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Poincaré Duality for Logarithmic Crystalline Cohomology

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Abstract. We prove Poincaré duality for logarithmic crystalline cohomology of log smooth schemes whose underlying schemes are reduced. This is a generalization of the result of P. Berthelot for usual smooth schemes and that of O. Hyodo for the special fibers of semi-stable families and trivial coefficients.

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Introduction

Let k be a perfect field of characteristic p and let W_m be the ring of Witt vectors of length m with coefficients in k for a positive integer m. Let N (resp. $W_m(N)$) be one of the following log. structures (in the sense of Fontaine–Illusie) on Spec(k) (resp. Spec(W_m)) ([K1]): (i) The trivial log. structure. (ii) The log. structure associated to $\mathbf{N} \to k$ (resp. W_m); $1 \mapsto 0$.

We consider an fs log. scheme (X, M) log. smooth ([K1] (3.3)) and universally saturated (Definition 2.17) over $(\mathrm{Spec}(k), N)$ whose underlying scheme is proper over k and of pure dimension n. Denote by f the structure morphism $(X, M) \to (\mathrm{Spec}(k), N)$. (In fact, for a smooth morphism of fs log. schemes g: $(Y, M_Y) \to (\mathrm{Spec}(k), N)$, g is universally saturated if and only if Y is reduced ([T]). Furthermore, when N is trivial, g is always universally saturated.) In the case (i) (resp. (ii)), a toric variety (resp. the special fiber of a semi-stable family over a discrete valuation ring A with residue field k) is a typical example ([K1] Example (3.7)). In this paper, we prove Poincaré duality for the log. crystalline cohomology of (X, M) with coefficients in a locally free $\mathcal{O}_{X/W_m}$-module of finite type. Since this can be applied to a proper smooth k-scheme with the trivial log. structure, this is a generalization of Poincaré duality for crystalline cohomology of a proper smooth scheme ([B] VII). For the special fiber of a semi-stable family and trivial coefficients, Poincaré duality has been proved by O. Hyodo, using de Rham–Witt complexes ([Hyo]). Since we want to treat twisted coefficients, we follow the method of [B] VII in this paper.

More precisely, we prove the following. Let γ be the canonical PD structure on the ideal pW_m and let $(X/W_m)^{\log}_{\text{crys}} = ((X, M)/(W_m, W_m(N), pW_m, \gamma))_{\text{crys}}$ be the log. crystalline site defined in [K1] Section 5. For a locally free $\mathcal{O}_{X/W_m}$-module of finite type, we define the log. crystalline cohomology and the log. crystalline cohomology with compact supports by

$$H^i_{\text{log-crys}}(X/W_m, E) := H^i((X/W_m)^{\log}_{\text{crys}}, E),$$
$$H^i_{\text{log-crys,c}}(X/W_m, E) := H^i((X/W_m)^{\log}_{\text{crys}}, K_{X/W_m}E),$$

where K_{X/W_m} is the ideal of $\mathcal{O}_{X/W_m}$ defined in Section 5. In the case of the special fiber of a semi-stable family, $K_{X/W_m} = \mathcal{O}_{X/W_m}$. Hence $H^i_{\text{log-crys,c}} = H^i_{\text{log-crys}}$.

THEOREM. *Let m be a positive integer. Then there exists a canonical homomorphism* Tr_f: $H^{2n}_{\text{log-crys,c}}(X/W_m, \mathcal{O}_{X/W_m}) \to W_m$ *called trace morphism such that, for any locally free $\mathcal{O}_{X/W_m}$-module of finite type, the pairing induced by the cup product and the trace morphism*

$$H^i_{\text{log-crys}}(X/W_m, E) \times H^{2n-i}_{\text{log-crys,c}}(X/W_m, \check{E}) \to W_m$$

is perfect, where $\check{E} = \mathcal{H}om_{\mathcal{O}_{X/W_m}}(E, \mathcal{O}_{X/W_m})$.

We remark here that the existence of Poincaré duality (with some compatibility of the trace map with Frobenius) implies the bijectivity of Frobenius on $\mathbb{Q} \otimes H^i_{\text{log-crys}}(X/W)$, which is not true in general when f is not universally saturated. See Remark 5.7 for details.

This paper is organized as follows. In Section 1, we review the theory of dualizing and residual complexes, and Grothendieck–Serre duality. In Section 2, we calculate $f^!\mathcal{O}_S$ for a certain kind of log. smooth morphism $(X, M) \to (S, N)$. In Section 3, we state Poincaré duality for de Rham cohomology of a log. smooth variety. The proof is easy once we construct the trace morphism. In the construction of the trace morphism, we need vanishing of the 'residues' of 'exact n-forms' for a log. smooth variety (Proposition 4.1), which is proved in Section 4. In Section 5, we state Poincaré duality for log. crystalline cohomology together with the definition of crystalline cohomology with compact supports. The main problem is again the construction of the trace morphism. After some preliminaries on local cohomology and Cousin complex for a log. crystalline topos in Section 6 and Section 7, we construct the trace morphism in Section 8.

In this paper, we use freely the notation and the terminology of [K1]. Especially log. structures are always considered in the étale topology. An fs log. structure M on a scheme X is a fine log. structure M on X such that all stalks are saturated ([K2] (1.1)) or, equivalently, there exists, étale locally on X, a chart $P \to \Gamma(X, M)$ with P saturated.

1. Review of Grothendieck–Serre Duality

We will briefly review the results of [Ha] which we will use in the following sections. In this section, we assume that schemes are Noetherian and morphisms of schemes are of finite type. Sheaves are considered in the Zariski topology.

For a scheme X, we will denote by $D(X)$ the derived category of the category of $\mathcal{O}_X$-modules and denote by $D_c(X)$ the full subcategory of $D(X)$ consisting of complexes with coherent cohomology. We will denote by D^+ (resp. D^-, resp. D^b) the full subcategory consisting of complexes bounded below (resp. bounded above resp. bounded).

For $R^\cdot \in D^+(X)$ and $F^\cdot \in D(X)$, we say that $F^\cdot$ is *reflexive* with respect to $R^\cdot$ if the natural morphism

$$F^\cdot \to R\mathcal{H}om^\cdot_X(R\mathcal{H}om^\cdot_X(F^\cdot, R^\cdot), R^\cdot)$$

is an isomorphism ([Ha] V Section 2). A *dualizing complex* on X is an object $R^\cdot \in D_c^+(X)$ of finite injective dimension such that every $F^\cdot \in D_c(X)$ or equivalently the structure sheaf $\mathcal{O}_X$ is reflexive with respect to $R^\cdot$. For a regular scheme of finite Krull dimension X, the structure sheaf $\mathcal{O}_X$ is a dualizing complex.

PROPOSITION 1.1 ([Ha] V Corollary 2.3, Proposition 3.4, and Proposition 7.1). *For a scheme X and $R^\cdot \in D_c^+(X)$ of finite injective dimension, $R^\cdot$ is dualizing if and only if, for every $x \in X$, there exists an integer $d(x)$ such that*

$$\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(k(x), R^\cdot_x) = \begin{cases} 0 & \text{for } i \neq d(x) \\ k(x) & \text{for } i = d(x). \end{cases}$$

Furthermore, when $R^\cdot$ is dualizing, d is a codimension function on X, that is, for any immediate specialization $x \to y$, $d(y) = d(x) + 1$.

For a dualizing complex $R^\cdot$ on X, we call d the codimension function associated with $R^\cdot$. It follows from this proposition that if X admits a dualizing complex, X is catenary and of finite Krull dimension ([Ha] V Corollary 7.2). When X is a Cohen–Macaulay connected scheme of finite type over a regular scheme, there exists d such that $\mathcal{H}^i(R^\cdot) = 0$ $(i \neq d)$ for any dualizing complex $R^\cdot$ on X.

For a smooth morphism $f\colon X \to Y$ of relative dimension n, we define the functor $f^\#\colon D_c^+(Y) \to D_c^+(X)$ to be $f^*(-) \otimes^{\mathbb{L}} \omega_{X/Y}[n]$, where $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ ([Ha] III Section 2). $f^\#$ preserves dualizing complexes ([Ha] V Theorem 8.3). For a finite morphism $f\colon X \to Y$, we define the functor $f^\flat\colon D_c^+(Y) \to D_c^+(X)$ to be $\overline{f}^* R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, -)$, where $\overline{f}$ is the morphism of ringed spaces $\overline{f}\colon (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ ([Ha] III Section 6). $f^\flat$ preserves dualizing complexes ([Ha] V Proposition 2.4). When f is a regular closed immersion of codimension n, we have a canonical isomorphism $f^\flat \cong Lf^*(-) \otimes^{\mathbb{L}} \omega_{X/Y}[-n]$ with $\omega_{X/Y} = (\wedge^n \mathcal{N}_{X/Y})^\vee$ ([Ha] III Corollary 7.3).

For a scheme X and $x \in X$, define the quasi-coherent injective $\mathcal{O}_X$-module $J(x)$ to be $i_*(\tilde{I})$, where i is a morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ and I is an injective hull of $k(x)$ as an $\mathcal{O}_{X,x}$-module ([Ha] II Section 7). A *residual complex* $K^\cdot$ on X is a complex of quasi-coherent injective $\mathcal{O}_X$-modules bounded below with coherent cohomology such that there is an isomorphism $\oplus_{i\in\mathbb{Z}} K^i \cong \oplus_{x\in X} J(x)$ ([Ha] VI Section 1). Denote by $\mathrm{Res}(X)$ the category of residual complexes on X. For a dualizing complex $R^\cdot$, define the Cousin complex $E(R^\cdot)$ by

$$\cdots \to 0 \to \mathcal{H}^0_{Z^0/Z^1}(R^\cdot) \to \mathcal{H}^1_{Z^1/Z^2}(R^\cdot) \to \cdots \to \mathcal{H}^i_{Z^i/Z^{i+1}}(R^\cdot) \to \cdots,$$

where $\mathcal{H}^i_{Z^i/Z^{i+1}}(R^\cdot)$ is placed in degree i and $Z^i = \{x \in X | d(x) \geqslant i\}$ with d the codimension function associated with $R^\cdot$. Then $E(R^\cdot)$ is a residual complex. If X admits a dualizing complex, E gives an equivalence of the category of dualizing complexes on X and $\mathrm{Res}(X)$. Its quasi-inverse is given by the restriction of the canonical functor $C^+(X) \to D^+(X)$ ([Ha] VI Proposition 1.1).

In the following, we assume that all schemes considered admit dualizing complexes.

For a finite (resp. smooth) morphism $f\colon X \to Y$, define the functor f^y (resp. f^z): $\mathrm{Res}(Y) \to \mathrm{Res}(X)$ by $f^y(K^\cdot) = E(f^\flat Q(K^\cdot))$ (resp. $f^z(K^\cdot) = E(f^\# Q(K^\cdot))$), where Q is the functor $K^+_c(Y) \to D^+_c(Y)$ ([Ha] VI Section 2). This is well-defined since $f^\flat$(resp. $f^\#$) preserves dualizing complexes.

By gluing these functors, we can construct a morphism $f^\Delta\colon \mathrm{Res}(Y) \to \mathrm{Res}(X)$ for any morphism $f\colon X \to Y$ which is canonically isomorphic to f^y (resp. f^z) when f is finite (resp. smooth) and satisfies various compatibilities such as $(gf)^\Delta \cong f^\Delta g^\Delta$. ([Ha] VI Theorem 3.1.)

When f is finite, there is a canonical isomorphism $f^y(K^\cdot) \cong \overline{f}^* \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K^\cdot)$ and hence an isomorphism $f_* f^y(K^\cdot) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K^\cdot)$. The evaluation at 1 gives an $\mathcal{O}_Y$-linear morphism of complexes $\rho_f\colon f_* f^y(K^\cdot) \to K^\cdot$.

THEOREM 1.2 ([Ha] VI Theorem 4.2). *For each morphism $f\colon X \to Y$, there exists a morphism*

$$\mathrm{Tr}_f\colon f_* f^\Delta \to 1 \tag{1.3}$$

of functors from $\mathrm{Res}(Y)$ *to the category of graded $\mathcal{O}_Y$-modules* (*where* 1 *denotes the forgetful functor*), *which satisfies*

(1) *For any morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the following diagram commutes.*

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$$\begin{tikzcd} g_* f_* f^\Delta g^\Delta \arrow[r, "\mathrm{Tr}_f"] \arrow[d, equal] & g_* g^\Delta \arrow[d, "\mathrm{Tr}_g"] \\ (gf)_*(gf)^\Delta \arrow[r, "\mathrm{Tr}_{gf}"] & 1. \end{tikzcd}$$

(2) Tr_f *coincides with* ρ_f *when* f *is finite.*

Furthermore, these Tr_f *are unique.*

THEOREM 1.4 (Residue theorem) ([Ha] VII Theorem 2.1). *Let* $f\colon\ X \to Y$ *be a proper morphism, and let* $K^\cdot$ *be a residual complex on* Y. *Then the trace map* $\mathrm{Tr}_f\colon\ f_* f^\Delta(K^\cdot) \to K^\cdot$ *is a morphism of complexes.*

We can define a functor $f^!\colon\ D_c^+(Y) \to D_c^+(X)$ for any morphism $f\colon\ X \to Y$ by

$$f^!(F^\cdot) = R\mathcal{H}om^\cdot_X(Lf^*(R\mathcal{H}om^\cdot_Y(F^\cdot, K^\cdot)), f^\Delta(K^\cdot))),$$

where $K^\cdot$ is a residual complex on Y ([Ha] VII Corollary 3.4(a)). Recall that we assume all schemes considered admit dualizing complexes. The functor $f^!$ is canonically isomorphic to $f^\flat$ (resp. $f^\#$) when f is finite (resp. smooth) and satisfies various compatibilities. When f is proper, we obtain from (1.3) and Theorem 1.4, the trace morphism $\mathrm{Tr}_f\colon\ Rf_* f^! \to 1$ ([Ha] VII Corollary 3.4(b)).

THEOREM 1.5 (Duality) ([Ha] VII Corollary 3.4(c)). *For a proper morphism* $f\colon$ $X \to Y$, *the composite*

$$\begin{aligned} Rf_* R\mathcal{H}om^\cdot_X(F^\cdot, f^! G^\cdot) &\longrightarrow R\mathcal{H}om^\cdot_Y(Rf_* F^\cdot, Rf_* f^! G^\cdot) \\ &\xrightarrow{\mathrm{Tr}_f} R\mathcal{H}om^\cdot_Y(Rf_* F^\cdot, G^\cdot) \end{aligned}$$

is an isomorphism for $F^\cdot \in D^-_{qc}(X)$ *and* $G^\cdot \in D_c^+(Y)$.

2. Relative Dualizing Sheaf for a Log. Smooth Morphism

We will calculate $f^! \mathcal{O}_S$ for a smooth morphism of fs log. schemes $f\colon\ (X, M) \to (S, N)$ satisfying certain conditions.

DEFINITION 2.1 ([K2], Def. (5.1)). An ideal I of a monoid P is a subset of P satisfying $PI \subseteq I$. A prime ideal $\mathfrak{p}$ of a monoid P is an ideal whose complement $P \backslash \mathfrak{p}$ in P is a submonoid. We will denote by $\mathrm{Spec}(P)$ the set of all primes of P.

It is easy to see $\mathrm{Spec}(P) = \mathrm{Spec}(P/P^*)$.

DEFINITION 2.2. Let P be a monoid.

(1) ([K2] Definition (5.4)). We define the dimension $\dim(P)$ to be the maximal length of a sequence of prime ideals $\mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \cdots \supsetneqq \mathfrak{p}_r$ of P. If the maximum does not exist, we define $\dim(P) = \infty$.

(2) For a prime ideal $\mathfrak{p}$ of P, we define the height $\mathrm{ht}(\mathfrak{p})$ to be the maximal length of a sequence of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \cdots \supsetneqq \mathfrak{p}_r$ of P. If the maximum does not exist, we define $\mathrm{ht}(\mathfrak{p}) = \infty$.

PROPOSITION 2.3 ([K2] Proposition (5.5)). *Let P be a finitely generated integral monoid.*

(1) $\mathrm{Spec}(P)$ *is a finite set.*

(2) $\dim(P) = \mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}}/P^*)$.

(3) *For* $\mathfrak{p} \in \mathrm{Spec}(P)$, *we have* $\dim(P\backslash\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim(P)$.

DEFINITION 2.4. (1) Let $h\colon Q \to P$ be a morphism of monoids. We say a prime $\mathfrak{p}$ of P is *horizontal* with respect to h if $h(Q) \subset P\backslash\mathfrak{p}$.

(2) For a morphism $f\colon (X, M) \to (Y, N)$ of log. schemes and $x \in X$, we say a prime $\mathfrak{p}$ of $M_{\overline{x}}$ is *horizontal* with respect to f if it is horizontal with respect to $f_{\overline{x}}^*\colon N_{\overline{f(x)}} \to M_{\overline{x}}$.

Let $f\colon (X, M) \to (S, N)$ be a smooth morphism of fs log. schemes. Define the sheaf of ideals I_f of the sheaf of monoids M by:

$\Gamma(U, I_f) = \{a \in \Gamma(U, M) |$ The image of a in $M_{\overline{x}}$ is contained in $\mathfrak{p}$ for all points $x \in U$ and all primes $\mathfrak{p} \in \mathrm{Spec}(M_{\overline{x}})$ of height 1 horizontal with respect to $f\}$.

LEMMA 2.5. *For $x \in X$, $(I_f)_{\overline{x}} = \{a \in M_{\overline{x}} | a$ is contained in $\mathfrak{p}$ for all primes $\mathfrak{p} \in \mathrm{Spec}(M_{\overline{x}})$ of height 1 horizontal with respect to $f\}$.*

Proof. The inclusion $\subset$ is trivial. We will prove the converse. Put $s = f(x)$. Since the question is étale local on X and on S, we may assume that we have charts $P \to \Gamma(X, M)$ and $Q \to \Gamma(S, N)$ such that $P \to M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ and $Q \to N_{\overline{s}}/\mathcal{O}^*_{S,\overline{s}}$ are isomorphisms ([K1] Lemma (2.10)). Since Q is finitely generated, we may also assume that the following diagram commutes, where h denotes the composite $Q \xrightarrow{\sim} N_{\overline{s}}/\mathcal{O}^*_{S,\overline{s}} \xrightarrow{f_{\overline{x}}^*} M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \xleftarrow{\sim} P$.

$$\begin{array}{ccc} P & \longrightarrow & \Gamma(X, M/\mathcal{O}_X^*) \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle f^*} \\ Q & \longrightarrow & \Gamma(S, N/\mathcal{O}_S^*). \end{array}$$

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Put $P_{\overline{y}} = M_{\overline{y}}/\mathcal{O}^*_{X,\overline{y}}$ and $Q_{\overline{t}} = N_{\overline{t}}/\mathcal{O}^*_{S,\overline{t}}$ for $y \in X$ and $t \in S$. Let φ_y (resp. φ_t) denote the composite $P \to \Gamma(X, M) \to P_{\overline{y}}$ (resp. $Q \to \Gamma(S, N) \to Q_{\overline{t}}$). Then the following diagram commutes when $t = f(y)$.

$$\begin{tikzcd} P \arrow[r, "\varphi_y"] & P_{\overline{y}} \\ Q \arrow[u, "h"] \arrow[r, "\varphi_t"] & Q_{\overline{t}}, \arrow[u, "h_{\overline{y}}"'] \end{tikzcd}$$

where $h_{\overline{y}}$ denotes the morphism induced by f.

Let a be an element of the right-hand set of the lemma. By replacing X by a suitable étale neighborhood of x, we may assume that there exists $b \in \Gamma(X, M)$ and $c \in P$ such that the stalk of b at x is a and the images of b and c in $\Gamma(X, M/\mathcal{O}^*_X)$ coincide. We assert $b \in \Gamma(X, I_f)$. It suffices to prove $\varphi_y(c) \in \mathfrak{p}$ for all $y \in Y$ and all $\mathfrak{p} \in \mathrm{Spec}(P_{\overline{y}})$ of height 1 horizontal with respect to $h_{\overline{y}}$. We have an injective map $\mathrm{Spec}(\varphi_y)\colon\ \mathrm{Spec}(P_{\overline{y}}) \to \mathrm{Spec}(P)$; $\mathfrak{q} \mapsto \varphi_y^{-1}(\mathfrak{q})$. Since $P^*_{\overline{y}} = \{1\}$ and φ_y induces an isomorphism $P/\varphi_y^{-1}(\{1\}) \xrightarrow{\sim} P_{\overline{y}}$, we have $P_{\overline{y}}\backslash\{1\} \in \mathrm{Spec}(P_{\overline{y}})$ and the image of $\mathrm{Spec}(\varphi_y)$ is the set of all primes of P contained in $\varphi_y^{-1}(P_{\overline{y}}\backslash\{1\})$. Hence $\varphi_y^{-1}(\mathfrak{p})$ is a prime of P of height 1 horizontal with respect to P and it suffices to prove $c \in \varphi_y^{-1}(\mathfrak{p})$. Since the image of c under the isomorphism $\varphi_x\colon\ P \xrightarrow{\sim} P_{\overline{x}}$ coincides with the image of $a \in M_{\overline{x}}$, this follows from the assumption on a. □

COROLLARY 2.6. *Assume that we have a chart* $P \to \Gamma(X, M)$, $Q \to \Gamma(S, N)$, $h\colon\ Q \to P$ *of the morphism* f *such that* P *and* Q *are saturated. Define the ideal* $I \subset P$ *by* $\{a \in P \mid a$ *is contained in* $\mathfrak{p}$ *for all primes* $\mathfrak{p}$ *of* P *of height* 1 *horizontal with respect to* $h\}$.

Then the sheaf of ideals I_f *is generated by the image of* I.

Proof. Let $x \in X$. By Lemma 2.5, $(I_f)_{\overline{x}}$ is the inverse image of the ideal $\{a \in M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \mid a \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}})$ of height 1 horizontal with respect to $f^*_{\overline{x}}\colon\ N_{\overline{f(x)}}/\mathcal{O}^*_{S,\overline{f(x)}} \to M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}}\}$ of $M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$. It is easy to see that the latter ideal coincides with the image of I. □

COROLLARY 2.7. *The ideal* $I_f\mathcal{O}_X$ *of* $\mathcal{O}_X$ *is quasi-coherent.*

DEFINITION 2.8. An fs log. scheme (X, M) is called *regular* if the underlying scheme X is locally Noetherian and, for each point $x \in X$, $\mathcal{O}_{X,\overline{x}}/I(\overline{x}, M)$ is a regular local ring and $\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I(\overline{x}, M)) + \mathrm{rank}_{\mathbb{Z}}(M^{\mathrm{gp}}_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}})$. Here $I(\overline{x}, M)$ denotes the ideal of $\mathcal{O}_{X,\overline{x}}$ generated by the image of $M_{\overline{x}}\backslash\mathcal{O}^*_{X,\overline{x}}$.

An fs log. scheme (X, M) is regular if and only if, étale locally on X, there exists a chart $h\colon\ P \to \Gamma(X, M)$ such that P is saturated and (X, M_{Zar}) is regular in the sense of [K2] Definition (2.1), where M_{Zar} is the log. structure in the Zariski topology associated to h. Furthermore, if (X, M) is regular, the latter statement holds

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for every chart $P \to \Gamma(X, M)$ with P saturated. Hence, we obtain the following propositions from the corresponding ones in the Zariski case.

PROPOSITION 2.9 (cf. [K2] Theorem (4.1)). *The underlying scheme of a regular fs log. scheme is Cohen–Macaulay and normal.*

PROPOSITION 2.10 (cf. [K2] Theorem (8.2)). *For a smooth morphism $(X, M) \to (S, N)$ of fs log. schemes, (X, M) is regular if (S, N) is regular.*

PROPOSITION 2.11 (cf. [K2] Theorem (11.6)). *Let (X, M) be a regular fs log. scheme. Then the set $U = \{x \in X | M_{\overline{x}} = \mathcal{O}^*_{X,\overline{x}}\}$ is dense open in X and $M = \mathcal{O}_X \cap j_*\mathcal{O}^*_U$, where $j\colon U \hookrightarrow X$ denotes the inclusion morphism.*

PROPOSITION 2.12 (cf. [K2] Corollary (7.3)). *Let (X, M) be a regular fs log. scheme. Then for $x \in X$ and $\mathfrak{p} \in \mathrm{Spec}(M_{\overline{x}})$, there exists a unique point $y \in X$ such that*

(1) $x \in \overline{\{y\}}$.
(2) *The cospecialization map $M_{\overline{x}} \to M_{\overline{y}}$ induces an isomorphism $M_{\overline{x}}/(M_{\overline{x}} \backslash \mathfrak{p}) \cong M_{\overline{y}}/\mathcal{O}^*_{X,\overline{y}}$.*
(3) $\dim(\mathcal{O}_{X,\overline{y}}) = \mathrm{ht}(\mathfrak{p})$ *or equivalently $I(\overline{y}, M)$ is the maximal ideal of $\mathcal{O}_{X,\overline{y}}$. Here $I(\overline{y}, M)$ is the same as in Definition* 2.8.

Let us consider a smooth morphism $f\colon (X, M) \to (S, N)$ of fs log. schemes again.

LEMMA 2.13. *Assume that (S, N) is regular. Let U be any étale X-scheme and let S_U be the set of points $x \in U$ of codimension 1 such that $M_{\overline{x}} \neq \mathcal{O}^*_{X,\overline{x}}$ and $N_{\overline{f(x)}} = \mathcal{O}^*_{S,\overline{f(x)}}$. Then $\Gamma(U, I_f) = \{a \in \Gamma(U, M) | a_{\overline{x}} \notin \mathcal{O}^*_{X,\overline{x}}$* for all $x \in S_U\}$.

Proof. Use Proposition 2.10 and Proposition 2.12. □

PROPOSITION 2.14. *Assume that (S, N) is regular. Define the set S_X in the same way as in Lemma* 2.13. *Let J be the ideal of $\mathcal{O}_X$ corresponding to the closed subset $\bigcup_{x \in S_X} \overline{\{x\}} \subset X$ with the reduced induced structure. Then we have $I_f\mathcal{O}_X = J$.*

Proof. This follows from Lemma 2.13 and [K2] Corollary (11.8). □

EXAMPLE 2.15. If $S = \mathrm{Spec}(k)$ with k a field, $X \to S$ is a usual smooth morphism, M is a log. structure defined by a reduced divisor D on X with relative normal crossings, and N is the trivial log. structure, then $I_f\mathcal{O}_X = \mathcal{O}_X(-D)$.

EXAMPLE 2.16. Let $S = \mathrm{Spec}(A)$ with A a discrete valuation ring and let X be a semi-stable family over S, that is, a regular scheme of finite type over S whose special fiber is a reduced divisor with normal crossings on X. Define the fs log. structure M on X (resp. N on S) by the log. structure given by the special fiber

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(resp. the closed point). Then we obtain a smooth morphism $f\colon (X, M) \to (S, N)$ ([K1] Examples (3.7) (2)). Let $f_n\colon (X_n, M_n) \to (S_n, N_n)$ be the reduction of f modulo $\mathfrak{m}^n$, where $\mathfrak{m}$ is the maximal ideal of A. Then we have $I_{f_n} = M_n$, and hence $I_{f_n}\mathcal{O}_{X_n} = \mathcal{O}_{X_n}$.

DEFINITION 2.17. Let $f\colon (X, M) \to (Y, N)$ be a morphism of fs log. schemes. We say that f is *universally saturated* if f is integral ([K1] Definition (4.3)) and, for any morphism $g\colon (Y', N') \to (Y, N)$ of fs log. schemes, the base change of (X, M) by g in the category of log. schemes is an fs log. scheme.

The universally saturatedness can be checked fiber by fiber, that is, the morphism f is universally saturated if and only if, for every geometric point $\overline{y}$ of Y with the inverse image log. structure $\overline{N}$ of N, the base change $f_{\overline{y}}$ of f by the canonical morphism $(\overline{y}, \overline{N}) \to (Y, N)$ is universally saturated. If the log. structure N is trivial, f is always universally saturated. The following two facts proven in [T] will be helpful to understand universally saturated morphisms although we will not use them in this paper: When Y is an $\mathbb{F}_p$-scheme for a prime p, f is universally saturated if and only if f is of Cartier type ([K1] (4.8)). When f is smooth and integral, f is universally saturated if and only if every fiber of the underlying morphism of schemes of f is reduced.

Let $f\colon (X, M) \to (S, N)$ be a smooth morphism of fs log. schemes again.

LEMMA 2.18. *Assume $S = \mathrm{Spec}(k)$ with k a field, N is the log. structure associated to $\mathbb{N} \to k; 1 \mapsto 0$, and $f\colon (X, M) \to (S, N)$ is universally saturated. Then the underlying morphism of schemes $X \to S$ is smooth in codimension 0.*

Proof. Let $x \in X$ be of codimension 0. By taking a smooth lifting of (X, M) to $\mathrm{Spec}(k[\mathbb{N}])$ with the canonical log. structure and using Proposition 2.10, we can verify that $M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \cong \mathbb{N}$. Since the morphism $\mathbb{N} \cong N_{\overline{s}}/\mathcal{O}^*_{S,\overline{s}} \to M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is universally saturated by assumption, this is an isomorphism. Hence, there exists a chart $(\mathbb{N} \to k; 1 \mapsto 0, P \to \Gamma(U, M), h\colon \mathbb{N} \to P)$ for an étale neighborhood U of x such that the order of the torsion part of the cokernel of h^{gp} is invertible in k, the morphism $U \to \mathrm{Spec}(k[P] \otimes_{k[\mathbb{N}]} k)$ is étale, and the morphism $\mathbb{N} \oplus P^* \to P$ induced by h and the inclusion map is an isomorphism. The lemma follows from this. $\square$

LEMMA 2.19. *Assume that (S, N) is regular and f is universally saturated. Furthermore assume that $X \to S$ is smooth of relative dimension n, and S is regular. Define the set S_X in the same way as in Lemma 2.13 and define the divisor D by $\Sigma_{x \in S_X}\overline{\{x\}}$. Then we have a canonical isomorphism $\Omega^n_{X/S} \cong \Omega^n_{X/S}(\log(M/N))(-D)$.*

Proof. Since X is regular, it suffices to prove that, for every point x of codimension 1 in X, the cokernel of the canonical morphism

$$\Omega^n_{X/S,x} \to \Omega^n_{X/S}(\log(M/N))_x \tag{2.20}$$

has length 1 as an $\mathcal{O}_{X,x}$-module when $x \in S_X$ and is 0 otherwise. It is trivial if $M_{\overline{x}} = \mathcal{O}^*_{X,\overline{x}}$. If $M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \cong \mathbb{N}$ and $N_{\overline{f(x)}} = \mathcal{O}^*_{S,\overline{f(x)}}$, there exists a chart $P \to \Gamma(U, M)$ with an isomorphism $P \cong \mathbb{N} \oplus P^*$ for an étale neighborhood U of x such that the order of the torsion part of P^{gp} is invertible on X and the morphism $U \to S \times \mathrm{Spec}(\mathbb{Z}[P])$ is étale. It follows that the cokernel of (2.20) is of length 1. If $M_{\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \cong \mathbb{N}$ and $N_{\overline{f(x)}}/\mathcal{O}^*_{S,\overline{f(x)}} \cong \mathbb{N}$, by the same argument as the proof of Lemma 2.18, there exists a chart $(\mathbb{N} \to \Gamma(S, N), P \to \Gamma(U, M), h\colon \mathbb{N} \to P)$ for an étale neighborhood U of x such that the order of the torsion part of the cokernel of h^{gp} is invertible on U, the morphism $U \to S \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])} \mathrm{Spec}(\mathbb{Z}[P])$ is étale, and $\mathbb{N} \oplus P^* \to P$ induced by h and the inclusion map is an isomorphism. It follows that (2.20) is an isomorphism. $\square$

THEOREM 2.21. *Assume that* $\Omega_{X/S}(\log(M/N))$ *has a constant rank* n*, the morphism* f *is universally saturated, and the underlying morphism of schemes of* f *is of finite type. Furthermore, assume one of the following conditions.*

(i) (S, N) *is regular and the underlying scheme* S *is Noetherian, regular and of finite Krull dimension.*

(ii) $S = \mathrm{Spec}(A/\pi^m A)$*, for a discrete valuation ring* A*, a prime* $\pi \in A$ *and* $m \geqslant 1$*, and the log. structure* N *is isomorphic to the one associated to* $\mathbb{N} \to A_m; 1 \mapsto a$ *for some* $a \in A$.

Then we have a canonical isomorphism $f^!\mathcal{O}_S \cong I_f\Omega^n_{X/S}(\log(M/N))[n]$.

Here we regard the coherent sheaf $I_f\Omega^n_{X/S}(\log(M/N))$ *as a sheaf on the Zariski site.*

Proof. (cf. [K2] Theorem (11.2)) In the first case, since the scheme X is Cohen–Macaulay (Proposition 2.9 and Proposition 2.10) and $f^!\mathcal{O}_S$ is a dualizing complex on X, $f^!\mathcal{O}_S$ has the form $\omega_{X/S}[n]$. Since the smooth locus $U \subset X$ of the morphism f contains all points of codimension $\leqslant 1$ (Lemma 2.18), the homomorphism $\omega_{X/Y} \to j_*j^*\omega_{X/Y}$ is an isomorphism by Proposition 1.1 and EGA IV Theorem 5.10.5. Here j denotes the canonical inclusion $U \hookrightarrow X$. From $j^*f^!\mathcal{O}_S \cong (f \circ j)^!\mathcal{O}_S \cong \Omega^n_{U/S}[n]$, we obtain $\omega_{X/S} \cong j_*\Omega^n_{U/S}$. By Lemma 2.19 and Proposition 2.14, we have

$$\begin{aligned} j_*\Omega^n_{U/S} &\cong j_*(j^*\Omega^n_{X/S}(\log(M/N))(-D)) \\ &\cong J\Omega^n_{X/S}(\log(M/N)) \cong I_f\Omega^n_{X/S}(\log(M/N)), \end{aligned}$$

where J is the same as in Proposition 2.14. The second isomorphism follows from the fact that X is Cohen–Macaulay and U contains all points of codimension $\leqslant 1$.

Next consider the second case. Let $S' = \mathrm{Spec}(A[T])$ and let N' be the log. structure on S' associated to $\mathbb{N} \to A[T]; 1 \mapsto T$. Define the exact closed immersion $i\colon (S, N) \hookrightarrow (S', N')$ by $T \mapsto a$ and the identity on $\mathbb{N}$. First assume that there exists globally a smooth lifting $f'\colon (X', M') \to (S', N')$ which is universally saturated. Let j be the morphism $(X, M) \to (X', M')$. By the first case, we have an isomorphism $f'^{!}\mathcal{O}_{S'} \cong I_{f'}\Omega^n_{X'/S'}(\log(M'/N'))[n]$. Hence by [Ha] III Corollary 7.3 (cf. Section 1), we have an isomorphism

$$\begin{aligned} f^! i^! \mathcal{O}_{S'} &\cong j^! f'^! \mathcal{O}_{S'} \\ &\cong I_{f'}\Omega^n_{X'/S'}(\log(M'/N')) \otimes_{\mathcal{O}_{X'}} (\wedge^2 \mathcal{N}_{X/X'})^{\check{}}[n-2]. \end{aligned}$$

Note that X is Cohen–Macaulay. On the other hand, we have an isomorphism $i^!\mathcal{O}_{S'} \cong (\wedge^2\mathcal{N}_{S/S'})^{\check{}}[-2]$. Hence we obtain an isomorphism $f^!\mathcal{O}_X \cong I_{f'}\Omega^n_{X'/S'}(\log(M'/N')) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X[n]$. Since $I_{f'}\mathcal{O}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \cong I_f \mathcal{O}_X$ by Lemma 2.22 and Lemma 2.5, we get the required isomorphism.

The restriction of this isomorphism to the smooth locus $U (\overset{u}{\hookrightarrow} X)$ of f induces the canonical isomorphism $u^* f^! \mathcal{O}_S \cong \Omega^n_{U/S}[n]$, and the homomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(I_f\Omega^n_{X/S}(\log(M/N)), I_f\Omega^n_{X/S}(\log(M/N))) \to \mathrm{Hom}_{\mathcal{O}_U}(\Omega^n_{U/S}, \Omega^n_{U/S})$$

is injective since U contains all points of codimension 0 (Lemma 2.18) and X is Cohen–Macaulay. Hence we can glue these isomorphisms in the general case. □

LEMMA 2.22. *Let $f\colon (X, M) \to (S, N)$ be a smooth integral morphism of fs log. schemes. Then $\mathcal{O}_X / I_f \mathcal{O}_X$ is flat over S.*

Proof. By taking a chart and using Corollary 2.6, we can reduce to the following fact, which is proved by a similar method of [K1] (4.1) (v)⇒(ii). Let $Q \to P$ be an injective morphism of finitely generated saturated monoids satisfying the condition [K1] (4.1)(iv). Define the ideal I of P in the same way as Corollary 2.6. Then $\mathbb{Z}[P]/I\mathbb{Z}[P]$ is flat over $\mathbb{Z}[Q]$. □

3. Poincaré Duality for de Rham Cohomology with Log. Poles

Let A be a discrete valuation ring, and let $\pi \in A$ be a prime element. For a positive integer m, let A_m be the reduction mod π^m of A. In the following, we fix m and put $S = \mathrm{Spec}(A_m)$. Let N be the log. structure on S associated to $\mathbb{N} \to A_m$; $1 \mapsto a$ for some $a \in A_m$. We allow the trivial log. structure.

Let $f\colon (X, M) \to (S, N)$ be a smooth universally saturated morphism. Assume that X is of constant dimension n, that is, $\Omega^{\log}_{X/S} := \Omega_{X/S}(\log(M/N))$ has constant rank n, and X is proper over S. By Theorem 2.21, we obtain a homomorphism $\mathrm{Tr}_f\colon H^n(X, I_f(\Omega^{\log}_{X/S})^n) \to A_m$ from the trace map $Rf_* f^! \mathcal{O}_S \to \mathcal{O}_S$ ([Ha] VII Corollary 3.4, cf. Section 1).

PROPOSITION 3.1. *The composite*

$$H^n(X, I_f(\Omega^{\log}_{X/S})^{n-1}) \xrightarrow{d} H^n(X, I_f(\Omega^{\log}_{X/S})^n) \xrightarrow{\mathrm{Tr}_f} A_m$$

is 0.

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We will prove this proposition in the next section.

DEFINITION 3.2. For a locally free $\mathcal{O}_X$-module E of finite rank with an integrable connection (with log. poles) $\nabla\colon\ E \to E \otimes_{\mathcal{O}_X} \Omega^{\log}_{X/S}$, we define the de Rham cohomology and the de Rham cohomology with compact supports by

$$H^m_{\text{log-dR}}(X/S, E) := H^m_{\text{Zar}}(X, E \otimes_{\mathcal{O}_X} (\Omega^{\log}_{X/S})^{\cdot}),$$

$$H^m_{\text{log-dR,c}}(X/S, E) := H^m_{\text{Zar}}(X, I_f E \otimes_{\mathcal{O}_X} (\Omega^{\log}_{X/S})^{\cdot}).$$

Remark 3.3. Since all terms of the de Rham complexes are coherent $\mathcal{O}_X$-modules, the cohomology groups in the right-hand side do not change if we replace the Zariski cohomology by the étale cohomology.

We will omit E when $E = \mathcal{O}_X$. We obtain from Proposition 3.1 a homomorphism $\mathrm{Tr}_f\colon\ H^{2n}_{\text{log-dR,c}}(X/S) \to A_m$.

THEOREM 3.4. *Let E be a locally free $\mathcal{O}_X$-module of finite type with an integrable connection $\nabla\colon\ E \to E \otimes_{\mathcal{O}_X} \Omega^{\log}_{X/S}$. Then the pairing induced by the cup product and the trace map $H^i_{\text{log-dR}}(X/S, E) \times H^{2n-i}_{\text{log-dR,c}}(X/S, \check{E}) \to A_m$ is perfect. Here $\check{E} = \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$.*

Proof. By Theorem 2.21 and Theorem 1.5, we can prove this in the same way as [B] VII 2.1.5. □

4. Proof of Proposition 3.1

In this section, we always work on Zariski sites. We keep the notation of Section 3. Here we do not assume that f is proper, but we assume that f is of finite type. The object $f^!(\mathcal{O}_S)$ is represented by the Cousin complex $E(f^!(\mathcal{O}_S)) \cong f^{\Delta}(\mathcal{O}_S)$ associated with the codimension function on X defined by the dualizing complex $f^!(\mathcal{O}_S)$ ([Ha] VI Proposition 1.1, cf. Section 1). By Theorem 2.21, $f^{\Delta}(\mathcal{O}_S)$ is canonically isomorphic to the complex

$$\begin{aligned}\cdots \ \to\ & 0 \to \mathcal{H}^0_{X/X^1}(I_f(\Omega^{\log}_{X/S})^n) \to \mathcal{H}^1_{X^1/X^2}(I_f(\Omega^{\log}_{X/S})^n) \to \cdots \\ \to\ & \mathcal{H}^{n-1}_{X^{n-1}/X^n}(I_f(\Omega^{\log}_{X/S})^n) \to \mathcal{H}^n_{X^n}(I_f(\Omega^{\log}_{X/S})^n) \to 0 \to \cdots,\end{aligned}$$

where X^d denotes the set of points of codimension d in X, and the term $\mathcal{H}^n_{X^n}(I_f(\Omega^{\log}_{X/S})^n)$ is placed in degree 0. Hence we obtain from the trace map (1.3) a morphism $\mathrm{Tr}_{f,x}\colon\ f_*(\mathcal{H}^n_x(I_f(\Omega^{\log}_{X/S})^n)) \to \mathcal{O}_S$, for each $x \in X^n$. Since the Cousin complex of $I_f(\Omega^{\log}_{X/S})^{n-1}$ gives an injective resolution of $I_f(\Omega^{\log}_{X/S})^{n-1}$, Proposition 3.1 follows from the following proposition. Note that the trace map $\mathrm{Tr}_f\colon\ Rf_*f^! \to 1$ is deduced from the trace map $\mathrm{Tr}_f\colon\ f_*f^{\Delta} \to 1$ ([Ha] VII Corollary 3.4 (b)).

PROPOSITION 4.1 (cf. [B] VII Proposition 1.2.6). *For every closed point x of X, the composite* $f_*(\mathcal{H}^n_x(I_f(\Omega^{\log}_{X/S})^{n-1})) \xrightarrow{d} f_*(\mathcal{H}^n_x(I_f(\Omega^{\log}_{X/S})^n)) \xrightarrow{\mathrm{Tr}_{f,x}} \mathcal{O}_S$ *is* 0.

LEMMA 4.2. *Let $u\colon U \to X$ be an étale morphism of finite type, and let x be a closed point of X such that $k(x') = k(x)$ for all $x' \in u^{-1}(x)$. Put $g = f \circ u$. By Theorem* 2.21, *we have isomorphisms*

$$f^\Delta \mathcal{O}_S \cong E(I_f(\Omega^{\log}_{X/S})^n), \qquad g^\Delta \mathcal{O}_S \cong E(I_g(\Omega^{\log}_{U/S})^n).$$

Hence, from the trace map $\mathrm{Tr}_u\colon u_* g^\Delta(\mathcal{O}_S) \cong u_* u^\Delta f^\Delta \mathcal{O}_S \to f^\Delta \mathcal{O}_S$, we obtain a morphism $\mathrm{Tr}_{u,x'}\colon u_*\mathcal{H}^n_{x'}(I_g(\Omega^{\log}_{X/S})^n) \to \mathcal{H}^n_x(I_f(\Omega^{\log}_{X/S})^n)$, for $x' \in u^{-1}(x)$. Let $t_1, \dots, t_n \in \mathfrak{m}_{X,x}$ be a regular sequence, let $\omega \in I_f(\Omega^{\log}_{X/S})^n_x$, and let $t'_1, \dots, t'_n \in \mathfrak{m}_{U,x'}$ and $\omega' \in I_g(\Omega^{\log}_{U/S})^n_{x'}$ be their inverse images. Then

$$\mathrm{Tr}_{u,x'}\left(\frac{\omega'}{t'_1 \cdots t'_d}\right) = \frac{\omega}{t_1 \cdots t_d}.$$

Proof. Put $Z = \mathrm{Spec}(\mathcal{O}_{X,x}/(t_1, \dots, t_d))$ and $Z_U = Z \times_X U$. Let v (resp. i, resp. j) be the morphism $Z_U \to Z$ (resp. $Z \hookrightarrow X$, resp. $Z_U \hookrightarrow U$). By assumption, Z_U is isomorphic to a finite disjoint union of copies of Z. We have a canonical isomorphism $i^\Delta f^\Delta \mathcal{O}_S \cong I_f(\Omega^{\log}_{X/S})^n \otimes_{\mathcal{O}_X} \omega_{Z/X}$ ([Ha] III Corollary 7.3, cf. Section 1). Hence, the degree 0 part of the trace map $\mathrm{Tr}_i\colon i_* i^\Delta f^\Delta \mathcal{O}_S \to f^\Delta \mathcal{O}_S$ gives a homomorphism $i_*(I_f(\Omega^{\log}_{X/S})^n \otimes_{\mathcal{O}_X} \omega_{Z/X}) \to \mathcal{H}^n_x(I_f(\Omega^{\log}_{X/S})^n)$. The image of $\omega \otimes (t_1 \wedge \cdots \wedge t_d)^\vee$ under this morphism is $\omega/(t_1 \cdots t_d)$ (cf. [B] VII The proof of Lemma 1.2.5). The same fact holds for $g^\Delta \mathcal{O}_S$ and j. Hence the lemma follows from $\mathrm{Tr}_u \mathrm{Tr}_j = \mathrm{Tr}_i \mathrm{Tr}_v$. $\square$

Proof of Proposition 4.1. (Part I) We first prove the claim in the case N is trivial. By [Ha] VI Theorem 5.6, we can reduce to the case where the residue field of A is separably closed. Then, by Lemma 4.2, we may assume that X is of the form $\mathrm{Spec}(A_m[P])$, where P is a finitely generated saturated monoid such that $P_{\mathrm{tor}} = \{1\}$. Using a finite rational polyhedral decomposition of $(P^{\mathrm{gp}})_{\mathbb{R}}$ ([KKMS] I Section 1, 2) which contains P, we obtain a 'compactification' of (X, M), that is, an S-morphism $(X, M) \hookrightarrow (\overline{X}, \overline{M})$ of fs log. schemes whose underlying morphism is an open immersion such that $\overline{f}\colon (\overline{X}, \overline{M}) \to S$ is smooth, its underlying morphism of schemes is proper, and M is the inverse image of $\overline{M}$. Furthermore, by [K2] (10.4), we obtain a log. étale morphism $\overline{g}\colon (\overline{X}', \overline{M}') \to (\overline{X}, \overline{M})$ whose underlying morphism of schemes is proper such that $(\overline{X}', \overline{M}')$ is log. smooth over S, $\overline{X}'$ is smooth over S, and

$$I_{\overline{f}} \mathcal{O}_{\overline{X}} \cong R\overline{g}_*(I_{\overline{f'}} \mathcal{O}_{\overline{X}'}), \tag{4.3}$$

([K2] Theorem (11.3), Lemma 2.13 and Lemma 2.22). Here $\overline{f'} = \overline{f}\,\overline{g}$.

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Since we do not treat X in the following, we omit the upper bar of the symbols for simplicity. Since X is proper over S, it is enough to prove that the homomorphism

$$H^n(X, I_f(\Omega^{\log}_{X/S})^{n-1}) \xrightarrow{d} H^n(X, I_f(\Omega^{\log}_{X/S})^n) \tag{4.4}$$

is 0. By (4.3), we have a commutative diagram

$$\begin{array}{ccc}
H^n(X, I_f(\Omega^{\log}_{X/S})^{n-1}) & \xrightarrow{d} & H^n(X, I_f(\Omega^{\log}_{X/S})^n) \\
\wr\downarrow & & \wr\downarrow \\
H^n(X', I_{f'}(\Omega^{\log}_{X'/S})^{n-1}) & \xrightarrow{d} & H^n(X', I_{f'}(\Omega^{\log}_{X'/S})^n) \\
\downarrow & & \| \\
H^n(X', \Omega^{n-1}_{X'/S}) & \xrightarrow{d} & H^n(X', \Omega^n_{X'/S}).
\end{array}$$

The bottom arrow is 0 because the composite

$$H^n(X', \Omega^{n-1}_{X'/S}) \xrightarrow{d} H^n(X', \Omega^n_{X'/S}) \xrightarrow{\mathrm{Tr}_{f'}} A_m$$

is 0 ([B] VII 1.2.6) and the trace map is an isomorphism. Hence, the homomorphism (4.4) is 0. □

LEMMA 4.5. *Let S' be* $\mathrm{Spec}(A_m[t])$ *and let N' be the log. structure on S' defined by $\mathbb{N} \to A_m[t]$; $1 \mapsto t$. Define the exact closed immersion $(S, N) \hookrightarrow (S', N')$ by $t \mapsto a$ and the identity of $\mathbb{N}$. Assume that we are given a smooth lifting f': $(X', M') \to (S', N')$ of the morphism f which is universally saturated. Let g be the composite of f' and $(S', N') \to S$. Let i be the exact closed immersion $(X, M) \hookrightarrow (X', M')$. By Theorem* 2.21, *we have canonical isomorphisms*

$$f^{\triangle}\mathcal{O}_S \cong E(I_f\Omega^n_{X/S}(\log(M/N))[n]),$$

$$g^{\triangle}\mathcal{O}_S \cong E(I_g\Omega^{n+1}_{X'/S}(\log(M'))[n+1]).$$

From the degree 0 *part of the trace morphism* Tr_i: $i_* f^{\triangle}\mathcal{O}_S \cong i_* i^{\triangle} g^{\triangle}\mathcal{O}_S \to g^{\triangle}\mathcal{O}_S$, *we obtain a morphism*

$$\mathrm{Tr}_{i,x}\colon\ i_*\mathcal{H}^n_x(I_f\Omega^n_{X/S}(\log(M/N))) \to \mathcal{H}^{n+1}_x(I_g\Omega^{n+1}_{X'/S}(\log(M'))),$$

for each closed point x of X.

Let x be a closed point of X, let $t'_1, \ldots, t'_n$ be elements of $\mathfrak{m}_{X',x}$ such that the images $t_1, \ldots, t_n$ in $\mathfrak{m}_{X,x}$ form a regular sequence, and let $\omega \in I_{f'}\Omega^n_{X'/S'}(\log (M'/N'))_x$. Then we have

$$\mathrm{Tr}_{i,x}\left(\frac{\overline{\omega}}{t_1\cdots t_n}\right) = \frac{\mathrm{d}t \wedge \omega}{(t-a)t'_1\cdots t'_n}.$$

Here $\overline{\omega}$ *denotes the image of* ω *under the homomorphism* $I_{f'}\Omega^n_{X'/S'}(\log(M'/N'))_x \to I_f\Omega^n_{X/S}(\log(M/N))_x$.

Remark 4.6. By applying Proposition 2.14 to a lifting of f' to $\mathrm{Spec}(A[t])$ with the log. structure associated to $\mathbb{N} \to A[t]$; $1 \mapsto t$ and using Lemma 2.18, we can verify that the homomorphism $\Omega^n_{X'/S'}(\log(M'/N')) \to \Omega^{n+1}_{X'/S}(\log(M'))$; $\omega \mapsto \mathrm{d}t \wedge \omega$ induces an isomorphism $I_{f'}\Omega^n_{X'/S'}(\log(M'/N')) \cong I_g\Omega^{n+1}_{X'/S}(\log(M'))$. Hence the right-hand side of the above equation makes sense.

SUBLEMMA 4.7. *Under the notation and assumption of Lemma* 4.5, *the isomorphism*

$$\begin{aligned} I_f\Omega^n_{X/S}(\log(M/N))[n] &\cong f^!\mathcal{O}_S \cong i^!g^!\mathcal{O}_S \\ &\cong i^!(I_g\Omega^{n+1}_{X'/S}(\log(M'))[n+1]) \\ &\cong I_g\Omega^{n+1}_{X'/S}(\log(M')) \otimes_{\mathcal{O}_{X'}} \check{\mathcal{N}}_{X/X'}[n] \end{aligned}$$

is given by $\overline{\omega} \mapsto (\mathrm{d}t \wedge \omega) \otimes (t-a)^{\vee}$ *for* $\omega \in I_{f'}\Omega^n_{X'/S'}(\log(M'/N'))$.

Proof. By restricting to the smooth loci of the morphisms f' and f, we can reduce to the corresponding fact for usual schemes. See the argument at the end of the proof of Theorem 2.21. □

Proof of Lemma 4.5. By Sublemma 4.7, it is enough to prove that the degree 0 part of the morphism

$$\begin{aligned} &i_*E_X(I_g\Omega^{n+1}_{X'/S}(\log(M')) \otimes_{\mathcal{O}_{X'}} \check{\mathcal{N}}_{X/X'}[n]) \\ &\quad \cong i_*i^{\Delta}E'_X(I_g\Omega^{n+1}_{X'/S}(\log(M'))[n+1]) \xrightarrow{\mathrm{Tr}_i} E'_X(I_g\Omega^{n+1}_{X'/S}(\log(M'))[n+1]) \end{aligned}$$

is given by

$$\begin{aligned} &\mathcal{H}^n_x(I_g\Omega^{n+1}_{X'/S}(\log(M')) \otimes_{\mathcal{O}_{X'}} \check{\mathcal{N}}_{X/X'}) \to \mathcal{H}^{n+1}_x(I_g\Omega^{n+1}_{X'/S}(\log(M'))) \\ &\frac{\omega \otimes (t-a)^{\vee}}{t_1 \cdots t_n} \mapsto \frac{\omega}{(t-a)t'_1 \cdots t'_n}. \end{aligned} \tag{4.8}$$

Here x is a closed point of X, $t'_1, \ldots, t'_n \in \mathfrak{m}_{X',x}$ is a lifting of a regular sequence $t_1, \ldots, t_n \in \mathfrak{m}_{X,x}$ and $\omega \in I_g\Omega^{n+1}_{X'/S}(\log(M'))_x$.

Let Z be the closed subscheme of X defined by a regular sequence $t_1, \ldots, t_n \in \mathfrak{m}_{X,x}$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{E}\varepsilon xt^n_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{E}xt^1_{\mathcal{O}_{X'}}(\mathcal{O}_X, \mathcal{K})) \arrow[d, equal, "\wr"'] \arrow[r, phantom, "\cong"] & \mathcal{E}xt^{n+1}_{\mathcal{O}_{X'}}(\mathcal{O}_Z, \mathcal{K}) \arrow[dd] \\
\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{K} \otimes_{\mathcal{O}_{X'}} \check{\mathcal{N}}_{X/X'}) \arrow[d, equal, "\wr"'] & \\
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{H}^n_x(\mathcal{K} \otimes_{\mathcal{O}_{X'}} \check{\mathcal{N}}_{X/X'})) \arrow[r] & \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_Z, \mathcal{H}^{n+1}_x(\mathcal{K})).
\end{tikzcd}$$

Here $\mathcal{K} = I_g\Omega^{n+1}_{X'/S}(\log(M'))$ and the bottom arrow is the morphism induced by (4.8). The image of

$$\omega \otimes ((t-a) \wedge t'_1 \wedge \cdots \wedge t'_n)^{\check{}} \in \mathcal{K} \otimes (\wedge^{n+1}\mathcal{N}_{Z/X'})^{\check{}} \cong \mathcal{E}xt^{n+1}_{\mathcal{O}_{X'}}(\mathcal{O}_Z, \mathcal{K}),$$

under the composite of the left arrow (resp. right arrow) is $(\omega \otimes (t-a)^{\check{}})/(t_1 \cdots t_n)$ (resp. $\omega/((t-a)t'_1 \cdots t'_n)$) (cf. [B] VII The proof of Lemma 1.2.5). This implies the claim. □

Proof of Proposition 4.1. (Part II) We will prove the proposition in the general case. By the same reason as in Part I, we may assume that the residue field of A is separably closed. By Lemma 4.2, we may assume that the assumption of Lemma 4.5 is satisfied. It suffices to prove that, for $\omega \in I_f\Omega^{n-1}_{X/S}(\log(M/N))_x$ and a regular sequence $t_1, \ldots, t_n \in \mathfrak{m}_{X,x}$, $\mathrm{Tr}_{f,x}(\mathrm{d}(\omega/t_1 \cdots t_n)) = 0$. Let $\omega' \in I_{f'}\Omega^{n-1}_{X'/S'}(\log(M'/N'))_x$ and $t'_i \in \mathfrak{m}_{X',x}$ be liftings of ω and t_i. Then we have

$$\mathrm{d}\left(\frac{\omega}{t_1 \cdots t_n}\right) = \frac{\mathrm{d}\omega}{t_1 \cdots t_n} - \sum_{1 \leqslant i \leqslant n} \frac{\mathrm{d}t_i \wedge \omega}{t_1 \cdots t_i^2 \cdots t_n},$$

$$\mathrm{d}\left(\frac{-\mathrm{d}t \wedge \omega'}{(t-a)t'_1 \cdots t'_n}\right) = \frac{\mathrm{d}t \wedge \mathrm{d}\omega'}{(t-a)t'_1 \cdots t'_n} - \sum_{1 \leqslant i \leqslant n} \frac{\mathrm{d}t \wedge \mathrm{d}t'_i \wedge \omega'}{(t-a)t'_1 \cdots t'^2_i \cdots t'_n}.$$

By Lemma 4.5, the latter is the image of the former under $\mathrm{Tr}_{i,x}$. Hence

$$\mathrm{Tr}_{f,x}\left(\mathrm{d}\left(\frac{\omega}{t_1 \cdots t_n}\right)\right) = 0,$$

since $\mathrm{Tr}_{g,x}\,\mathrm{Tr}_{i,x} = \mathrm{Tr}_{f,x}$ and the proposition has already been proved when N is trivial. □

5. Poincaré Duality for Log. Crystalline Cohomology

Let k be a perfect field of characteristic p, and let W be the ring of Witt vectors with coefficients in k. Fix an integer $m \geqslant 1$ and put $W_m = W/p^mW$. Let S be the scheme $\mathrm{Spec}(W_m)$, and let N be the log. structure on S associated to $\mathbb{N} \to W_m$; $1 \mapsto$

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a for some $a \in W_m$. Note that we allow the trivial log. structure. Let $S_0 = \mathrm{Spec}(k)$ and let N_0 be the inverse image of N. Let $f\colon (X, M) \to (S_0, N_0)$ be a smooth universally saturated morphism (Definition 2.17) of relative dimension n. Let γ be the canonical PD-structure on the ideal $p\mathcal{O}_S$.

We will define an ideal $K_{X/S}$ of $\mathcal{O}_{X/S}$ on $(X/S)^{\log}_{\mathrm{crys}} := ((X, M)/(S, N, p\mathcal{O}_S, \gamma))_{\mathrm{crys}}$ and on $(X/S)^{\log}_{\mathrm{Rcrys}} := ((X, M)/(S, N, p\mathcal{O}_S, \gamma))_{\mathrm{Rcrys}}$. (See Definition 6.2 for the definition of the restricted crystalline site.) Define the sheaf $M_{X/S}$ of monoids on the crystalline sites by $\Gamma((U, T, M_T, \delta), M_{X/S}) = \Gamma(T, M_T)$, and define the ideal $I_{X/S}$ of $M_{X/S}$ by $\Gamma((U, T, M_T, \delta), I_{X/S}) = \{a \in \Gamma((U, T, M_T, \delta), M_{X/S})\mid$ the image of a in $M_{T,\bar{x}}/\mathcal{O}^*_{T,\bar{x}} \cong M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ is contained in $\mathfrak{p}$ for all points $x \in T$ and all $\mathfrak{p} \in \mathrm{Spec}(M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}})$ of height 1 horizontal with respect to $N_{\overline{f(x)}}/\mathcal{O}^*_{S,\overline{f(x)}} \to M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}\}$.

LEMMA 5.1. *For each PD-thickening* (U, T, M_T, δ), *the stalk of* $I_T := (I_{X/S})_{(U,T,M_T,\delta)}$ *at* $x \in T$, *is* $\{a \in M_{T,\bar{x}}\mid$ *The image of* a *in* $M_{T,\bar{x}} = M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ *is contained in* $\mathfrak{p}$ *for all primes* $\mathfrak{p} \in \mathrm{Spec}(M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}})$ *of height* 1 *horizontal with respect to* $N_{\overline{f(x)}}/\mathcal{O}^*_{S,\overline{f(x)}} \to M_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}\}$.

Proof. The same as Lemma 2.5. □

COROLLARY 5.2. *Let* (U, T, M_T, δ) *and* I_T *be the same as in Lemma* 5.1. *Assume that we are given a chart* $P \to \Gamma(T, M_T)$ *with* P *saturated, and an element* $\pi \in P$ *whose image in* $\Gamma(X, M/\mathcal{O}^*_X)$ *coincides with the image of* 1 *under* $\mathbb{N} \to \Gamma(S, N/\mathcal{O}^*_S) \to \Gamma(X, M/\mathcal{O}^*_X)$. *Define the ideal* I *of* P *by:*

$I = \{a \in P\mid a$ *is contained in all primes* $\mathfrak{p} \in \mathrm{Spec}(P)$ *of height* 1 *which does not contain* $\pi\}$.

Then we have $I_T = I\mathcal{O}^*_T$.

We define the ideal $K_{X/S}$ of $\mathcal{O}_{X/S}$ by $I_{X/S}\mathcal{O}_{X/S}$.

LEMMA 5.3. *Let* E *be a quasi-coherent* $\mathcal{O}_{X/S}$*-module on* $(X/S)^{\log}_{\mathrm{Rcrys}}$. *Then* $K_{X/S}E$ *is a quasi-coherent* $\mathcal{O}_{X/S}$*-module. Especially,* $K_{X/S}E$ *is a crystal.*

Proof. By Corollary 5.2, $(K_{X/S}E)_T$ is a quasi-coherent $\mathcal{O}_T$-module for every T. Hence, it suffices to prove that $K_{X/S}E$ is a crystal. Since the question is étale local on X, we may assume that we have an exact closed immersion of (X, M) into a smooth fs log. scheme (Y, L) over (S, N). Let (D, M_D) be the PD-envelope of this exact closed immersion with respect to γ. Then we can verify that the connection $\nabla\colon E_D \to E_D \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ induces that on $(K_{X/S}E)_D$, which defines a crystal F. We claim $K_{X/S}E = F$. Since we work on the restricted crystalline site, F is an $\mathcal{O}_{X/S}$-submodule of E. Hence, it suffices to show that, for any morphism $f\colon T \to D$ in $(X/S)^{\log}_{\mathrm{Rcrys}}$, the morphism $f^*(K_{X/S})_D \to (K_{X/S})_T$ is surjective. This follows from Lemma 5.1. □

Let $Q_{X/S}\colon ((X/S)^{\log}_{\mathrm{Rcrys}})^{\sim} \to ((X/S)^{\log}_{\mathrm{crys}})^{\sim}$ be the canonical morphism of topoi.

DEFINITION 5.4. Let E be a locally free $\mathcal{O}_{X/S}$-module of finite rank on $(X/S)^{\log}_{\mathrm{crys}}$. We define the crystalline cohomology and the crystalline cohomology with compact supports by:

$$H^i_{\text{log-crys}}(X/S, E) = H^i((X/S)^{\log}_{\mathrm{crys}}, E) \cong H^i((X/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{X/S}E),$$

$$\begin{aligned} H^i_{\text{log-crys,c}}(X/S, E) &= H^i((X/S)^{\log}_{\mathrm{crys}}, K_{X/S}E) \\ &\cong H^i((X/S)^{\log}_{\mathrm{Rcrys}}, K_{X/S}(Q^*_{X/S}E)). \end{aligned}$$

We omit E when $E = \mathcal{O}_{X/S}$. By reducing to the case $S = S_0$, we can prove that these cohomology groups are finitely generated over W_m when X is proper over k. (cf. [B] VII Theorem 1.1.)

PROPOSITION 5.5. *Assume that X is proper over k. Then there is a canonical morphism called trace morphism* $\mathrm{Tr}_f\colon\ H^{2n}_{\text{log-crys},c}(X/S) \to W_m$. We will prove this proposition in Section 8.

THEOREM 5.6. *Assume that X is proper over k. For a locally free $\mathcal{O}_{X/S}$-module E of finite rank on $(X/S)^{\log}_{\mathrm{crys}}$, the pairing induced by the cup product and the trace morphism $H^i_{\text{log-crys}}(X/S, E) \times H^{2n-i}_{\text{log-crys},c}(X/S, \check{E}) \to W_m$ is perfect. Here $\check{E} = \mathcal{H}om_{\mathcal{O}_{X/S}}(E, \mathcal{O}_{X/S})$.*

Proof. We can reduce to Poincaré duality for de Rham cohomology of $(X, M)/(S_0, N_0)$ in the same way as [B] VII 2.1. □

This theorem can be applied to the special fiber of a semi-stable family (Example 2.16 with $n = 1$). In this case, $H^i_{\text{log-crys}} = H^i_{\text{log-crys},c}$.

Remark 5.7. Suppose that there still exists a trace morphism $\mathrm{Tr}_f\colon\ H^{2n}_{\text{log-crys},c}(X/S) \to W_m$, compatible with m which induces Poincaré duality and satisfies $\mathrm{Tr}_f(\varphi(x)) = p^n\varphi(\mathrm{Tr}_f(x))$ without assuming that f is universally saturated, or equivalently, that f is of Cartier type. Then, it implies that the Frobenius endomorphism on $\mathbb{Q} \otimes \varprojlim_m H^i_{\text{log-crys}}(X/W_m)$ is bijective, which is not true in general as follows.

Choose an integer e prime to p and let (X, M) be the scheme $\mathrm{Spec}(k[T]/(T^e))$ endowed with the log. structure associated to $\mathbb{N} \to k[T]/(T^e); 1 \mapsto T$. Assume that N_0 is non-trivial and let $f\colon\ (X, M) \to (S_0, N_0)$ be the k-morphism induced by $e\colon\ \mathbb{N} \to \mathbb{N}$. Then f is étale and X is finite over S_0. However, $H^0_{\text{log-crys}}(X/W_m) \cong W_m[T]/(T^e)$ and the Frobenius endomorphism is given by $T \mapsto T^p$.

In the case f is of Cartier type, O. Hyodo and K. Kato proved in [Hyo-K] (2.23), (2.24) that the Frobenius endomorphism on $\mathbb{Q} \otimes H^i_{\text{log-crys}}(X/W)$ is bijective.

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6. Local Cohomology for Log. Crystalline Topos

Since the definitions and fundamental properties of local cohomology for log. crystalline topoi are completely parallel to those for classical crystalline topoi ([B] VI 1), we will just give a sketch of them.

In this section, a sheaf of families of supports is considered in the étale topology, that is, for a scheme X, a sheaf of families of supports Φ on X is a sheaf of sets on $X_{\text{ét}}$ such that for any étale X-scheme U, $\Phi(U)$ is a family of supports on U, and for any morphism of étale X-schemes $f\colon U' \to U$, the restriction map $\Phi(f)\colon \Phi(U) \to \Phi(U')$ is given by $Z \mapsto f^{-1}(Z)$.

Let (S, N) be a fine log. scheme with a PD-ideal (I, γ), and let (X, M) be a fine log. scheme over (S, N). Assume that γ extends to X and that there exists an integer $n > 0$ such that $n\mathcal{O}_S = 0$.

When $X \to S$ is locally of finite type, we define the restricted crystalline site as follows.

DEFINITION 6.1 (cf. [B] IV Definition 1.7.1). Assume that X is locally of finite type over S. Let (U, T, M_T, δ) be an object of $(X/S)^{\log}_{\text{crys}} = ((X, M)/(S, N, I, \gamma))_{\text{crys}}$. We say that (U, T, M_T, δ) is a *fundamental thickening* if there exists an (S, N)-closed immersion $i\colon\ (U, M|U) \hookrightarrow (Y, L)$ into a smooth fine log. scheme over (S, N) such that (U, T, M_T, δ) is isomorphic to the PD-envelope of i with respect to γ.

DEFINITION 6.2 (cf. [B] IV Definition 2.2.1). Assume that X is locally of finite type over S. We define the *restricted crystalline site*

$$(X/S)^{\log}_{\text{Rcrys}} = ((X, M)/(S, N, I, \gamma))_{\text{Rcrys}}$$

to be the full-subcategory of $(X/S)^{\log}_{\text{crys}}$ consisting of fundamental thickenings endowed with the induced topology.

Note that the above definition is slightly different from that of [B] IV 2. When we consider the restricted crystalline site $(X/S)^{\log}_{\text{Rcrys}}$, we always assume that $X \to S$ is locally of finite type. In the following, we write $\Gamma((X, M)/(S, N), F)$ for $\Gamma((X/S)^{\log}_{\text{crys}}, F)$ (resp. $\Gamma((X/S)^{\log}_{\text{Rcrys}}, F)$) for a sheaf of Abelian groups F on $(X/S)^{\log}_{\text{crys}}$ (resp. $(X/S)^{\log}_{\text{Rcrys}}$).

Let φ be a family of supports on X. For a sheaf of Abelian groups F on $(X/S)^{\log}_{\text{crys}}$ (resp. $(X/S)^{\log}_{\text{Rcrys}}$), we define $\Gamma_\varphi((X, M)/(S, N), F)$ to be $\{s \in \Gamma((X, M)/(S, N), F) |$ For each (U, T, M_T, δ), there exists $Z \in \varphi$ such that the support of the section s_T of F_T defined by s is contained in $u^{-1}(Z)$, where u is the morphism $U \to X\}$. This is left exact on F. We denote by $H^i_\varphi((X, M)/(S, N), F)$ its derived functors.

For two families of supports $\psi \subset \varphi$ on X, we define $\Gamma_{\varphi/\psi}((X, M)/(S, N)), F)$ to be the quotient $\Gamma_\varphi((X, M)/(S, N), F)/\Gamma_\psi((X, M)/(S, N), F)$, and we denote by $H^i_{\varphi/\psi}((X, M)/(S, N), F)$ its derived functors.

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For a sheaf of families of supports Φ on X and a sheaf of Abelian groups F on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$), we define the subsheaf $\underline{\Gamma}_{\Phi}(F)\subset F$ by $\underline{\Gamma}_{\Phi}(F)((U,T,M_T,\delta))=\{s\in F((U,T,M_T,\delta))|$ The section s has its support in $\Phi(U)$ as a section of $F_T\}$. This is left exact on F. We denote by $\mathcal{H}^i_{\Phi}(F)$ its derived functors.

Finally, for two sheaves of families of supports $\Psi\subset\Phi$, we define $\underline{\Gamma}_{\Phi/\Psi}(F)$ to be the quotient $\underline{\Gamma}_{\Phi}(F)/\underline{\Gamma}_{\Psi}(F)$, and denote by $\mathcal{H}^i_{\Phi/\Psi}(F)$ its derived functors.

For a family of supports φ on X, we define the sheaf of families of supports $\tilde{\varphi}$ by $\tilde{\varphi}(U)=\{Z|Z$ is a closed subset of U such that there exists an open covering $U=\cup_iU_i$ and $Z_i\in\varphi$ such that $U_i\cap Z\subset u^{-1}(Z_i)\}$. Here $u\colon\ U\to X$ denotes the structure morphism.

For a closed subset $Z\subset X$, we define φ_Z to be the set of closed subsets of Z and put

$$\Gamma_Z:=\Gamma_{\varphi_Z},\qquad \underline{\Gamma}_Z:=\underline{\Gamma}_{\widetilde{\varphi_Z}},\qquad H^i_Z:=H^i_{\varphi_Z},\qquad \mathcal{H}^i_Z=\mathcal{H}^i_{\widetilde{\varphi_Z}}.$$

For two closed subsets $Z\subset Y\subset X$, we define $\Gamma_{Y/Z},\underline{\Gamma}_{Y/Z},H^i_{Y/Z}$ and $\mathcal{H}^i_{Y/Z}$ similarly.

For a family of supports φ on X, we have

$$H^i_{\varphi}(F)\cong\varinjlim_{Z\in\varphi}H^i_Z(F),\qquad \mathcal{H}^i_{\tilde{\varphi}}(F)\cong\varinjlim_{Z\in\varphi}\mathcal{H}^i_Z(F)\tag{6.3}$$

and similar isomorphisms for $H^i_{\varphi/\psi}$ and $\mathcal{H}^i_{\tilde{\varphi}/\tilde{\psi}}$ (cf. [Ha] IV Section 1).

More generally, when X is locally Noetherian, for a subset $Z\subset X$ stable under specialization (i.e. $x\in Z$ and $x'\in\overline{\{x\}}$ implies $x'\in Z$), we define the family of supports φ_Z on X by $\{Y|Y$ is a closed subset of X and there exists an open covering $X=\cup_iU_i$ and a finite set $\{x_\lambda;\lambda\in\Lambda_i\}\subset Z$ for each i such that $Y\cap U_i$ is contained in $\cup_{\lambda\in\Lambda_i}\overline{\{x_\lambda\}}\}$ and define Γ_Z, H^i_Z, $\underline{\Gamma}_Z$, $\mathcal{H}^i_Z$, $\Gamma_{Z/Z'}$, $H^i_{Z/Z'}$, $\underline{\Gamma}_{Z/Z'}$, and $\mathcal{H}^i_{Z/Z'}$ in the same way.

Flasque sheaves are acyclic for the four functors defined above (cf. [B] VI 1.1.6). Hence, for a sheaf of rings A on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$) and an A-module F, the local cohomology of F considered in the category of A-modules coincides with that of F as sheaves of Abelian groups since injective A-modules are flasque.

For a flasque sheaf F on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$) and an object $T\in(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$), F_T is flasque on $T_{\text{ét}}$ (cf. [B] VI Proposition 1.1.5). From this, we obtain the following proposition.

PROPOSITION 6.4 (cf. [B] VI Proposition 1.1.7). *Let $\Psi\subset\Phi$ be two sheaves of families of supports on X and let F be a sheaf of Abelian groups on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$). Then, for all $(U,T,M_T,\delta)\in(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$), we have isomorphisms*

$$\mathcal{H}^i_{\Phi}(F)_T\cong\mathcal{H}^i_{\text{ét},\Phi|U}(F_T),\qquad \mathcal{H}^i_{\Phi/\Psi}(F)_T\cong\mathcal{H}^i_{\text{ét},\Phi|U/\Psi|U}(F_T).$$

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We need the following propositions.

DEFINITION 6.5. Assume that X is locally noetherian. For a sheaf of Abelian groups F on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$) and a point $x \in X$, we define $\mathcal{H}^i_x(F) := \mathcal{H}^i_{\overline{\{x\}}/(\overline{\{x\}}\setminus\{x\})}(F)$.

PROPOSITION 6.6. *Assume that X is locally noetherian and let $Z' \subset Z \subset X$ be two subsets stable under specialization. Assume that $z \in Z\setminus Z'$, $z' \in \overline{\{z\}}$, $z' \neq z$ implies $z' \in Z'$. Then, for any sheaf of rings A on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$) and an A-module F, there exists a canonical A-linear isomorphism $\bigoplus_{z\in Z\setminus Z'} \mathcal{H}^i_z(F) \cong \mathcal{H}^i_{Z/Z'}(F)$.*

Proof. Using Proposition 6.4, we can reduce to the corresponding claim for étale cohomology. □

PROPOSITION 6.7 (cf. [B] VI Proposition 1.1.11). *Let $(\Phi_i)_{i\in\mathbb{N}}$ be a family of sheaves of families of supports on X such that $\Phi_{i+1} \subset \Phi_i$ and $\Phi_i = 0 (i \gg 0)$. Then, for a sheaf of rings A on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$) and an A-module F, there is a spectral sequence of A-modules $E_1^{p,q} = \mathcal{H}^{p+q}_{\Phi_p/\Phi_{p+1}}(F) \Rightarrow \mathcal{H}^n_{\Phi_0}(F)$.*

For an open immersion $j\colon V \hookrightarrow X$, we denote by j_{crys} the morphism of topoi

$$((V/S)^{\log}_{\mathrm{crys}})^{\sim} \xrightarrow{\sim} ((X/S)^{\log}_{\mathrm{crys}})^{\sim}/V_{\mathrm{crys}} \to ((X/S)^{\log}_{\mathrm{crys}})^{\sim}$$

$$(\text{resp. } ((V/S)^{\log}_{\mathrm{Rcrys}})^{\sim} \xrightarrow{\sim} ((X/S)^{\log}_{\mathrm{Rcrys}})^{\sim}/V_{\mathrm{crys}} \to ((X/S)^{\log}_{\mathrm{Rcrys}})^{\sim}),$$

where V_{crys} denotes the sheaf of sets defined by $\Gamma((U, T, M_T, \delta), V_{\mathrm{crys}}) = \mathrm{Hom}_X(U, V)$. Let Y be the closed subscheme of X defined by the ideal generated by some sections $t_1, \dots, t_d \in \Gamma(X, \mathcal{O}_X)$. Let $U_{i_0,\dots,i_k} (1 \leqslant i_0 < \cdots < i_k \leqslant d)$ be the locus where $t_{i_0}, \dots, t_{i_k}$ are invertible and let $j_{i_0,\dots,i_k}$ be the inclusion $U_{i_0,\dots,i_k} \hookrightarrow X$. For a sheaf of Abelian groups F on $(X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$), we define the complex $C^{\cdot}_{t_1,\dots,t_d}(F)$ to be

$$C^k_{t_1,\dots,t_d}(F) = \bigoplus_{1\leqslant i_0<\cdots<i_k\leqslant d} (j_{i_0,\dots,i_k\mathrm{crys}})_*(j_{i_0,\dots i_k\mathrm{crys}})^* F$$

with the differential homomorphisms defined by the usual alternating sums.

PROPOSITION 6.8 (cf. [B] VI Proposition 1.2.5). *Let F be an $\mathcal{O}_{X/S}$-module such that, for each $(V, T, M_T, \delta) \in (X/S)^{\log}_{\mathrm{crys}}$ (resp. $(X/S)^{\log}_{\mathrm{Rcrys}}$), $F_{(V,T,M_T,\delta)}$ is a quasi-coherent $\mathcal{O}_T$-module. Then $R\underline{\Gamma}_Y(F)$ is canonically isomorphic to the complex $0 \to F \to C^{\cdot}_{t_1,\dots,t_d}(F) \to 0$, where F is placed in degree 0. Especially, if F is quasi-coherent, $\mathcal{H}^d_Y(F)$ is quasi-coherent, and if F is a quasi-coherent $\mathcal{O}_{X/S}$-module on $(X/S)^{\log}_{\mathrm{Rcrys}}$, then $\mathcal{H}^i_Y(F)$ are quasi-coherent for all i.*

7. Cousin Complex on Log. Crystalline Topos

LEMMA 7.1. *Let X be a locally noetherian scheme and let $\psi \subset \varphi$ be two families of supports on X. Let F be a quasi-coherent $\mathcal{O}_X$-module. Let ε denote the projection $X^{\sim}_{\text{ét}} \to X^{\sim}_{\text{Zar}}$. Then $R\varepsilon_*(\mathcal{H}^i_{\text{ét},\tilde{\varphi}/\tilde{\psi}}(F)) \cong \mathcal{H}^i_{\text{Zar},\tilde{\varphi}/\tilde{\psi}}(F)$ and, for any étale morphism $u\colon U \to X$,*

$$\mathcal{H}^i_{\text{ét},\tilde{\varphi}/\tilde{\psi}}(F)|U_{\text{Zar}} \cong (u_{\text{Zar}})^*\mathcal{H}^i_{\text{Zar},\tilde{\varphi}/\tilde{\psi}}(F),$$

where the left-hand side is the sheaf on U_{Zar} obtained by restriction.

Proof. We prove the claim for $\mathcal{H}^i_{\bullet,Y/Z}(F)(\bullet = \text{ét}, \text{Zar})$ for closed subschemes Y and Z of X such that $Z \subset Y$. The general case follows from this and the isomorphism

$$\mathcal{H}^i_{\bullet,\tilde{\varphi}/\tilde{\psi}}(F) \cong \varinjlim_{\substack{Y\in\varphi, Z\in\psi\\ Z\subset Y}} \mathcal{H}^i_{\bullet,Y/Z}(F) \quad (\bullet = \text{ét}, \text{Zar}).$$

For any $\mathcal{O}_X$-module F on $X_{\text{ét}}$, we have a canonical homomorphism

$$\underline{\Gamma}_{\text{Zar},Y/Z}(\varepsilon_*F) \to \varepsilon_*\underline{\Gamma}_{\text{ét},Y/Z}(F), \tag{7.2}$$

which makes the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \varepsilon_*\underline{\Gamma}_{\text{ét},Z}(F) \arrow[r] & \varepsilon_*\underline{\Gamma}_{\text{ét},Y}(F) \arrow[r] & \varepsilon_*\underline{\Gamma}_{\text{ét},Y/Z}(F) & \\
0 \arrow[r] & \underline{\Gamma}_{\text{Zar},Z}(\varepsilon_*F) \arrow[r] \arrow[u, equal] & \underline{\Gamma}_{\text{Zar},Y}(\varepsilon_*F) \arrow[r] \arrow[u, equal] & \underline{\Gamma}_{\text{Zar},Y/Z}(\varepsilon_*F) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Since the functors $\underline{\Gamma}_{\text{Zar},Y/Z}$, $\underline{\Gamma}_{\text{ét},Y/Z}$, and ε_* preserve flasque sheaves and flasque sheaves are acyclic for these functors, the homomorphism (7.2) induces a morphism of functors

$$R\underline{\Gamma}_{\text{Zar},Y/Z}R\varepsilon_* \to R\varepsilon_*R\underline{\Gamma}_{\text{ét},Y/Z}. \tag{7.3}$$

If F is flasque, the homomorphism (7.2) is an isomorphism since $\underline{\Gamma}_{\text{ét},Z}(F)$ is flasque and hence $R^1\varepsilon_*(\underline{\Gamma}_{\text{ét},Z}(F)) = 0$. This implies that the morphism (7.3) is an isomorphism.

Since $R\varepsilon_*\mathcal{G} = \mathcal{G}$ for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ on $X_{\text{ét}}$, it remains to prove that $\mathcal{H}^i_{Y/Z,\text{ét}}(F)$ is quasi-coherent if F is quasi-coherent. This follows from the following sublemma. □

SUBLEMMA 7.4. *Let X be a scheme and let $Y \subset X$ be the closed subscheme defined by the ideal generated by a sequence of sections $t_1, \ldots, t_d \in \Gamma(X, \mathcal{O}_X)$. Let $U_{i_0,\ldots,i_k} \subset X$ be the maximal open subset of X where $t_{i_0}, \ldots, t_{i_k}$ is invertible*

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for $1 \leqslant i_0 < \cdots < i_k \leqslant d$ *and let* $j_{i_0,\ldots,i_k}$ *be the inclusion* $U_{i_0,\ldots,i_k} \hookrightarrow X$. *Let* F *be a quasi-coherent* $\mathcal{O}_X$*-module on* $X_{\text{ét}}$ *and let* $C^{\cdot}_{t_1,\ldots,t_d}(F)$ *be the complex whose* k*th term is* $\oplus_{1\leqslant i_0<\cdots<i_k\leqslant d}(j_{i_0,\ldots,i_k})_*(j_{i_0,\ldots,i_k})^*F$ *and whose differential* $C^k \to C^{k+1}$ *is the usual alternating sum of the* $k+2$ *homomorphisms. Then* $R\underline{\Gamma}_{\text{ét},Y}(F)$ *is canonically isomorphic to the complex* $F \to C^{\cdot}_{t_1,\ldots,t_d}(F)$, *where* F *is placed in degree* 0.

Let (S, N) be as in the beginning of Section 5.

COROLLARY 7.5. *Let* $g\colon (Y, L) \to (S, N)$ *be a smooth universally saturated morphism, and let* I_g *be as in Section* 2. *Then, for any locally free* $\mathcal{O}_Y$*-module of finite rank* F *and any integer* $d \geqslant 0$, *we have* $\mathcal{H}^i_{\text{ét},X^d/X^{d+1}}(I_g F) = 0$ $(i \neq d)$, *where* X^d *denotes the set of points of codimension* d *in* X.

Proof. By Lemma 7.1, it suffices to prove the corresponding claim for Zariski cohomology. We may assume Y is quasi-compact. Then, since $I_g\mathcal{O}_Y$ is a dualizing complex on Y by Theorem 2.21, the claim follows from [Ha] V Proposition 7.3. □

Let $f\colon (X, M) \to (S_0, N_0)$ and $K_{X/S}$ be as in Section 5.

PROPOSITION 7.6. *For any locally free* $\mathcal{O}_{X/S}$*-module* E *of finite rank on* $(X/S)^{\log}_{\text{Rcrys}}$ *and any integer* $d \geqslant 0$ $\mathcal{H}^i_{X^d/X^{d+1}}(K_{X/S}E) = 0$ $(i \neq d)$ *and* $\mathcal{H}^d_{X^d/X^{d+1}}(K_{X/S}E)$ *is a crystal of* $\mathcal{O}_{X/S}$*-module, where* X^d *denotes the set of points of codimension* d *in* X.

Proof. By Proposition 6.8 and (6.3), $\mathcal{H}^i_{X^d}(K_{X/S}E)$ is a crystal. Hence $\mathcal{H}^i_{X^d/X^{d+1}}(K_{X/S}E)$ is a crystal. So the vanishing of the cohomology of degree $i \neq d$ follows from Corollary 7.5 and Proposition 6.4. □

By Proposition 6.7, we obtain the following corollary.

COROLLARY 7.7. *Under the assumption of Proposition* 7.6, *we have a resolution of* $K_{X/S}E$

$$\begin{aligned} K_{X/S}E \;&\to\; \mathcal{H}^0_{X^0/X^1}(K_{X/S}E) \to \mathcal{H}^1_{X^1/X^2}(K_{X/S}E) \\ &\to\; \cdots \to \mathcal{H}^d_{X^d/X^{d+1}}(K_{X/S}E) \to \cdots. \end{aligned}$$

8. Construction of the Trace Morphism

DEFINITION 8.1. For a locally Noetherian scheme X, a sheaf of Abelian groups F on $X_\bullet$ ($\bullet =$ ét, Zar), and a point x of X, we define $\mathcal{H}^i_{\bullet,x}(F) := \mathcal{H}^i_{\bullet,\overline{\{x\}}/(\overline{\{x\}}\setminus\{x\})}(F)$.

We have an isomorphism $\mathcal{H}^i_{\text{Zar},x}(F) \cong i_{x_*}H^i_{\text{Zar},x}(F)$, where i_x is the morphism $x \to X$ ([Ha] IV Section 1 Variation 8 Motif F).

Let (S, N) be the same as in Section 5, and let $g\colon (Y, L) \to (S, N)$ be a smooth universally saturated morphism whose underlying morphism of schemes is of finite type. Assume that Y is of constant dimension n. We define the complex

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$\mathcal{H}^{\cdot i}_{\bullet,y}(I_g(\Omega^{\log}_{Y/S})^{\cdot})(\bullet = \text{ét, Zar})$ for $y \in Y$ to be

$$\cdots \to 0 \to \mathcal{H}^i_{\bullet,y}(I_g\mathcal{O}_Y) \to \mathcal{H}^i_{\bullet,y}(I_g(\Omega^{\log}_{Y/S})^1)$$
$$\to \cdots \to \mathcal{H}^i_{\bullet,y}(I_g(\Omega^{\log}_{Y/S})^{n-1}) \to \mathcal{H}^i_{\bullet,y}(I_g(\Omega^{\log}_{Y/S})^n) \to 0 \to \cdots,$$

where $\mathcal{H}^i_{\bullet,y}(I_g\mathcal{O}_Y)$ is placed in degree 0. Let ε_Y denote the morphism of topoi $Y^{\sim}_{\text{ét}} \to Y^{\sim}_{\text{Zar}}$. Then by Lemma 7.1,

$$R\varepsilon_{Y*}(\mathcal{H}^{\cdot i}_{\text{ét},y}(I_g(\Omega^{\log}_{Y/S})^{\cdot})) \cong \mathcal{H}^{\cdot i}_{\text{Zar},y}(I_g(\Omega^{\log}_{Y/S})^{\cdot}). \tag{8.2}$$

On the other hand, we obtain from Proposition 4.1 a morphism

$$\mathrm{Res}_{g,y}\colon\ R^n g_{\text{Zar}*}(\mathcal{H}^{\cdot n}_{\text{Zar},y}(I_g(\Omega^{\log}_{Y/S})^{\cdot})) \to \mathcal{O}_S,$$

for each closed point $y \in Y$.

Let (S_0, N_0) be the same as in Section 5 and let $f\colon\ (X, M) \to (S_0, N_0)$ be a smooth universally saturated morphism whose underlying morphism of schemes is of finite type. Assume that X is of constant dimension n. Let ε_X denote the morphism of topoi $X^{\sim}_{\text{ét}} \to X^{\sim}_{\text{Zar}}$ and let $\overline{u}_{X/S}$ be the projection $((X/S)^{\log}_{\text{Rcrys}})^{\sim} \to X^{\sim}_{\text{ét}}$.

LEMMA 8.3. *Let $x \in X$ be a point. Let $g\colon\ (Y, L) \to (S, N)$ be a smooth lifting of an open neighborhood $U \subset X$ of x. Then we have a canonical isomorphism*

$$R(\varepsilon_X\overline{u}_{X/S})_*\mathcal{H}^i_x(K_{X/S}) \cong i_{x*}H^{\cdot i}_{\text{Zar},x}(I_g(\Omega^{\log}_{Y/S})^{\cdot}),$$

where i_x denotes the morphism $i_x\colon\ x \to X$.

Proof. Let j denote the morphism $U \hookrightarrow X$. Since we have an isomorphism

$$j^{-1}R(\varepsilon_X\overline{u}_{X/S})_*\mathcal{H}^i_x(K_{X/S}) \cong i'_{x*}H^{\cdot i}_{\text{Zar},x}(I_g(\Omega^{\log}_{Y/S})^{\cdot})$$

by Proposition 7.6, Proposition 6.4, and (8.2), where i'_x denotes the morphism $x \to U$, it is sufficient to prove that the morphism

$$R(\varepsilon_X\overline{u}_{X/S})_*\mathcal{H}^i_x(K_{X/S}) \to Rj_*j^{-1}(R(\varepsilon_X\overline{u}_{X/S})_*\mathcal{H}^i_x(K_{X/S}))$$

is an isomorphism. Since the question is Zariski local on X, we may assume that there exists a lifting $g'\colon\ (Y', L') \to (S, N)$ of f globally. Then we have an isomorphism

$$R(\varepsilon_X\overline{u}_{X/S})_*\mathcal{H}^i_x(K_{X/S}) \cong i_{x*}H^{\cdot i}_{\text{Zar},x}(I_{g'}(\Omega^{\log}_{Y'/S})^{\cdot}).$$

The claim follows from this. □

Let $f_{X/S}\colon\ ((X/S)^{\log}_{\text{Rcrys}})^{\sim} \to (S_{\text{Zar}})^{\sim}$ be the composite $f_{\text{Zar}}\varepsilon_X\overline{u}_{X/S}$.

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PROPOSITION 8.4 (cf. [B] VII Proposition 1.2.8). *For a closed point $x \in X$, there exists a unique homomorphism of $\mathcal{O}_S$-modules* $\mathrm{Res}_{f,x}\colon R^n f_{X/S*}(\mathcal{H}_x^n(K_{X/S})) \to \mathcal{O}_S$ *such that, for any smooth universally saturated lifting $g\colon (Y, L) \to (S, N)$ of an open neighborhood U of x, the following diagram commutes*

$$\begin{tikzcd} R^n f_{X/S*}(\mathcal{H}_x^n(K_{X/S})) \arrow[r, "\mathrm{Res}_{f,x}"] \arrow[d, equal, "\wr"', "\text{Lemma 8.3}"] & \mathcal{O}_S \arrow[d, equal] \\ R^n g_{\mathrm{Zar}*}(\mathcal{H}^{\cdot n}_{\mathrm{Zar},y}(I_g(\Omega^{\log}_{Y/S})^{\cdot})) \arrow[r, "\mathrm{Res}_{g,x}"] & \mathcal{O}_S \end{tikzcd} \tag{8.5}$$

Proof. The existence of a neighborhood and a lifting follows from [K1] Proposition (3.14). Choose U and g and define $\mathrm{Res}_{f,x}$ by the commutative diagram (8.5). Since, for any affine open neighborhood U of X and any two liftings (Y, L) and (Y', L'), there is an (S, N)-isomorphism $(Y, L) \cong (Y', L')$ compatible with the embeddings of $(U, M|U)$ (loc. cit.), this definition does not depend on the choice of U and g. □

LEMMA 8.6 (cf. [B] VII Lemma 1.4.1). *$R^i\overline{u}_{X/S*}$ commutes with direct sums.*

Proof. Since the question is étale local on X, we may assume that there exists a closed immersion of (X, M) into a smooth fs log. scheme (Y, L) over (S, N). Then there exists an isomorphism

$$R\overline{u}_{X/S*}\mathcal{F} \cong (\mathcal{F}_{(X,X)} \to \mathcal{F}_{(X,D_X(Y))} \to \mathcal{F}_{(X,D_X(Y^2))} \to \cdots),$$

(cf. [B] V Proposition 1.3.1) and each term of the complex commutes with direct sums. □

Let X^d be the set of points of codimension d of X.

LEMMA 8.7 (cf. [B] VII Proposition 1.4.2). *Assume that X is quasi-compact.*

(1) *For all i and j, the homomorphism*

$$\bigoplus_{x\in X^i\setminus X^{i+1}} R^j f_{X/S*}(\mathcal{H}_x^i(K_{X/S})) \to R^j f_{X/S*}(\mathcal{H}^i_{X^i/X^{i+1}}(K_{X/S}))$$

is an isomorphism.

(2) *For all i, $R^j f_{X/S*}(\mathcal{H}^i_{X^i/X^{i+1}}(K_{X/S})) = 0, \quad j > n.$*

Proof. (1) This follows from Proposition 6.6, Lemma 8.6 and the fact that $R^i f_{\mathrm{Zar}*}$ and $R\varepsilon_{X*}$ commute with direct sums. Note that X is Noetherian.

(2) By (1), it is enough to prove $R^j f_{X/S*}(\mathcal{H}_x^i(K_{X/S})) = 0$ for $x \in X$ and $j > n$. This follows from Lemma 8.3. □

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By Lemma 8.7 and Proposition 8.4, we obtain an $\mathcal{O}_S$-linear homomorphism

$$\mathrm{Res}_f\colon\ R^n f_{X/S*}(\mathcal{H}^n_{X^n}(K_{X/S})) \to \mathcal{O}_S. \tag{8.8}$$

PROPOSITION 8.9 (Residue Theorem) (cf. [B] VII Proposition 1.4.5) *Assume that X is proper over k. Then the composite*

$$R^n f_{X/S*}(\mathcal{H}^n_{X^{n-1}/X^n}(K_{X/S})) \to R^n f_{X/S*}(\mathcal{H}^n_{X^n}(K_{X/S})) \xrightarrow{\mathrm{Res}_f} \mathcal{O}_S$$

is 0.

In the following until the end of the proof of Proposition 8.9, we simply write $X, Y, \ldots,$ instead of $(X, M), (Y, L), \ldots,$ for log. schemes. We will need an analogue of [B] VII 1.3 for fine log. schemes.

LEMMA 8.10. *Let X and S be fine log. schemes, let $f\colon X \to S$ be a log. smooth morphism, let I be an ideal of $\mathcal{O}_S$ such that $I^2 = (0)$, and let $S_0 \hookrightarrow S$ be the exact closed immersion defined by I. Let $X_0 := X \times_S S_0$ and let i be the exact closed immersion $X_0 \hookrightarrow X$.*

(1) *Define the presheaf $\mathcal{A}ut_S(X/X_0)$ on $(X_0)_{\text{ét}}$ by*

$$\Gamma(U, \mathcal{A}ut_S(X/X_0)) = \{\sigma \in \mathrm{Aut}_S(X|U) |\ \sigma \circ i|U = i|U\}.$$

Then there is a canonical isomorphism

$$\mathcal{A}ut_S(X/X_0) \cong \mathcal{H}om_{\mathcal{O}_{X_0}}(i^*\Omega^{\log}_{X/S}, I\mathcal{O}_X).$$

Especially $\mathcal{A}ut_S(X/X_0)$ can be regarded as a quasi-coherent $\mathcal{O}_{X_0}$-module.

Let $Z_0 \hookrightarrow X_0$ be an exact closed immersion and let k be an arbitrary positive integer. Let $u_0\colon W_0 \hookrightarrow X_0$ (resp. $u\colon W \hookrightarrow X$) be the exact closed immersion whose underlying morphism of schemes gives the kth infinitesimal neighborhood of Z_0 in X_0 (resp. X). Let j be the morphism $W_0 \hookrightarrow W$.

(2) *Define the presheaf $\mathcal{A}ut_S(W/W_0)$ on $(W_0)_{\text{ét}}$ by*

$$\Gamma(V, \mathcal{A}ut_S(W/W_0)) = \{\tau \in \mathrm{Aut}_S(W|V) |\ \tau \circ j|V = j|V\}.$$

Then there is a canonical isomorphism

$$\mathcal{A}ut_S(W/W_0) \cong \mathcal{H}om_{\mathcal{O}_{W_0}}(j^*\Omega^{\log}_{W/S}, I\mathcal{O}_W).$$

Especially $\mathcal{A}ut_S(W/W_0)$ can be regarded as a quasi-coherent $\mathcal{O}_{W_0}$-module.

(3) *The homomorphism $\mathcal{A}ut_S(X/X_0) \to u_{0*}\mathcal{A}ut_S(W/W_0)$ defined by restriction is a homomorphism of $\mathcal{O}_{X_0}$-modules. If f is integral, this is surjective as a morphism of sheaves.*

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Proof. (1) and (2) follows from [K1] Proposition (3.9). The surjectivity in (3) follows from an analogue of [B] VII Lemma 1.3.2 for fine log. schemes in the same way as [B] VII Corollary 1.3.3. Left to the reader for details. □

DEFINITION 8.11 (cf. [B] VII Definition 1.3.1). Let X and S be fine log. schemes, let $f\colon X \to S$ be a log. smooth morphism, let $Z \hookrightarrow X$ be an exact closed immersion, and let k be an arbitrary positive integer. Let $W \hookrightarrow X$ be the exact closed immersion whose underlying morphism of schemes gives the kth infinitesimal neighborhood of Z in X. Let $S \hookrightarrow S'$ be an exact closed immersion, let I be the ideal of $\mathcal{O}_S$ defining S and let $S_n \hookrightarrow S'$ be the exact closed immersion defined by I^{n+1}. We say that an S_n-fine log. scheme W_n with an S-isomorphism $W \xrightarrow{\sim} W_n \times_{S_n} S$ is a *deformation of order* n if it satisfies the following condition.

For each $x \in Z$, there exist an étale neighborhood U of x in X as a scheme and a log. smooth lifting $U_n \to S_n$ of $f|U$ such that there is an S_n-isomorphism between $W_n|U$ and W_n' which induces the identity of $W|U$. Here W_n' is the kth infinitesimal neighborhood of $Z|U$ in U_n endowed with the inverse image log. structure.

LEMMA 8.12. *Keep the notation and assumption of Definition* 8.11. *Furthermore assume that f is integral and the underlying scheme of X is affine. Let n be a positive integer and let W_{n-1} be a deformation of order $n-1$ of W. Then, there is a deformation of order n W_n with an S_{n-1}-isomorphism $W_{n-1} \cong W_n \times {}_{S_n}S_{n-1}$ which induces the identity on W. Furthermore such a deformation is unique up to isomorphisms.*

Proof. There exists W_n étale locally on X, and two deformations W_n and W_n' are isomorphic étale locally on X by Lemma 8.10 (3) (cf. [B] VII Proposition 1.3.4). Hence the lemma follows from Lemma 8.10 (2). □

COROLLARY 8.13 (cf. [B] VII Corollary 1.3.7). *Keep the notation and assumption of Definition* 8.11. *Assume that f is integral, the underlying scheme of X is separated, and the underlying scheme of Z is noetherian of dimension* 1. *Then, for all n, there exists a deformation of order n of W.*

Proof of Proposition 8.9. The proof is essentially the same as that of [B] VII Proposition 1.4.5. By Lemma 8.7, it is sufficient to prove that for any $z \in X^{n-1}\backslash X^n$, the composite

$$\begin{aligned} R^n f_{X/S*}\mathcal{H}_z^{n-1}(K_{X/S}) &\to R^n f_{X/S*}\mathcal{H}^n_{\overline{\{z\}}\backslash\{z\}}(K_{X/S}) \\ &\cong \bigoplus_{x\in\overline{\{z\}}\backslash\{z\}} R^n f_{X/S*}\mathcal{H}^n_x(K_{X/S}) \to \mathcal{O}_S \end{aligned}$$

is 0, where the last morphism is the sum of $\mathrm{Res}_{f,x}$.

Fix $z \in X^{n-1}\backslash X^n$ and $a \in R^n f_{X/S*}\mathcal{H}_z^{n-1}(K_{X/S})$. Choose and fix an affine open neighborhood $U \subset X$ of z and a log. smooth lifting $g\colon V \to S$ of $f|U$ such that V

has an fs. log. structure. By Lemma 8.3, there is a surjection $H^{n-1}_{\mathrm{Zar},z}(I_g(\Omega^{\log}_{V/S})^n) \to R^n f_{X/S*}\mathcal{H}^{n-1}_z(K_{X/S})$. Let b be a lifting of a under this surjection.

Let i_z be the morphism $z \to V$ and let Z be the closure of $\{z\}$ in X with the reduced induced closed subscheme structure. Let k be a positive integer such that b is killed by the kth power of the defining ideal of the closed immersion $Z \cap U \hookrightarrow V$ as a section of the quasi-coherent $\mathcal{O}_V$-module $i_{z*}H^{n-1}_{\mathrm{Zar},z}(I_g(\Omega^{\log}_{V/S})^n)$.

Let $h\colon W \to S$ be a deformation of order $m-1$ (Definition 8.11) of the kth infinitesimal neighborhood $Z^{(k)}$ of $Z \subset X$. Since the dimension of Z is 1, such a deformation exists (Corollary 8.13). Furthermore, by Lemma 8.12, we may assume that we are given an S-exact closed immersion $u\colon W \cap U \hookrightarrow V$ such that the following diagram commutes and $W \cap U$ becomes a kth infinitesimal neighborhood of $Z \cap U \hookrightarrow V$ by this morphism u.

$$\begin{array}{ccc} Z^{(k)} \cap U & \longrightarrow & W \cap U \\ \downarrow & & \downarrow{\scriptstyle u} \\ U & \longrightarrow & V. \end{array}$$

By Theorem 2.21, we have isomorphisms

$$\begin{aligned} (h|W\cap U)^{\Delta}\mathcal{O}_S &\cong u^{\Delta}g^{\Delta}\mathcal{O}_S \\ &\cong \overline{u}^*\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_{W\cap U}, g^{\Delta}\mathcal{O}_S) \\ &\cong \overline{u}^*\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_{W\cap U}, E(I_g(\Omega^{\log}_{V/S})^n[n])) \end{aligned}$$

and, hence, an isomorphism

$$((h|W\cap U)^{\Delta}\mathcal{O}_S)^{-1} \cong \overline{u}^*\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_{W\cap U}, i_{z*}H^{n-1}_{\mathrm{Zar},z}(I_g(\Omega^{\log}_{V/S})^n)). \tag{8.14}$$

Here $\overline{u}$ denotes the morphism of ringed spaces $(W\cap U, \mathcal{O}_{W\cap U}) \to (V, u_*(\mathcal{O}_{W\cap U}))$.

Since the codimension function d associated to the dualizing complex $Q(h^{\Delta}\mathcal{O}_S)$ is $d(x) = \mathrm{codim}_Z(x) - 1$, we have isomorphisms

$$(h^{\Delta}\mathcal{O}_S)^0 \cong \oplus_{x\in Z\setminus\{z\}}(h^{\Delta}\mathcal{O}_S)^0_x, \tag{8.15}$$

$$(h_*h^{\Delta}\mathcal{O}_S)^{-1} \cong ((h|W\cap U)_*(h|W\cap U)^{\Delta}\mathcal{O}_S)^{-1}. \tag{8.16}$$

By the choice of k and the isomorphisms (8.14) and (8.16), we can regard b as an element of $h_*(h^{\Delta}\mathcal{O}_S)^{-1}$, which we denote by a'.

Claim. For any $x \in Z\setminus\{z\}$, the image of a under the composite

$$R^n f_{X/S*}\mathcal{H}^{n-1}_z(K_{X/S}) \xrightarrow{\delta} R^n f_{X/S*}\mathcal{H}^n_x(K_{X/S}) \xrightarrow{\mathrm{Res}_{f,x}} \mathcal{O}_S \tag{8.17}$$

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coincides with the image of a' under the composite

$$h_*(h^{\Delta}\mathcal{O}_S)^{-1} \xrightarrow{\delta} h_*(h^{\Delta}\mathcal{O}_S)^0_x \xrightarrow{\mathrm{Res}_{h,x}} \mathcal{O}_S. \tag{8.18}$$

Here δ denotes the x-component of the homomorphism $R^n f_{Y/S*}\mathcal{H}^{n-1}_z(K_{X/S}) \to R^n f_{X/S*}\mathcal{H}^n_{Z\setminus\{z\}}(K_{X/S})$ (resp. $h_*(h^{\Delta}\mathcal{O}_S)^{-1} \to h_*(h^{\Delta}\mathcal{O}_S)^0$) and $\mathrm{Res}_{h,x}$ denotes the restriction of the trace map to the x-component.

Since h^{red} is proper, h is also proper. Hence, by the Residue Theorem (Theorem 1.4), the sum of the images of a' under the homomorphisms (8.18) is 0. Hence the proposition follows from this claim.

We will prove the claim in the following. Take an affine open neighborhood U' of x in X and a log. smooth lifting $g'\colon V' \to S$ of $f|U'$ such that V' has an fs log. structure. Let W' be the kth infinitesimal neighborhood of $Z \cap U'$ in V' endowed with the inverse image of the log. structure of V', and let h' be the morphism $W' \to S$. Put $U'' = U \cap U'$. Then, by Lemma 8.12, [K1] Proposition (3.14) (1) and Lemma 8.10 (3), there exist an S-isomorphism $\varepsilon_W\colon W' \xrightarrow{\sim} W \cap U'$ compatible with the embeddings of $Z \cap U'$ and an S-isomorphism $\varepsilon_V\colon V' \cap U'' \xrightarrow{\sim} V \cap U''$ compatible with the embeddings of U'' such that the restriction of ε_W to $W' \cap U''$ coincides with the morphism induced by ε_V.

Let $\varepsilon(b)$ be the image of b under the isomorphism

$$\begin{aligned} H^{n-1}_{\mathrm{Zar},z}(I_g(\Omega^{\log}_{V/S})^n) &\cong H^{n-1}_{\mathrm{Zar},z}(I_{g|V\cap U''}(\Omega^{\log}_{V\cap U''/S})^n) \\ &\xrightarrow{\varepsilon_V^*} H^{n-1}_{\mathrm{Zar},z}(I_{g'|V'\cap U''}(\Omega^{\log}_{V'\cap U''/S})^n) \\ &\cong H^{n-1}_{\mathrm{Zar},z}(I_{g'}(\Omega^{\log}_{V'/S})^n). \end{aligned}$$

Then the image of a under the morphism (8.17) coincides with the image of $\varepsilon(b)$ under the composite

$$H^{n-1}_{\mathrm{Zar},z}(I_{g'}(\Omega^{\log}_{V'/S})^n) \xrightarrow{\delta} H^n_{\mathrm{Zar},x}(I_{g'}(\Omega^{\log}_{V'/S})^n) \xrightarrow{\mathrm{Res}_{g',x}} \mathcal{O}_S.$$

Let $\varepsilon(a')$ be the image of a' under the isomorphism

$$h_*(h^{\Delta}\mathcal{O}_S)^{-1} \cong (h|W \cap U')_*((h|W \cap U')^{\Delta}\mathcal{O}_S)^{-1} \xrightarrow{\varepsilon_W^*} h'_*(h'^{\Delta}\mathcal{O}_S)^{-1}.$$

Then the image of a' under the morphism (8.18) coincides with the image of $\varepsilon(a')$ under the composite $h'_*(h'^{\Delta}\mathcal{O}_S)^{-1} \xrightarrow{\delta} h'_*(h'^{\Delta}\mathcal{O}_S)^0_x \xrightarrow{\mathrm{Res}_{h',x}} \mathcal{O}_S$.

The image of $\varepsilon(a')$ under the isomorphism

$$h'_*(h'^{\Delta}\mathcal{O}_S)^{-1} \cong \mathrm{Hom}_{\mathcal{O}_{V'}}(\mathcal{O}_{W'}, i'_{z*}H^{n-1}_{\mathrm{Zar},z}(I_{g'}(\Omega^{\log}_{V'/S})^n))$$

coincides with the element defined by $\varepsilon(b)$, where i'_z is the morphism $i'_z\colon z \to V'$. Hence the image of $\varepsilon(a')$ under the trace map $\mathrm{Tr}_{u'}\colon h'_*(h'^{\Delta}\mathcal{O}_S)^{-1} \to g'_*(g'^{\Delta}\mathcal{O}_S)^{-1}$,

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where u' is the exact closed immersion $W' \hookrightarrow V'$, coincides with the image of $\varepsilon(b)$ under the inclusion $H^{n-1}_{\mathrm{Zar},z}(I_{g'}(\Omega^{\log}_{V'/S})^n) \hookrightarrow g'_*(g'^{\Delta}\mathcal{O}_S)^{-1}$. Therefore the claim follows from the following commutative diagram (Theorem 1.2 (1) and Theorem 1.4).

$$\begin{tikzcd}
h'_*(h'^{\Delta}\mathcal{O}_S)^{-1} \arrow[r, "\delta"] \arrow[d, "\mathrm{Tr}_{u'}"] & h'_*(h'^{\Delta}\mathcal{O}_S)^0_x \arrow[r, "\mathrm{Res}_{h',x}"] \arrow[d, "\mathrm{Tr}_{u'}"] & \mathcal{O}_S \arrow[d, equal] \\
g'_*(g'^{\Delta}\mathcal{O}_S)^{-1} \arrow[r, "\delta"] & g'_*(g'^{\Delta}\mathcal{O}_S)^0_x \arrow[r, "\mathrm{Res}_{g',x}"] & \mathcal{O}_S.
\end{tikzcd}$$

Let $E(K_{X/S})$ be the Cousin complex

$$\cdots \to 0 \to \mathcal{H}^0_{X^0/X^1}(K_{X/S}) \to \mathcal{H}^1_{X^1/X^2}(K_{X/S})$$
$$\to \cdots \to \mathcal{H}^{n-1}_{X^{n-1}/X^n}(K_{X/S}) \to \mathcal{H}^n_{X^n}(K_{X/S}) \to 0 \to \cdots.$$

Then the morphism $K_{X/S} \to E(K_{X/S})$ gives a resolution of $K_{X/S}$ (Corollary 7.7), and we obtain the spectral sequence

$$E_1^{i,j} = R^j f_{X/S*}(\mathcal{H}^i_{X^i/X^{i+1}}(K_{X/S})) \Rightarrow R^k f_{X/S*} K_{X/S}.$$

By Lemma 8.7 (2), we obtain an exact sequence

$$R^n f_{X/S*}(\mathcal{H}^{n-1}_{X^{n-1}/X^n}(K_{X/S})) \to R^n f_{X/S*}(\mathcal{H}^n_{X^n}(K_{X/S}))$$
$$\to R^{2n} f_{X/S*}(K_{X/S}) \to 0.$$

Hence, by Proposition 8.9, we obtain the following theorem.

THEOREM 8.19 (cf. [B] VII Theorem 1.4.6). *Assume that X is proper over k. Then there exists a unique $\mathcal{O}_S$-linear homomorphism (called trace morphism)* Tr_f: $R^{2n} f_{X/S*}(K_{X/S}) \to \mathcal{O}_S$ *such that, for all closed points x, the following diagrams commute.*

$$\begin{tikzcd}
R^n f_{X/S*}(\mathcal{H}^n_x(K_{X/S})) \arrow[r, "\mathrm{Res}_{f,x}"] \arrow[d] & \mathcal{O}_S \arrow[d, equal] \\
R^{2n} f_{X/S*}(K_{X/S}) \arrow[r, "\mathrm{Tr}_f"] & \mathcal{O}_S.
\end{tikzcd}$$

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