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MINIMAL RELATIONS FOR CERTAIN WREATH PRODUCTS OF GROUPS

D. L. JOHNSON

1. Introduction. Let p be a rational prime, G a non-trivial finite p group, and K the field of p elements, regarded as a trivial G-module according to context; then we define:

$d(G) = \dim_K H^1(G, K)$, the minimal number of generators of G,

$r(G) = \dim_K H^2(G, K)$,

$r'(G) =$ the minimal number of relations required to define G,

where, in the last equation, it is sufficient to take the minimum over those presentations of G with $d(G)$ generators. It is well known (see § 2) that the following inequalities hold:

$$r'(G) \geqq r(G) \geqq d(G).$$

We shall consider only finite p-groups, so that the class of groups with $r = d$ coincides with that consisting of those groups whose Schur multiplicator is trivial. Very little seems to be known (see [**3**, p. 103]) about the extent of the class $\mathscr{G}_p$ of p-groups G for which $r'(G) = r(G)$. We shall be interested in a particular aspect of this problem here, and hope to publish a more comprehensive treatment at some future time.†

In this article, we first prove the elementary fact that $\mathscr{G}_p$ is closed under direct products and then use this to establish the main theorem which asserts that, for odd p, $\mathscr{G}_p$ is closed under standard wreath products, providing that the second factor has trivial multiplicator. The method of proof consists simply of writing down a set of relations for the wreath product and then deducing their minimality by restricting to a "known" subgroup. As an immediate consequence, we observe that for any odd prime p and any natural number n, $\mathscr{G}_p$ contains the Sylow p-subgroup of the symmetric group of degree n.

It seems reasonable to suppose that the application of more powerful techniques might effect the extension of this result to cover any or all of the following cases: $p = 2$, general wreath products, no restriction on the multiplicator of the second factor.

2. Resolutions. Throughout this section, p is a fixed prime, G is a non-trivial finite p-group and $K = \mathrm{GF}(p)$. Our first lemma is a special case of [**2**, Theorem 10].

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†*Added in proof.* See D. L. Johnson and J. W. Wamsley, *Minimal relations for certain finite p-groups* (to appear in Israel J. Math.) and the references therein.

LEMMA 1. *There exists a free resolution F of G over K such that F_0, F_1, and F_2 have KG-ranks* 1, $d(G)$, $r(G)$, *respectively.*

With the notation of the lemma, the exact sequence

$$F_2 \to F_1 \to F_0 \to K \to 0$$

yields: $\dim_K F_2 - \dim_K F_1 + \dim_K F_0 - \dim_K K \geqq 0$, i.e.,

$$(r(G) - d(G) + 1)|G| - 1 \geqq 0,$$

and hence: $(r(G) - d(G) + 1 > 0$, which proves the following result.

COROLLARY. $r(G) \geqq d(G)$.

An immediate consequence of [**5**, Lemma 5.1] is the following result.

LEMMA 2. *If G has a presentation with $\bar{d}$ generators and $\bar{r}$ relations, then there exists a free resolution $\bar{F}$ of G over K with $\bar{F}_0$, $\bar{F}_1$, $\bar{F}_2$ of KG-ranks* 1, $\bar{d}$, $\bar{r}$, *respectively.*

Applying this with $\bar{d} = d(G)$, $\bar{r} = r'(G)$, together with the obvious minimality of the resolution F of Lemma 1, we obtain the following result.

COROLLARY. $r'(G) \geqq r(G)$.

The final result of this section follows from [**2**, Theorems 2 and 3].

LEMMA 3. *Let F, F' be free resolutions of G over K and let f_i, f_i' be the KG-ranks of F_i, F_i', respectively, $i \geqq 0$. If*

$$f_i = f_i' \quad 0 \leqq i \leqq n-1, \qquad f_n \geqq f_n' \quad (\textit{some } n \geqq 0),$$

then there exists a free resolution F'' of G over K, with the KG-rank of F_i'' being equal to

$$f_i', \quad 0 \leqq i \leqq n; \qquad f_{n+1} - (f_n - f_n'), \quad i = n+1; \qquad f_i, \quad i \geqq n+2.$$

This lemma is roughly to the effect that superfluous copies of KG can be cancelled from consecutive pairs of terms in a free resolution of G over K.

3. Direct products. In this section, n is a natural number and G and H are non-trivial finite p-groups, regarded as subgroups of $G \times H$ in the usual way. Throughout the paper, the direct product of n copies of G will be denoted by G^n.

LEMMA 4. (i) $d(G \times H) = d(G) + d(H)$.
(ii) $r(G \times H) = r(G) + r(H) + d(G)d(H)$,
(iii) $r'(G \times H) \leqq r'(G) + r'(H) + d(G)d(H)$,
(iv) $G, H \in \mathcal{G}_p \Rightarrow G \times H \in \mathcal{G}_p$,
(v) $d(G^n) = nd(G)$,
(vi) $r(G^n) = nr(G) + \frac{1}{2}n(n-1)d(G)^2$.

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Proof. (i) and (ii) are well known (see [**4**]).

(iii) If $G = Gp\{D(G); R(G)\}$, $H = Gp\{D(H); R(H)\}$, then clearly,

$$G \times H = Gp\{D(G), D(H); R(G), R(H), [D(G), D(H)]\},$$

where the notation $[\cdot, \cdot]$ denotes a commutator. The number of relations in this presentation is just the right-hand side of (iii).

(iv) If $G, H \in \mathscr{G}_p$, it follows at once from (ii) and (iii) that

$$r'(G \times H) \leqq r(G \times H).$$

The reverse inequality is given by the corollary to Lemma 2.

(v) Induction applied to (i).

(vi) Induction applied to (ii), together with (v).

4. Wreath products. Since the notation $G \wr H$ will signify the standard wreath product of the groups G and H, we have a short exact sequence of groups

$$1 \rightarrow G^h \xrightarrow{i} G \wr H \underset{s}{\rightleftarrows} H \rightarrow 1, \tag{1}$$

where $h = |H|$, i denotes inclusion, and s is a splitting. We regard G and H as subgroups of $G \wr H$ via the embeddings

$$\begin{array}{ll} G \rightarrow G \wr H, & H \rightarrow G \wr H, \\ \alpha \mapsto i(\alpha, 1, \ldots, 1), & \beta \mapsto s(\beta). \end{array}$$

LEMMA 5. *Let $G = Gp\{D(G); R(G)\}$, $H = Gp\{D(H); R(H)\}$ be finite groups and choose a subset X of H minimal with respect to the property that every nonidentity element of H, or its inverse, is in X; then*

$$G \wr H = Gp\{D(G), D(H); R(G), R(H), [D(G), D(G)^X]\}, \tag{2}$$

where the notation $\cdot^X$ denotes conjugation.

Proof. Denoting the group on the right-hand side of (2) by $\bar{G}$, we outline the proof in a number of stages.

(i) H is generated by $D(H)$, G by $D(G)$, and G^h by $D(G)^H$, and so, because of (1), the set $\{D(G), D(H)\}$ generates $G \wr H$.

(ii) The relations defining $\bar{G}$ all being satisfied in $G \wr H$, we have an epimorphism: $\bar{G} \rightarrow G \wr H$.

(iii) The subgroups $\langle D(H)\rangle$, $\langle D(G)\rangle$ of $\bar{G}$ are homomorphic images of H, G, respectively, and so the normal closure of $\langle D(G)\rangle$ in $\bar{G}$ is a homomorphic image of G^h.

(iv) The factor group of $\bar{G}$ by the normal closure of $\langle D(G)\rangle$ being a homomorphic image of $\langle D(H)\rangle$, we have that $|\bar{G}| \leqq |G \wr H|$, and the result now follows from step (ii).

THEOREM. *Let p be an odd prime and $G, H \in \mathscr{G}_p$; then, if H has trivial multiplicator, $G \wr H \in \mathscr{G}_p$.*

Proof. First note that, since $G \times H$ is a homomorphic image of $G \wr H$, the generators for $G \wr H$ given in Lemma 5 are minimal; hence

$$d(G \wr H) = d(G) + d(H).$$

Furthermore,

$$(*) \begin{cases} r(G \wr H) \leqq r'(G \wr H), & \text{corollary to Lemma 2,} \\ \qquad\quad \leqq r'(G) + r'(H) + \tfrac{1}{2}(h-1)d(G)^2, & \text{Lemma 5 and since } p \text{ odd,} \\ \qquad\quad = r(G) + d(H) + \tfrac{1}{2}(h-1)d(G)^2, & \text{by hypothesis,} \end{cases}$$

where $h = |H|$.

In accordance with Lemma 1, choose a free resolution F of $G \wr H$ over K with:

$$f_0 = 1, \qquad f_1 = d(G) + d(H), \qquad f_2 = r(G \wr H).$$

Restricting to G^h, we obtain a free resolution F' with:

$$f_0' = h, \qquad f_1' = h(d(G) + d(H)), \qquad f_2' = hr(G \wr H).$$

Now, by Lemmas 1 and 4, there exists a free resolution F^m of G^h over K with

$$f_0^m = 1, \qquad f_1^m = hd(G), \qquad f_2^m = hr(G) + \tfrac{1}{2}h(h-1)d(G)^2.$$

Now apply Lemma 3 to F' and F^m (with $n = 0$) to obtain a resolution F'' with:

$$f_0'' = 1, \qquad f_1'' = h(d(G) + d(H)) - (h-1), \qquad f_2'' = hr(G \wr H).$$

Applying Lemma 3 to F'', F^m (with $n = 1$), we have a resolution F''' with:

$$f_0''' = 1, \qquad f_1''' = hd(G), \qquad f_2''' = hr(G \wr H) - [hd(H) - (h-1)].$$

Now since the resolution F^m is minimal, we have:

$$(4) \qquad hr(G \wr H) - [hd(H) - (h-1)] \geqq hr(G) + \tfrac{1}{2}h(h-1)d(G)^2.$$

But from above, we have

$$(5) \qquad r(G \wr H) \leqq r(G) + d(H) + \tfrac{1}{2}(h-1)d(G)^2.$$

Combining (4) and (5) and cancelling, we obtain:

$$0 \leqq r(G) + d(H) + \tfrac{1}{2}(h-1)d(G)^2 - r(G \wr H) \leqq 1 - 1/h,$$

and since the middle member is an integer, it must be zero. Hence, the inequalities in (*) become equalities and the theorem is proved.

5. Example. We use the above theorem to prove the following result.

COROLLARY. *For any natural number n and any odd prime p, the Sylow p-subgroup of the symmetric group of degree n is in the class $\mathscr{G}_p$.*

Proof. Let p be an odd prime, and define a collection $\mathscr{G} = \{G_s | s \geqq 0\}$ of groups by:

$$G_0 = \{1\}, \text{ the trivial group, and}$$
$$G_s = G_{s-1} \wr Z_p, \quad s \geqq 1.$$

Then the Sylow p-subgroup of the symmetric group of degree n is a direct product of groups from $\mathscr{G}$; thus, by Lemma 4, it is sufficient to prove that $\mathscr{G} \subseteq \mathscr{G}_p$. This is achieved by induction on s, the cases $s = 0, 1$ being obvious and the inductive step being a simple application of the theorem.

Note. We find that

$$\left.\begin{aligned} d(G_s) &= s \quad \text{and} \\ r(G_s) &= s + \frac{1}{12}(p-1)(s-1)s(2s-1) \end{aligned}\right\}, \qquad s \geqq 0,$$

which agrees with a result of Bogačenko [**1**].

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University of Nottingham,
Nottingham, England