3.1.

3.3 $(A^2B^2)^4$ *commutes with* A *and* B, i.e. $afge$ commutes with A and B. Since $afge$ commutes with B, we have

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$$afge = bgaf = \dots ,$$

and, since $afge$ commutes with af, we have

$$afge = geaf,$$

so that $$bg = ge = \dots .$$

In particular $$af = ec,$$

whence $$A^{-1}afA = A^{-1}ecA,$$

i.e. $$c^{-1} = d^{-1},$$

and again $$a = E$$

by 3.1.

3.4. $(A^3B^4)^3$ *commutes with* A *and* B, i.e. $ae^{-1}b$ commutes with A and B. Since $ae^{-1}b$ commutes with B, we have

$$ae^{-1}b = bf^{-1}c,$$

so that $$A^{-1}ae^{-1}bA = A^{-1}bf^{-1}cA,$$

i.e. $$afda^{-1}g^{-1}f^{-1} = a^{-1}g^{-1}f^{-1}caf = cafa^{-1}g^{-1}f^{-1},$$

since each expression equals $ae^{-1}b$ which commutes with caf. We thus have

$$afd = caf = \dots ,$$

and so $$a = E,$$

by 3.2.

It follows at once that the corresponding " stronger " relations

$$(A^2B^4)^3 = E, \quad (A^2B^2)^4 = E, \quad (A^3B^4)^3 = E$$

yield Klein's group (the first of these was given by Sinkov [4]). It may be remarked that if the group $[3, 7]^+$ is defined in terms of generators R, S, satisfying

$$R^2 = S^7 = (RS)^3 = E,$$

then A, A^2B^4, A^2B^2 are conjugates of $RS^{-1}RS$, $RS^{-2}RS^2$, $RS^{-3}RS^3$ respectively, so that Klein's group is determined when the period of any commutator of R with a power of S is prescribed, or when this power of a commutator is set to commute with R and S.

It is also remarkable that if we replace $(A^2B^2)^4$ by $(A^2B^2)^3$ in 3.3 or $(A^3B^4)^3$ by $(A^3B^4)^2$ in 3.4 we obtain larger groups, namely $LF(2, 13)$ of order 1092 and $LF(2, 2^3)$ of order 504 respectively.

4. Two groups discussed by Sinkov. Sinkov [4] proved that the relations $A^8 = (A^2B^4)^6 = E$ yield a factor group of $[3, 7]^+$ of order 10752. Mennicke and I [3] showed that the relation $(A^2B^4)^6 = E$ can be omitted, being implied by $A^8 = E$. I now show that this can be weakened further.

4.1. A^8 *commutes with B.* This implies that

$$a^2 = g^2 = A^{-1}g^2A = e^{-2} = a^{-2},$$

so that
$$a^4 = \dots = E,$$

and that
$$a^2 = a^{-2} = b^{-2} = A^{-1}b^{-2}A = (fga)^2 = \dots$$

and
$$a^2 = a^{-2} = f^{-2} = A^{-1}f^{-2}A = (ca)^2 = \dots .$$

The common square of a and its conjugates is thus of period 2 and commutes with every element of the group, so that the group is of order at most twice that given by $a^2 = E$. Further, every relation implied by $a^2 = E$ will be valid either as it stands or with an extra multiplier a^2 on one side.

We have
$$\begin{aligned} a^2 &= abcabc \\ &= a^2 \,.\, aba^{-1}c^{-1}bc, \qquad \text{since } a^2 = (ca)^2, \\ &= a^2 \,.\, abacbc \\ &= abab \,.\, bcbc \\ &= (ab)^2 \,.\, (bc)^2, \end{aligned}$$

so that
$$(ab)^2 = a^2(bc)^{-2} = (cd)^2 = a^2(de)^{-2} = ef^2$$
$$= a^2(fg)^{-2} = (ga)^2 = a^2(ab)^{-2},$$

whence $(ab)^2 = \dots$ and $(ab)^4 = \dots = a^2$.

Now $a^2 = E$ implies $(ab)^2 = E$, so that when a^2 commutes with B we have

$$\text{either} \quad (ab)^2 = E \quad \text{or} \quad (ab)^2 = a^2.$$

In the presence of $(ab)^4 = a^2$, each of these implies the other, and so

$$a^2 = (ab)^2 = E,$$

and we have the group $(8, 7 \mid 2, 3)$.

4.2 Sinkov also showed that the sets of relations

$$A^8 = A^4BA^4B^2A^4B^4 = E,$$
$$A^8 = A^4B^4A^4B^2A^4B = E$$

yield factor groups (distinct but isomorphic) of $(8, 7 \mid 2, 3)$ of order 1344. I now show that these relations can be weakened by the omission of the relation $A^8 = E$. In each case, since A^4 and A^{-4} are no longer equal by definition, the remaining relation generalizes to several alternatives, of the forms

$$A^{\pm 4}BA^{\pm 4}B^2A^{\pm 4}B^4 = E, \quad A^{\pm 4}B^4A^{\pm 4}B^2A^{\pm 4}B = E$$

respectively. Actually each of these relations can be further weakened to read that the product commutes with A and B, and this is all that is assumed below. I give the proofs for two combinations of signs. Proofs for the other cases can be readily constructed; most are as easy as 4.22, the case 4.21 being conspicuously more difficult and given in full for that reason.

4.21. *$A^{-4}BA^4B^2A^4B^4$ commutes with A and B*, i.e. $a^{-1}fb$ commutes with A and B. First we have

$$a^{-1}fb = c^{-1}ad = f^{-1}dg,$$

so that
$$A^{-1}c^{-1}adA = A^{-1}f^{-1}dgA,$$
i.e.
$$f^{-1}ab^{-1} = cab^{-1}e^{-1} = e^{-1}cab^{-1},$$

since each expression equals $a^{-1}fb$ which commutes with e^{-1}; thus

$$e^{-1}cf = E,$$
and so
$$a^{-1}fb = \dots = E.$$
We now have
$$g^{-1}ea = E,$$
so that
$$A^{-1}g^{-1}eaA = E,$$
i.e.
$$ed^{-1}f^{-1}a = E,$$
i.e.
$$ae = fd.$$
Again,
$$e^{-1}cf = E,$$
so that
$$A^{-1}e^{-1}cfA = E,$$
i.e.
$$fdfa^{-1}c^{-1} = E,$$
giving
$$aefa^{-1}c^{-1} = E,$$
i.e.
$$a^{-1}d^{-1}ef = E,$$
since $c^{-1}ad = E$, i.e.
$$ef = da,$$
so that
$$bc = ae$$
by B conjugation. Thus

$$bc = fd,$$
i.e.
$$bcd^{-1}f^{-1} = E,$$
i.e.
$$baf^{-1} = E,$$
since $c^{-1}ad = E$, i.e.
$$af^{-1}b = E.$$

Combining this with $a^{-1}fb = E$, we get
$$a^2 = f^2 = \dots,$$
and
$$a^2 = E,$$

by 4.1. We have thus proved both of Sinkov's relations, in the forms

$$a^2 = a^{-1}fb = E,$$

so that the group is that of order 1344 shown by him to be determined by these relations.

4.22. *$A^4BA^4B^2A^4B^4$ commutes with A and B*, i.e. afb commutes with A and B. We have

$$afb = ecf = A^{-1}ecfA = d^{-1}a^{-1}c^{-1} = A^{-1}d^{-1}a^{-1}c^{-1}A = ba^{-1}f^{-1} = a^{-1}f^{-1}b,$$

since afb commutes with b. We thus have

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$$a^2 = f^{-2} = \dots ,$$
and
$$a^2 = E$$

by 4.1. It follows at once that $afb = E$; for example, since $a = a^{-1}$, we may use the first part of 4.21, proving that $a^{-1}fb = E$, and again we have a group of order 1344.

With similar proofs in the other cases, we find that if any one of the products

$$A^{\pm 4}BA^{\pm 4}B^2A^{\pm 4}B^4, \quad \text{i.e.} \quad a^{\pm 1}f^{\pm 1}b^{\pm 1},$$
or
$$A^{\pm 4}B^4A^{\pm 4}B^2A^{\pm 4}B, \quad \text{i.e.} \quad a^{\pm 1}g^{\pm 1}c^{\pm 1},$$

commutes with both A and B, then both that product and A^8 reduce to E, and we have one of the groups of order 1344 shown by Sinkov to be determined by these relations. In each case the subgroup $\{a, \dots\}$ consists of the 8 elements E, a, b, c, d, e, f, g.

It may be shown, by the method of the appendix or otherwise, that $(A^2B^{-2}AB^2)^3 = abf^{-1}$. This yields the alternative forms $(A^2B^{-2}AB^2)^3 = E$ or $(A^2B^{-2}AB^2)^3$ commutes with A and B for these definitions.

5. Groups satisfying $(A^2B^4)^6 = E$. The relation $(A^2B^4)^6 = E$ does not imply $A^8 = E$, as it is consistent with the relation $A^7 = E$ in the group $(7, 7 \mid 2, 3)$ of order 1092; indeed it does not appear to imply that either the order of the group or the period of A is finite. It is of interest, however, to develop certain consequences of this relation, by means of which it is found to be incompatible with the period of A being 12 or 16.

We have
$$\begin{aligned}(A^2B^4)^3 &= afdb \\ &= b^{-1}d^{-1}f^{-1}a^{-1} \\ &= fdba^{-1},\end{aligned}$$

since $afdbgec = E$ implies that $(fdb)^{-1}$ is a conjugate of $(A^2B^4)^3$. Similarly

$$(A^2B^4)^3 = afdb = d^{-1}f^{-1}a^{-1}b,$$
so that
$$fdba^{-1} = d^{-1}f^{-1}a^{-1}b$$
and
$$(fd)^2 = a^{-1}bab^{-1}.$$
Now
$$(A^2B^4)^6 = E,$$
so that
$$(afdb)^2 = E, \quad (gec)^{-2} = E, \quad (dbg)^{-2} = E,$$
$$A^{-1}(dbg)^{-2}A = E, \quad (cd)^{-2} = E, \quad \text{and} \quad (ba)^{-2} = E;$$
hence
$$(fd)^2 = a^{-1}(ba)^{-1}b = a^{-2}b^{-2},$$
so that
$$(fd)^2b^2a^2 = E, \quad (af)^2d^2c^2 = E, \quad A^{-1}(af)^2d^2c^2A = E, \quad \text{and} \quad c^{-2}b^{-2}f^2 = E.$$
We thus have
$$f^2 = b^2c^2, \quad g^2 = c^2d^2, \quad \dots ,$$
and so
$$a^2f^2 = a^2b^2c^2 = e^2c^2 = e^2f^2g^2 = \dots ,$$
i.e.
$$a^2f^2 = b^2g^2 = \dots = a^2b^2c^2 = b^2c^2d^2 = \dots .$$

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These same conclusions also follow from the weaker assumption that $(A^2B^4)^6$ commutes with A and B; the proof is similar. We now show that these relations are incompatible with $A^{12} = E$ and with $A^{16} = E$.

5.1. $A^{12} = E$, i.e. $a^3 = E$. This implies $a^2 = a^{-1}$, $b^2 = b^{-1}, \ldots$, so that $e^2 = a^2b^2$ yields $e^{-1} = a^{-1}b^{-1}$, i.e. $e = ba$. But ba is a conjugate of $(A^2B^4)^3$ and so of period 2, while e is a conjugate of a and so of period 3. The relations are therefore incompatible and we have Klein's group with $A^4 = (A^2B^4)^3 = E$.

5.2. $A^{16} = E$, i.e. $a^4 = E$. This implies that $a^2 = a^{-2}$, $b^2 = b^{-2}, \ldots$, so that $e^2 = a^2b^2$ yields $e^{-2} = a^{-2}b^{-2}$, i.e. $e^2 = b^2a^2$. Thus a^2 commutes with b^2, similarly b^2 with c^2, etc. From

$$a^2b^2c^2 = b^2c^2d^2$$

we thus get

$$b^2a^2c^2 = b^2d^2c^2,$$

so that

$$a^2 = d^2,$$

and so

$$a^2 = E,$$

by 4.1. This group therefore reduces to $(8, 7 \mid 2, 3)$ with $A^8 = E$.

By slight modifications of these proofs, we can show that corresponding incompatibilities obtain between the weaker relations that $(A^2B^4)^6$ commutes with A and B and that either A^{12} or A^{16} commutes with B.

APPENDIX

In this appendix the method of Todd and Coxeter [5, 2] for establishing the order of a finite group by enumeration of cosets is applied to proving relations between the elements of a group. Todd and Coxeter choose a suitable subgroup of the group, which they define to be coset 1, and then define other cosets 2, 3, ... and deduce relations between cosets until finally they obtain a closed set of cosets each of which is transformed into itself or another when multiplied by a generator of the group. (I follow [2] rather than [5] in regarding right multiplication as standard.) The results can be listed in a multiplication table showing the effect on each coset of each generator of the group, and also, for convenience in working, the effect of the inverse of each generator which is not involutory. With such a table one can easily find the effect on coset 1 of multiplication by a general element of the group expressed as a product of generators. If the resulting coset is coset 1 itself, it follows that the element chosen is an element of the subgroup (perhaps the identity element) and is therefore expressible as a product of the generators of the subgroup. Unfortunately the coset multiplication table does not of itself offer any means of finding such an expression. However, in the special case where the coset enumeration is straightforward and does not involve the definition and subsequent elimination of redundant cosets, it is possible to retrace the steps made in constructing the multiplication table in such a way as to derive an expression for the element as a product of the generators of the subgroup. The procedure is described in the following simple worked example.

The octahedral group is defined by the relations

$$A^4 = E, \qquad (\alpha)$$
$$B^3 = E, \qquad (\beta)$$
$$(AB)^2 = E, \qquad (\gamma)$$

and we shall enumerate the six cosets of the subgroup $\{A\}$ in this group, keeping a systematic record of the order in which relations between cosets are obtained. First define $1\,A = 1$ (this defines coset 1 to be the subgroup $\{A\}$), $1B = 2$, $1B^{-1} = 3$. Relation (γ) now yields

$$2A = 3 \qquad (1)$$

and (β) yields $$2B = 3. \qquad (2)$$

Now define $2A^{-1} = 4$, $3A = 5$. (α) now yields

$$5A = 4 \qquad (3)$$

and (γ) yields $$5B = 4. \qquad (4)$$

Finally define $4B = 6$. Then (β) yields

$$6B = 5 \qquad (5)$$

and (γ) yields $$6A = 6. \qquad (6)$$

The enumeration is now complete and we have the multiplication table given below. A subscript figure denotes that the entry is one made by deduction and corresponds to the reference number given above; entries without subscripts correspond to definitions of cosets.

	A	A^{-1}	B	B^{-1}
1	1	1	2	3
2	3_1	4	3_2	1
3	5	2_1	1	2_2
4	2	5_3	6	5_4
5	4_3	3	4_4	6_5
6	6_6	6_6	5_5	4

We are now in a position to carry out the reductions. Consider for example the element $(B^{-1}AB^{-1})^{-1}A(B^{-1}AB^{-1})$. First we write this in full with coset numbers beneath the spaces and subscripts beneath appropriate letters, thus:

$$\begin{array}{ccccccccccccccc} & B & & A^{-1} & & B & & A & & B^{-1} & & A & & B^{-1} & . \\ 1 & & 2 & & 4 & & 6 & {}_6 & 6 & & 4 & & 2 & & 1 \end{array}$$

This denotes that $1B = 2$, $2A^{-1} = 4$, etc., and that $6A = 6$ was the sixth deduced relation between cosets, the other relations between pairs of cosets being definitions. Since the final coset is coset 1, we infer that this element is an element of the subgroup, i.e. a power of A, and we shall find what power. Deduction (6) was made from relation (γ): $(AB)^2 = E$; we replace A by $B^{-1}A^{-1}B^{-1}$ and, after cancelling an adjacent pair BB^{-1}, we obtain

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$$\begin{array}{ccccccccccccccc} & B & & A^{-1} & & A^{-1} & & B^{-1} & & B^{-1} & & A & & B^{-1} & \\ 1 & & 2 & & 4 & {}_3 & 5 & {}_5 & 6 & & 4 & & 2 & & 1 \end{array}.$$

Here we have two pairs of cosets corresponding to deduced relations between cosets; we replace the later in order of deduction, namely $5B^{-1} = 6$ deduced from (β): $B^3 = E$, and obtain

$$\begin{array}{ccccccccccccc} & B & & A^{-1} & & A^{-1} & & B & & A & & B^{-1} & \\ 1 & & 2 & & 4 & {}_3 & 5 & {}_4 & 4 & & 2 & & 1 \end{array}.$$

Continuing similarly, always replacing the generator corresponding to the latest deduced relation between cosets, we find successively

$$\begin{array}{ccccccccccccc} & B & & A^{-1} & & A^{-1} & & A^{-1} & & B^{-1} & & B^{-1} & \\ 1 & & 2 & & 4 & {}_3 & 5 & & 3 & {}_2 & 2 & & 1 \end{array},$$

$$\begin{array}{ccccccccc} & B & & A & & B^{-1} & & B^{-1} & \\ 1 & & 2 & {}_1 & 3 & {}_2 & 2 & & 1 \end{array},$$

$$\begin{array}{ccccccc} & B & & A & & B & \\ 1 & & 2 & {}_1 & 3 & & 1 \end{array},$$

$$\begin{array}{ccc} & A^{-1} & \\ 1 & & 1 \end{array}.$$

The reduction is now complete, and we have shown that $(B^{-1}AB^{-1})^{-1}A(B^{-1}AB^{-1}) = A^{-1}$. If desired, the final reduction can be exhibited without reference to the method of derivation in the form

$$\begin{aligned}(B^{-1}AB^{-1})^{-1}A(B^{-1}AB^{-1}) &= BA^{-1}BAB^{-1}AB^{-1} = BA^{-1}A^{-1}B^{-1}B^{-1}AB^{-1} \\ &= BA^{-1}A^{-1}BAB^{-1} = BA^{-1}A^{-1}A^{-1}B^{-1}B^{-1} \\ &= BAB^{-1}B^{-1} = BAB = A^{-1},\end{aligned}$$

where each expression has been derived from that preceding it by a substitution according to one of the defining relations of the group, and this can be readily checked by inspection.

This method does not, then, give reductions not otherwise obtainable, as the sequence above could easily have been found by trial. What it does offer is a strictly determinate technique for finding reductions by following mechanically the rule of always substituting for the generator corresponding to the latest deduced relation between cosets in the original order of deduction. Since this generator is replaced by others corresponding to definitions or earlier deductions, every substitution makes positive progress towards eliminating deduced relations between cosets, and after a clearly bounded number of substitutions we are left with a sequence of generators relating pairs of cosets according to the original definitions specifying coset 1, i.e. a product of generators of the chosen subgroup.

Since the order of reduction is strictly determinate, the only variations being because the order of enumeration of cosets is not unique, it is possible to suppress the details entirely, as I have done several times in §2 of this note. A reader wishing to verify these reductions may do so either by enumerating the 168 cosets of the subgroup $\{B^{-n}A^4B^n\,;\, n = 0, 1, \dots, 6\}$ in

$[3, 7]^+$, or, more simply, by enumerating the 24 cosets of $\{A^4, B\}$ in $[3, 7]^+$, and following the method of this appendix. It is sufficient to work with the latter subgroup as the elements of the former are precisely those of the latter having indices of B whose sum is zero modulo 7, and the expression of such elements of $\{A^4, B\}$ as elements of $\{B^{-n}A^4B^n\}$ is immediate. It is of no consequence to the enumeration method of Todd and Coxeter, or to this development of it, that neither the group nor the subgroup is here of finite *order*. All that is needed is the finite *index*, given by the number of cosets. Even this may be unnecessary to the present development if, in an incomplete enumeration of cosets, one can define and deduce sufficient to exhibit that $1X = 1$ for the chosen element X of the group.

Some of the reductions quoted in § 2 of this note are long and far from straightforward, involving intermediate products of much greater length than either the initial or final product (and they would have been extremely difficult to find by trial). Indeed the reductions are commonly neither short nor elegant, and one can sometimes simplify them substantially by using other relations deducible from the given defining relations. This is of little importance, however, since the reductions are reproducible by simply following the procedure given here, and they can therefore be suppressed when the results are found.

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