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COMPLEMENTS OF MINIMAL IDEALS IN SOLVABLE LIE RINGS

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ABSTRACT. Conditions for the existence and conjugacy of complements of certain minimal ideals of solvable Lie algebras over a Noetherian ring R are considered. Let L be a solvable Lie algebra and A be a minimal ideal of L. If L/A is nilpotent and L is not nilpotent then A has a complement in L, all such complements are conjugate and self-normalizing and if C is a complement then there exists an $x \in L$ such that $C = \{y \in L;\ yad^n x = 0$ for some $n = 1, 2, \ldots\}$. A similar result holds if A is self-centralizing and a finitely generated R-module.

In this note conditions for the existence and conjugacy of complements of certain abelian ideals of solvable Lie algebras over a Noetherian ring are considered. Similar results have been found in group theory by Newell [5] and in finite dimensional Lie algebras by Barnes [4].

We call a ring Noetherian if it is commutative, has a unit and its ideals satisfy the ascending chain condition. Let R be a Noetherian ring. All algebras and modules over R are assumed unital. Let L be a Lie algebra over R, $x \in L$ and H be a subalgebra of L which is adx invariant. Let n be a non-negative integer. Let $E_H^n(x) = \{y \in H;\ yad^n x = 0\}$, $E_H(x) = \bigcup_{n=0}^{\infty} E_H^n(x)$, $H_n(x) = Had^n x$ and $H_\omega(x) = \bigcap_{n=0}^{\infty} H_n(x)$. Let K be another subalgebra of L and let $N_H(K) = \{a \in H;\ Kada \subseteq K\}$ and $C_H(K) = \{a \in H;\ Kada = 0\}$.

LEMMA 1. *Let L be a Lie algebra over a commutative ring and let $A \trianglelefteq L$. Let $N \trianglelefteq L$ such that $[L, N] \subseteq C_L(A)$. Then for each $x \in N$, $\sigma_x = adx$ restricted to A is an L-endomorphism of A. Assume that σ_x is onto. Then A is abelian and $E_{L/A}(A + x) = (A + E_A(x))/A$. Furthermore, if $E_N(x)$ is a complement of A in N, then all complements of A in N are conjugate.*

Proof. Let $y \in L$ and $a \in A$. Then $[y, a]\sigma_x = [[y, a], x] = [[y, x], a] + [y, [a, x]] = [y, [a, x]] = [y, (a)\sigma_x]$. Hence σ_x is an L-endomorphism of A.

For the remainder of the proof, assume that σ_x is onto. Then $A = A\sigma_x \subseteq [L, N] \subseteq C_L(A)$, hence A is abelian. Let $A + y \in E_{L/A}(A + x)$. Then there exists an n such that $(y)ad^n x \in A$ and $(y)ad^n x = (a)ad^n x$ for some $a \in A$. Therefore $(a - y)ad^n x = 0$ and $a - y = t \in E_L(x)$. Then $y = a - t \in A + E_L(x)$. Therefore

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$A+y \subseteq A+E_L(x)$ and $E_{L/A}(A+x) \subseteq (A+E_L(x))/A$. Since the reverse inclusion is clear, equality holds.

Assume that $N=A+E_N(x)$ and $A \cap E_N(x)=0$. Let C be another complement of A in N. Then $x=c+a$ for some $c \in C$ and $a \in A$. Since σ_x is onto, $a=[b,x]$ for some $b \in A$ and $c=x+[x,b]=(x)\exp(adb)$ and $A+x=A+c$. Since $N/A \subseteq E_{L/A}(A+x)$, it follows that for each $y \in C$, $yad^n c \in A \cap C$ for some n, depending on y. Therefore $C \subseteq E_N(c)$. Since $A \cap E_N(x)=0$ and $c=(x)\exp(adb)$, it follows that $A \cap E_N(c)=0$. Hence $C=E_N(c)=E_N(x)\exp(adb)$ is a conjugate of $E_N(x)$.

Recall that $x \in L$ is called left-Engel if $E_L(x)=L$.

THEOREM 1. *Let L be a Lie algebra over a commutative ring, let A be a minimal ideal of L and let N be an ideal in L which contains A such that $[L,N] \subseteq C_L(A)$ but $N \not\subseteq C_L(A)$. Suppose that N/A is generated by left-Engel elements. Then A has a complement in L, all such complements are conjugate and self-normalizing in L and each is of the form $E_L(z)$ for some $z \in N$.*

Proof. Choose $x \in N$ such that $x \notin C_L(A)$ and $A+x$ is left-Engel in N/A. Hence adx is a non-trivial L-endomorphism of A and, since A is minimal, must be an isomorphism. Since $N/A=E_{N/A}(A+x)$, it follows from Lemma 1 that $N=A+E_N(x)$. Since adx is an isomorphism of A, $A \cap E_N(x)=0$. Hence A has a complement in N. Now $E_L(x)$ is a complement of A in L, again using Lemma 1. Let H be another complement of A in L. Since both H and $E_L(x)$ complement A in L, the map $\psi: E_L(x) \to H$ defined by $(y)\psi=$ the unique element of $(A+y) \cap H$ is an isomorphism. Putting $(y+a)\alpha=(y)\psi+a$ for $y \in E_L(x)$ and $a \in A$ defines an automorphism α of L, with $(E_L(x))\alpha=H$. Thus $H=E_L((x)\alpha)$. But $(x)\alpha=(x)\psi=x+a$ for some $a \in A$ and since adx is an isomorphism on A, $a=[b,x]$ for some $b \in A$. Thus $(x)\alpha=x-[x,b]=(x)\theta$ where $\theta=\exp(ad(-b))$. Hence $H=E_L((x)\theta)=(E_L(x))\theta$ and complements to A are conjugate. Since $N_L(E_L(z))=E_L(z)$ for any $z \in L$, all complements are self-normalizing and the result holds.

As a consequence we have

THEOREM 2. *Let L be a Lie algebra over a commutative ring and let A be a minimal ideal of L. Assume that A is abelian, L/A is nilpotent, but L is not nilpotent. Then A has a complement in L, all such complements are conjugate and self-normalizing and each is of the form $E_L(z)$ for some $z \in L$.*

Proof. Since $C_L(A) \neq L$, $L/C_L(A)$ has a non-trivial center, $N/C_L(A)$. Then N satisfies the conditions on N in the previous theorem. Hence the conclusion follows from that result.

In order to obtain our other splitting result, a version of Engel's theorem similar to the one given in [2] is needed. However a somewhat different form is needed. (See the example [3, p. 417]).

Consequently we need

LEMMA 2. *Let R be a Noetherian ring and let L be a nilpotent Lie algebra over R. Let A be a non-zero L-module which is a finitely generated R-module. For $y \in L$, let $A_\omega(y) = \bigcap_{i=0}^{\infty} Ay^i$. Suppose that $A_\omega(y) = 0$ for all $y \in L$. Then AL is properly contained in A.*

Proof. We may assume that the module is a faithful L-module. Then L is isomorphic to an R-submodule of a finitely generated R-module and hence is Noetherian (as an R-module). We show that if the result holds for any subalgebra M properly contained in L, then there exists a subalgebra N of L such that $M \subset N \subseteq L$ and for which the result holds. Therefore assume that AM is properly contained in A. Since L is nilpotent, there exists $y \in N_L(M)$, $y \notin M$. Then AM is y-invariant. Let $\bar{A} = A/AM$. Then $\bar{A}y$ is properly contained in $\bar{A}$. Hence, letting $N = \langle M, y \rangle$, AN is properly contained in A. Since any one-dimensional subalgebra of L satisfies the conditions for M, an increasing chain of subalgebras of L is obtained, each of which satisfies the conclusion of the lemma. This chain must terminate at some subalgebra which, because of the above process, must be L. Hence AL is properly contained in A.

THEOREM 3. *Let L be a solvable Lie algebra over a Noetherian ring R. Let A be a minimal ideal of L such that A is a finitely generated R-module and $C_L(A) = A$. Then A is complemented in L, all such complements are conjugate and self-normalizing in L and of the form $E_L(z)$ for some $z \in L$.*

Proof. Since L is solvable, L/A contains an abelian ideal, M/A. Let $y \in M$. We claim that $A_\omega(y) \trianglelefteq M$. Let $x \in M$, $a \in A$. Then $((a)ad^n y)adx = ((a)adx)ad^n y$ since $M^2 \subseteq A$ and A is abelian. Hence $A_n(y)adx \subseteq A_n(y)$, $A_\omega(y)adx \subseteq A_\omega(y)$ and $A_\omega(y) \trianglelefteq M$. Now we claim that $A_\omega(y) \trianglelefteq L$. Let $z \in L$. Then $((a)ad^n y)adz = ((a)adz)ad^n y + \sum_{i=0}^{n-1} (a)ad^i yad[y, z]ad^{n-i-1} y$ using only that adz is a derivation. Clearly $((a)adz)ad^n y \in A_n(y)$ and everything under the summation is in $A_{n-1}(y)$ since $[y, z] \in M$ and $A_i(y)ad[y, z] \subseteq A_i(y)$. Therefore $((a)ad^n y)adz \in A_n(y) + A_{n-1}(y) = A_{n-1}(y)$. Hence $A_n(y)adz \subseteq A_{n-1}(y)$. Therefore $A_\omega(y)adz \subseteq A_\omega(y)$ and $A_\omega(y) \trianglelefteq L$. Since A is minimal, either $A_\omega(y) = A$ or $A_\omega(y) = 0$. We consider two cases

CASE I. $A_\omega(y) = 0$ for all $y \in M$. Since $[[A, M], L] \subseteq [M, A]$ and A is minimal, it follows that $[M, A] = 0$ or $[M, A] = A$. The former contradicts that $C_M(A) = A$. The latter contradicts Lemma 2.

CASE II. $A_\omega(y) = A$ for some $y \in M$. Since $Aady = A$, it follows that $A = Lad^n y$ for all $n \geq 2$. By [1, Lemma 2.6, p. 246], $L = E_L(y) + A_\omega(y) = E_L(y) + A$. Since $A \trianglelefteq L$, $E_L(y) \cap A \trianglelefteq E_L(y)$ and since A is abelian, $E_L(y) \cap A \trianglelefteq A$. Hence $E_L(y) \cap A \trianglelefteq L$. Then either $E_L(y) \supseteq A$ or $E_L(y) \cap A = 0$. Assume that $E_L(y) \supseteq A$. Then $E_A(y) = A$. By the ascending chain condition on submodules of A,

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there exists a positive integer n such that $E_A(y)=E_A^n(y)$. Hence $0=Aad^ny=A_\omega(y)=A$ which is a contradiction. Therefore $E_L(y)\cap A=0$ and $E_L(y)$ is a complement of A.

Let $U=E_L(y)$ and V be another complement of A in L. For $v_1\in V$, $v_1-y=v_2+c$ where $v_2\in V$, $c\in A$. Then $y+c=v_1-v_2=v\in V$. Since $A_\omega(y)=A$, there exists an $a\in A$ such that $[y, a]=c$. Let $\mathcal{T}_a=\exp(ada)$. Then $(y)\mathcal{T}_a=y+[y, a]=y+c=v$. Now $U\mathcal{T}_a=E_L(y)\mathcal{T}_a=E_L(v)$. We claim that $E_L(v)\supseteq V$. Let $x\in V$. Then $(x)ad^2v=[[x, y+c], y+c]\in A_2(y)+A=A$. Hence $(x)ad^2v\in A\cap V=0$, $x\in E_L(v)$ and $V\subseteq E_L(v)$. Since V and $E_L(v)=U\mathcal{T}_a$ are complements of A, $E_L(v)=V$. Hence U and V are conjugate. Also V is of the desired form and self-normalizing.

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