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MULTIPLICATION OPERATORS

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1. Introduction. Let $V(x) \geqq 0$ be given on $\mathbf{R}^n$ and define

$$(1.1) \qquad C_{s,p,q,\lambda}(V) = \sup_{u \in C_0^\infty} \|Vu\|_q / \|(\lambda^2 - \Delta)^{s/2} u\|_p.$$

This constant has played a role in many investigation. For $n = 3$ it was shown in Courant-Hilbert [**7**] p. 446 that

$$C_{1,2,2,0}(|x|^{-1}) \leqq 2.$$

In [**10**], Kato estimates $C_{2,2,2,\lambda}(V)$ in terms of the $L^2 + L^\infty$ norm of V in $\mathbf{R}^3$. Stummel [**22**] showed that $C_{2,2,2,1}(V)$ is bounded by

$$(1.2) \qquad C \sup_y \left(\int_{|x-y|<1} V(x)^2 |x-y|^{\alpha - n} dx \right)^{1/2}$$

in $\mathbf{R}^n$, $n > 2$, provided $\alpha < 4$. Browder [**6**] and Balslev [**3**] showed that $C_{s,q,q,\lambda}(V)$ is bounded by

$$(1.3) \qquad c \sup_y \left(\int_{|x-y|<r} V(x)^q |x-y|^{\alpha - n} dx \right)^{1/q}$$

in $\mathbf{R}^n$, $n > 2$, when s is a positive integer and $\alpha < sq$. This was extended to s any positive real number by Schechter [**18**]. In [**19**] it was shown that $C_{s,2,2,1}(V)$ is bounded by (1.2) if $\alpha = 2s$.

In [**12**] Mazja showed that for $n > 2$

$$C_{1,2,2,0}(V)^2 \leqq \sup_e 4 \int_e V(x)^2 dx / (n-2)\omega \operatorname{cap}(e)$$

and

$$C_{1,2,2,0}(V)^2 \geqq \sup_e \int_e V(x)^2 dx / (n-2)\omega \operatorname{cap}(e)$$

where ω is the surface area of the unit sphere in $\mathbf{R}^n$, cap (c) is the Green capacity of e and the supremum is taken over all compact sets $e \subset \mathbf{R}^n$.

Using the Sobolev inequality, Simon [**21**] and others observe that

$$C_{1,2,2,0}(V) \leqq C \|V\|_n.$$

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For $n = 3$ he showed that

$$C_{1,2,2,\lambda}(V) \leqq \left(\frac{1}{16\pi^2}\int\int V(x)^2 V(y)^2 |x-y|^{-2} e^{-2\lambda|x-y|} dx dy\right)^{1/4}$$

(cf. [**21**] for references to the earlier physics literature). Schechter [**17**] showed that

$$C_{1,2,2,\lambda}(V)^2 = \inf_{\psi>0} \sup_x \psi(x)^{-1} \int V(y)^2 \psi(y) G_{2,\lambda}(x-y) dy \tag{1.4}$$

where $G_{s,\lambda}(x)$ is the Bessel potential of order s (cf., e.g., [**2**]). It is the kernel of the operator

$$(\lambda^2 - \Delta)^{-s/2} f(x) = \int G_{s,\lambda}(x-y) f(y) dy. \tag{1.5}$$

The infimum in (1.4) is taken over all positive functions $\psi(x)$.

Berger-Schechter [**4**] were the first to consider $C_{s,p,q,\lambda}(V)$ for $p \neq q$. They studied the case

$$1 < p \leqq q < \infty, \quad 1/p < s/n + 1/q. \tag{1.6}$$

They showed that $C_{s,p,q,1}(V)$ is bounded by (1.3) provided

$$0 < \alpha/nq < s/n + 1/q - 1/p, \quad 2 < n.$$

Adams [**1**] showed that for $1 < p \leqq q < \infty$, $2 < n$, $C_{s,p,q,0}(V)$ is equivalent to

$$\sup_e \left(\int \left(\int_e |x-y|^{s-n} V(y)^q dy\right)^{p'} dx\right)^{1/p'} \Big/ \left(\int_e V(y)^q dy\right)^{1/q'}. \tag{1.7}$$

The strongest result to date is that of Kerman-Sawyer [**11**] who showed that $C_{s,p,q,\lambda}(V)$ is equivalent to

$$\sup_Q \left(\int \left(\int_Q G_{s,\lambda}(x-y) V(y)^q dy\right)^{p'} dx\right)^{1/p'} \Big/ \left(\int_Q V(y)^q dy\right)^{1/q'}. \tag{1.8}$$

Here the supremum is taken over all dyadic cubes Q. They showed that this in turn is equivalent to

$$\sup_Q \left(\int_Q (M_\Phi \chi_Q V^q)^{p'} dx\right)^{1/p'} \Big/ \left(\int_Q V(y)^q dy\right)^{1/q'}$$

where

$$M_\Phi w(x) = \sup_{\substack{Q \\ x\in Q}} \left[\frac{1}{|Q|}\int_{|y|<|Q|^{1/\lambda}} G_{s,\lambda}(y)dy\right]\int_Q w(y)dy.$$

(Here $|Q|$ denotes the volume of Q.)

Before the work of Kerman and Sawyer, the author **[14, 15]** announced the results of **[16]** in which the norms $M_{\alpha,r,t,\delta}(V)$ depending on four parameters were introduced. It was shown that

$$(1.9)\qquad C_{s,p,q,\lambda}(V) \leqq C\, M_{\alpha,r,t,1/\lambda}(V)$$

holds for suitable choices of the parameters (we could allow $q < p$). In the present paper we strengthen these results considerably. In particular, we streamline the norms $M_{\alpha,r,t,\delta}(V)$ to obtain stronger estimates. Details are given in the next section.

In **[18]** Fefferman and Phong showed that

$$(1.10)\qquad C_{1,2,2,0}(V) \leqq C_q \sup_{\delta,x}\left(\delta^{q-n}\int_{|x-y|<\delta} V(y)^q dy\right)^{1/q}$$

holds for any $q > 2$. They were unaware of the result (1.4) of the author **[17]**. The proof of (1.10) given in **[8]** is rather long and involved. In Section 6 we shall show that it is a simple consequence of (1.4). In fact we shall give a direct easy proof of (1.10) without involving the ideas of **[8]**.

In comparing our results with those of other authors, one should note that they are not as strong as those of Kerman-Sawyer and Adams. The norms on the right hand side of (1.9) are not equivalent to the expression (1.8), and theoretically (1.8) can be bounded by these norms. However the norms $M_{\alpha,r,t,\delta}(V)$ do have advantages over expressions such as (1.7) and (1.8). Firstly, they are norms, while the expressions are not. Secondly, they are more easily computed than the expressions. Moreover, the dependence of (1.1) on λ is more clearly expressed. In addition, inequality (1.9) holds even when $q < p$, while (1.1) is not bounded by expressions (1.7) or (1.8) in this case. In short, there are situations in which the norms $M_{\alpha,r,t,\delta}(V)$ have a distinct advantage over expressions such as (1.7) or (1.8).

2. The norms. For functions $V(x)$ defined on $\mathbf{R}^n$ we define a family of norms $M_{\alpha,r,t,\delta}(V)$ for $\alpha \in \mathbf{R}$, $1 \leqq r \leqq \infty$, $1 \leqq t \leqq \infty$, $0 < \delta \leqq \infty$. For $0 \leqq \alpha \leqq n$ we define

$$(2.1)\qquad M_{\alpha,\delta}V(x) = \sup_{\rho\leqq\delta} \rho^{\alpha-n}\int_{|y-x|<\rho} |V(y)|dy$$

and

$$(2.2)\qquad M_{\alpha,r,t,\delta}(V) = \|[M_{\alpha,\delta}(|V|^r)]^{1/r}\|_t, \quad 0 < \alpha < n,$$

where the norm is that of $L^t = L^t(\mathbf{R}^n)$. In defining the norms for other values of α we used the following abbreviations

$$\begin{aligned}(2.3)\qquad &\|V\|_{r,t,\delta} = \|[M_{n,\delta}(|V|^r)]^{1/r}\|_t, \quad 0<\delta<\infty,\\ &\|V\|_{r,t,\infty} = \|V\|_r\\ &\|V\|_{\infty,t,\delta} = \|\sup_{|y-x|<\delta}|V(y)|\|_t.\end{aligned}$$

We define

$$(2.4)\qquad M_{\alpha,r,t,\delta}(V) = \delta^{(\alpha-n)/r}\|V\|_{r,t,\delta}, \quad \alpha \geqq n$$

$$\begin{aligned}(2.5)\qquad M_{0,r,t,\delta}(V) &= \|[M_{0,\delta}(|V|^r)^{1/r}\|_t, \quad r<t\\ &= \delta^{-n/r}\|V\|_{r,t,\delta}, \quad t \leqq r\end{aligned}$$

$$(2.6)\qquad M_{\alpha,r,t,\delta}(V) = \delta^{(\alpha-n)/r}\|V\|_{r,t,\delta}, \quad \alpha < 0.$$

Some properties of these norms will be given in Section 3. At times we shall use the following abbreviations.

$$M_\alpha V = M_{\alpha,\infty}V, \quad MV = M_0 V$$

$$M_{\alpha,r,t}(V) = M_{\alpha,r,t,1}(V), \quad \|V\|_{r,t} = \|V\|_{r,t,1}.$$

Our first result is

THEOREM 2.1. *Assume that*

$$(2.7)\qquad 1<q<r, \quad 1<p<\infty, \quad 0<s<n, \quad p' \leqq t$$

and

$$(2.8)\qquad \min(0, 1/r-1/t) \leqq \alpha/nr = s/n + 1/q - 1/p - 1/t \leqq s/n.$$

Then there is a constant C independent of u, V and λ such that

$$(2.9)\qquad \|Vu\|_q \leqq CM_{\alpha,r,t,1/\lambda}(V)\|(\lambda^2-\Delta)^{s/2}u\|_p.$$

Notice that we take $r>q$ in Theorem 2.1. In general it is not true if $r=q$. However, there is a case in which we can prove (2.9) when $r=q$. A function $W(x)$ is said to satisfy condition A_∞ if there are constants $\sigma>1$ and C such that

$$(2.10)\qquad \left(\rho^{-n}\int_{B_\rho}|W(x)|^\sigma dx\right)^{1/\sigma} \leqq C\rho^{-n}\int_{B_\rho}|W(x)|dx$$

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holds for all $\rho > 0$ and all balls B_ρ of radius ρ (for another definition, cf. [**13**]). For such functions we can strengthen Theorem 2.1 in the following way.

THEOREM 2.2. *Assume that*

$$1 < p, q < \infty, \quad 0 < s < n, \quad p' \leqq t \tag{2.11}$$

and

$$\min(0, 1/q - 1/t) \leqq \alpha/nq = s/n + 1/q - 1/p - 1/t \leqq s/n. \tag{2.12}$$

Assume further that $W = |V|^q$ satisfies condition A_∞. Then

$$\|Vu\|_q \leqq CM_{\alpha,q,t,1/\lambda}(V)\|(\lambda^2 - \Delta)^{s/2}u\|_p \tag{2.13}$$

where the constant does not depend on u, V or λ, but depends on the constant in (2.10).

3. Properties of the norms. In this section we discuss some of the properties of the norms $M_{\alpha,r,t,\delta}(V)$.

LEMMA 3.1. *If $\alpha, \beta \geqq 0$, $r, t, \lambda, \mu, \sigma, \tau \geqq 1$, and*

$$1/\mu = 1/r + 1/\lambda, \quad 1/\tau = 1/t + 1/\sigma, \quad \alpha/r + \beta/\lambda = s/\mu$$

then

$$\|[M_{s,\delta}(|Vu|^\mu)]^{1/\mu}\|_\tau \leqq \|[M_{\alpha,\delta}(|V|^r)]^{1/r}\|_t \|[M_{\beta,\delta}(|u|^\lambda)]^{1/\lambda}\|_\sigma. \tag{3.1}$$

Proof. We have

$$\left(\rho^{s-n}\int_{|y-x|<\rho} |V(y)u(y)|^\mu\right)^{1/\mu}$$
$$\leqq \left(\rho^{\alpha-n}\int_{|y-x|<\rho} |V(y)|^r dy\right)^{1/r}\left(\rho^{\beta-n}\int_{|y-x|<\rho} |u(y)|^\lambda dy\right)^{1/\lambda}.$$

This implies (3.1).

LEMMA 3.2. *There is a constant C depending only on n such that*

$$\|v\|_{r,t,k\delta} \leqq Ck^n\|v\|_{r,t,\delta} \tag{3.2}$$

holds for all positive integers.

Proof. The ball $|x| < k$ can be covered by $N(k)$ balls of radius one and centers $z_1^{(k)}, \ldots, z_{N(k)}^{(k)}$. Then the ball $|x| < k\delta$ can be covered by $N(k)$ balls of radius δ with centers of $\delta z_1^{(k)}, \ldots, \delta z_{N(k)}^{(k)}$. Thus for $v(x) \geqq 0$

$$\left(\int_{|x-y|<k\delta} v(x)^r dx\right)^{1/r} \leqq \left(\sum_{j=1}^{N(k)} \int_{|x-y-\delta z_j^{(k)}|<\delta} v(x)^r dx\right)^{1/r}$$
$$\leqq \sum_{j=1}^{N(k)} \left(\int_{|x-y-\delta z_j^{(k)}|<\delta} v(x)^r dx\right)^{1/r}.$$

Hence

(3.3) $\quad \|v\|_{r,t,k\delta} \leqq N(k)\|v\|_{r,t,\delta}.$

Now we note that there is a constant C depending only on n such that $N(k) \leqq Ck^n$.

LEMMA 3.3. *If* $1 < \sigma \leqq \tau \leqq \infty$ *and*

(3.4) $\quad 1/\sigma \leqq \alpha/n + 1/\tau$

then

(3.5) $\quad \|M_{\alpha,\delta}v\|_{\tau,t,\delta} \leqq C\delta^{\alpha+(n/\tau-n/\delta)}\|v\|_{\delta,t,\delta}.$

The constant C *does not depend on* δ *or* v.

Proof. First assume that $\tau \neq \infty$. Let $B_\delta(y)$ be the ball of radius δ and center y. Let χ_N denote the characteristic function of the set N. Then for $v(x) \geqq 0$,

$$\left(\int_{B_1(y)} (M_{\alpha,1}v)^\tau dx\right)^{1/\tau} \leqq \left(\int \left[M_\alpha \left(\chi_{B_2(y)}v\right)\right]^\tau dx\right)^{1/\tau}$$
$$\leqq C\left(\int_{B_2(y)} v(x)^\sigma dx\right)^{1/\sigma}.$$

If we replace $v(x)$ by $u(\delta x)$, this becomes

$$\left(\int_{B_\delta(y)} (M_{\alpha,\delta}u)^\tau dx\right)^{1/\tau} \leqq C\delta^{\alpha+(n/\tau-n/\sigma)}\left(\int_{B_{2\delta}(y)} u(x)^\sigma dx\right)^{1/\sigma}$$

where the constant is independent of δ. Taking the L^t norm of both sides, we obtain (3.5) with δ replaced by 2δ on the right hand side. We restore the δ by making use of Lemma 3.2. If $\tau = \infty$ we note that

$$M_{\alpha,\delta}v(x) \leqq \sup_{\rho\leqq\delta} \rho^{\alpha-(n/\sigma)}\left(\int_{|z-x|<\rho} v(z)^\sigma dz\right)^{1/\sigma}$$
$$\leqq \delta^{\alpha-(n/\sigma)}\left(\int_{|z-x|<\delta} v(z)^\sigma dz\right)^{1/\sigma}$$

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since $n \leqq \alpha\sigma$. We proceed as before.

LEMMA 3.4. *If* $1 \leqq r < \sigma \leqq t \leqq \infty$ *and*

$$1/\sigma \leqq \alpha/nr + 1/t \tag{3.6}$$

then

$$\|[M_{\alpha,\delta}(|V|^r)]^{1/r}\|_t \leqq C\delta^{\alpha/r-n/\sigma}\|V\|_{\sigma,t\delta} \tag{3.7}$$

where the constant C does not depend on V or δ.

Proof. First we note that

$$\|v\|_{t,t,\delta} = C\delta^{n/t}\|v\|_t \tag{3.8}$$

where the constant does not depend on v or δ. Thus the left hand side of (3.7) equals

$$\begin{aligned} C\delta^{-n/t}\|M_{\alpha,\delta}(V^r)\|_{t/r,t/r}^{1/r} &\leqq C'\alpha^{\alpha/r+n/t-n/\sigma-n/t}\|V^r\|_{\sigma/r,t/r,\delta}^{1/r} \\ &= C'\delta^{\alpha/r-n/\sigma}\|V\|_{\sigma,t,\delta} \end{aligned}$$

by Lemma 3.3 since $r < t$.

LEMMA 3.5. *If inequality* (3.6) *is strict, we may take* $\sigma = r$ *in* (3.7).

Proof. We merely note that we may take $\sigma = 1$ in (3.5) if inequality (3.4) is strict.

LEMMA 3.6. *If* $t \leqq r$ *and* $0 < \alpha \leqq n$, *then*

$$\left\|\left[M_{\alpha,\delta}\left(|V|^r\right)\right]^{1/r}\right\|_t \leqq C\delta^{(\alpha-n)/r}\|V\|_{r,t,\delta} \tag{3.9}$$

where the constant C is independent of V and δ.

Before proving Lemma 3.6 we introduce another set of norms. Let $\{Q_{\delta i}\}$ be an enumeration of all cubes in $\mathbf{R}^n$ having volumes δ^n and vertices at points with coordinates which are integral multiples of δ. If $x = (x_1, \ldots, x_n)$ is a point in $Q_{\delta i}$, then each x_k satisfies an inequality of the form $j\delta \leqq x_k < (j+1)\delta$ for some integer j depending on k. We define

$$\begin{aligned} \|V\|_{L^{r,t,\delta}} &= \left(\sum_i \left(\int_{Q_{\delta i}} |V(x)|^r dx\right)^{t/r}\right)^{1/t}, \quad 1 \leqq t < \infty \\ &= \sup_i \left(\int_{Q_{\delta i}} |V(x)|^r dx\right)^{1/r}, \quad t = \infty \\ &= \|V\|_r, \quad \delta = \infty. \end{aligned} \tag{3.10}$$

For $r = \infty$ we replace

$$\left(\int_{Q_{\delta i}} |V(x)|^r dx\right)^{1/r}$$

by

$$\sup_{Q_{\delta i}} |V(x)|$$

in these definitions. We shall use

LEMMA 3.7. *The norms* $\|V\|_{r,t,\delta}$ *and* $\delta^{n/t}\|V\|_{L^{r,t,\delta}}$ *are equivalent.*

Proof. Assume first that r, t, δ are not ∞. We may take $V(x) \geqq 0$. Then

$$\begin{aligned}\|M_{n,\delta}(V^r)^{1/r}\|_t &= \left(\sum_i \int_{Q_{\delta i}} M_{n,\delta}(V^r)^{t/r} dx\right)^{1/t} \\ &= \left(\sum_i \int_{Q_{\delta i}} \left(\int_{B_\delta(x)} V(y)^r dy\right)^{t/r} dx\right)^{1/t}.\end{aligned}$$

For $k > 0$ and Q a cube, let kQ denote the cube having the same center as Q and edge length multiplied by k. If x is in $Q_{\delta i}$, then $B_\delta(x)$ is contained in $3Q_{\delta i}$. If $r \leqq t$, then

$$\begin{aligned}\int_{Q_{\delta i}} \left(\int_{B_\delta(x)} V(y)^r dy\right)^{t/r} dx &\leqq \left(\int_{3Q_{\delta i}} \left(\int_{Q_{\delta i}} V(y)^t dx\right)^{r/t} dy\right)^{t/r} \\ &\leqq \delta^n \left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{t/r}.\end{aligned} \tag{3.11}$$

If $t < r$, the left hand side of (3.11) is bounded by

$$\left(\int_{Q_{\delta i}} \int_{B_\delta(x)} V(y)^r dy\, dx\right)^{t/r} \left(\int_{Q_{\delta i}} dx\right)^{1-(t/r)} \leqq \delta^n \left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{t/r}.$$

Thus

$$\|M_{n,\delta}(V^r)^{1/r}\|_t \leqq \delta^{n/t} \left(\sum_i \left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{t/r}\right)^{1/t}. \tag{3.12}$$

The cube $3Q_{\delta i}$ is contained in the 3^n cubes $Q_{\delta j_k}$ adjacent to $Q_{\delta i}$. Thus

$$\left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{1/r} \leqq \sum_{k=1}^{3^n} \left(\int_{Q_{\delta j_k}} V(y)^r dy\right)^{1/r}. \tag{3.13}$$

Hence

$$(3.14)\qquad \|M_{n,\delta}(V^r)^{1/r}\|_t \leqq 3^n\delta^{n/t}\|V\|_{L^{r,t,\delta}}.$$

Conversely, let k be any number $\geqq \sqrt{n}$. Then any ball of radius $k\delta$ and center in $Q_{\delta i}$ contains $Q_{\delta i}$. Let y be a point in $Q_{\delta i}$ such that

$$\delta^n\left(M_{n,k\delta}V^r(y)\right)^{t/r} \leqq \int_{Q_{\delta i}} M_{n,k\delta}(V^r)^{t/r}.$$

Such a point exists since the integrand cannot always be greater than the average value of the integral. Thus

$$\left(\int_{Q_{\delta i}} V(x)^r dx\right)^{t/r} \leqq \delta^{-n}\int_{Q_{\delta i}} M_{n,k\delta}(V^r)^{t/r}.$$

Summing over i and taking the t-th root, we obtain

$$(3.15)\qquad \|V\|_{L^{r,t,\delta}} \leqq \delta^{-n/t}\|V\|_{r,t,k\delta}.$$

We now make use of Lemma 3.2 to reach the desired conclusion. When $t=\infty$ we have

$$\|M_{n,\delta}(V^r)^{1/r}\|_\infty = \sup_x\left(\int_{B_\delta(x)} V(y)^r dy\right)^{1/r} \leqq \sup_i\left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{1/r}$$

since $B_\delta(x)\subset 3Q_{\delta i}$ if $x\in Q_{\delta i}$. If we use (3.13) we obtain the desired estimate. The converse is obvious. The case $r=\infty$ is also simple and is omitted.

Now we give the

Proof of Lemma 3.6. Assume that r, t, δ are not ∞ and that $V(x)\geqq 0$. Note that

$$(3.16)\qquad M_{\alpha,\delta}v(x) \leqq \int_{|y-x|<\delta} |y-x|^{\alpha-n}|v(y)|dy.$$

Hence

$$\begin{aligned}
\|M_{\alpha,\delta}(V^r)^{1/r}\|_t &= \left(\sum_i\int_{Q_{\delta i}} M_{\alpha,\delta}\left(\chi_{3Q_{\delta i}}V^r\right)^{t/r}dx\right)^{1/t}\\
&\leqq \left(\sum\left(\int_{Q_{\delta i}} M_{\alpha,\delta}\left(\chi_{3Q_{\delta i}}V^r\right)dx\right)^{t/r}\left(\int_{Q_{\delta i}} dx\right)^{1-(t/r)}\right)^{1/t}\\
&\leqq \delta^{n/t-n/r}\left(\sum_i\left(\int_{Q_{\delta i}}\int_{3Q_{\delta i}} |y-x|^{\alpha-n}V(y)^r dydx\right)^{t/r}\right)^{1/t}\\
&\leqq C\delta^{n/t+(\alpha-n)/r}\left(\sum_i\left(\int_{3Q_{\delta i}} V(y)^r dy\right)^{t/r}\right)^{1/t}\\
&\leqq C'\delta^{n/t+(\alpha-n)/r}\|V\|_{L^{r,t,\delta}}
\end{aligned}$$

Now we apply Lemma 3.7.

COROLLARY 3.8. *If*

$$(3.17) \quad 1/r < \alpha/nr + 1/t$$

then the norms $M_{\alpha,r,t,\delta}(V)$ *and* $\delta^{(\alpha-n)/r}\|V\|_{r,t,\delta}$ *are equivalent.*

Proof. When $\alpha > 0$ and $t \leqq r$, inequality (3.9) holds by Lemma 3.6. On the other hand, we always have

$$(3.18) \quad \|V\|_{r,t,\delta} = \|M_{n,\delta}(V^r)^{1/r}\|_t \leqq \|M_{\alpha,\delta}(V^r)^{1/r}\|_t.$$

If $\alpha \leqq 0$ and $t \leqq r$, the two norms are equal by definition. When $\alpha > 0$ and $r < t$, inequality (3.9) follows from Lemma 3.5. The case $\alpha \leqq 0$ and $r \leqq t$ is excluded by (3.17).

COROLLARY 3.9 *If* $r \leqq \rho$ *and* $\tau \leqq t$, *then*

$$(3.19) \quad \|V\|_{r,t,\delta} \leqq C\delta^{n/t+n/r-n/\tau-n/\rho}\|V\|_{\rho,\tau,\delta}.$$

Proof. The inequality

$$(3.20) \quad \|V\|_{L^{r,t,\delta}} \leqq \delta^{n/r-n/\rho}\|V\|_{L^{\rho,\tau,\delta}}$$

is a simple consequence of the definition. We apply Lemma 3.7.

For $s > 0$, $\lambda \geqq 0$ we let $G_{s,\lambda}(x)$ be the Bessel potential of order s. It satisfies

$$(3.21) \quad \left(\lambda^2 - \Delta\right)^{-s/2} f = G_{s,\lambda} * f$$

$$(3.22) \quad G_{s,\lambda} * G_{t,\lambda} = G_{s+t,\lambda}$$

$$(3.23) \quad \begin{aligned} G_{s,\lambda} &\sim |x|^{s-n}, \lambda|x| \leqq 1, \quad 0 < s < n \\ &\sim |\lambda x|^{(s-n-1)/2} e^{-\lambda|x|}, \quad \lambda|x| > 1. \end{aligned}$$

(For properties of Bessel potentials, cf. e.g., [**2**]). It follows from (3.23) that

$$(3.24) \quad M_{s,\delta} f \leqq C_s G_{s,1/\delta} * |f|, \quad 0 < s < n.$$

The following was proved in [**20**] employing a method of Muckenhoupt-Wheeden [**13**].

LEMMA 3.10. *If* $1 \leqq t < \infty$ *and* $0 < s < n$, *then there is a constant* C *independent of* v *and* λ *such that*

$$(3.25) \quad \|G_{s,\lambda} * v\|_t \leqq C\|M_{s,1/\lambda} v\|_t.$$

We also have

LEMMA 3.11. *If*

$$1/r = 1/r_0 + 1/r_1, \quad 1/t = 1/t_0 + 1/t_1 \tag{3.26}$$

then

$$\|Vu\|_{r,t,\delta} \leqq \|V\|_{r_0,t_0,\delta}\|u\|_{r_1,t_1,\delta}. \tag{3.27}$$

Proof. By (3.26)

$$\left(\int_{B_\delta(x)} |V(y)u(y)|^r dy\right)^{1/r} \leqq \left(\int_{B_\delta(x)} |V(y)|^{r_0} dy\right)^{1/r_0} \left(\int_{B_\delta(x)} |u(y)|^{r_1} dy\right)^{1/r_1}.$$

If we take the L^t norm of both sides, we obtain (3.27).

LEMMA 3.12. *Suppose*

$$1/\tau \leqq 1/p \leqq 1/\sigma + s/n. \tag{3.28}$$

where $\sigma \neq \infty$ *if* $n = sp$. *Then*

$$\|u\|_{\sigma,\tau,1/\lambda} \leqq C\lambda^{(n/p-n/\sigma-n/\tau)-s}\|(\lambda^2 - \Delta)^{s/2}u\|_p \tag{3.29}$$

where the constant C *does not depend on* n *or* λ.

Proof. By the local Sobolev imbedding theorem there is a constant C independent of y such that

$$\left(\int_{|x-y|<1} |v(x)|^\sigma dx\right)^{1/\sigma} \leqq C\left(\int_{|x-y|<2} |(1-\Delta)^{s/2}v(x)|^p dx\right)^{1/p}. \tag{3.30}$$

Since $p \leqq \tau$, we have

$$\begin{aligned} \|v\|_{\sigma,\tau} &\leqq C\left(\int\left(\int_{|x-y|<2} |(1-\Delta)^{s/2}v(x)|^p dx\right)^{\tau/p} dy\right)^{1/\tau} \\ &\leqq C\left(\int\left(\int_{|y-x|<2} |(1-\Delta)^{s/2}v(x)|^\tau dy\right)^{p/\tau} dx\right)^{1/p} \\ &\leqq C'\left(\int |(1-\Delta)^{s/2}v(x)|^p dx\right)^{1/p}. \end{aligned} \tag{3.31}$$

This proves (3.29) for $\lambda = 1$. For the general case, substitute $v(x) = u(x/\lambda)$ in (3.32). The left hand side becomes

$$\left(\int\left(\int_{|x-y|<1}|u(x/\lambda)|^\sigma dx\right)^{\tau/\sigma} dy\right)^{1/\tau}$$
$$= \left(\int\left(\int_{\lambda|x'-y'|<1}|u(x')|^\sigma \lambda^n dx'\right)^{\tau/\sigma} \lambda^n dy'\right)^{1/\tau}$$
$$= \lambda^{n/\sigma+n/\tau}\|u\|_{\sigma,\tau,1/\lambda}$$

where we made the substitutions $x = \lambda x', y = \lambda y'$. The right hand side becomes

$$C\left(\int|(1-\Delta)^{s/2}u(x/\lambda)|^p dx\right)^{1/p}$$
$$= \left(\int|(1-\lambda^{-2}\Delta)^{s/2}u(x')|^p \lambda^n dx'\right)^{1/p}$$
$$= C\lambda^{n/p-s}\|(\lambda^2-\Delta)^{s/2}u\|_p.$$

This gives (3.29).

4. The proofs. In this section we give the proof of Theorem 2.1. We begin by considering the case $0 \leqq \alpha \leqq n$. Take

$$\mu = 1, \quad \tau = p', \quad \lambda = r', \quad \beta = (sr-\alpha)r'/r$$

in Lemma 3.1. Then we have

$$\|M_{s,\delta}(Vu)\|_{p'} \leqq \|[M_{\alpha,\delta}(|V|^r)]^{1/r}\|_t \|[M_{\beta,\delta}(|u|^{r'})]^{1/r'}\|_\sigma. \tag{4.1}$$

where

$$1/p' = 1/t + 1/\sigma. \tag{4.2}$$

Let

$$K = \|[M_{\alpha,\delta}(|V|^r)]^{1/r}\|_t. \tag{4.3}$$

Since $r' < q' \leqq \sigma$ and

$$1/q' \leqq \beta/nr' + 1/\sigma$$

we see by Lemma 3.4 that

$$\|[M_{\beta,\delta}(|u|^{r'})]^{1/r'}\| + \sigma \leqq C\|u\|_{q'}. \tag{4.4}$$

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Let $\delta = 1/\lambda$. In view of Lemma 3.10, inequalities (4.1) and (4.4) imply

$$\|G_{s,\lambda} * (Vu)\|_{p'} \leqq CK\|u\|_{q'}. \tag{4.5}$$

By duality we have

$$\|VG_{s,\lambda} * v\|_q \leqq CK\|v\|_p. \tag{4.6}$$

If we take $v = (\lambda^2 - \Delta)^{s/2}u$, we obtain (2.9).

Next we note that Lemma 3.11 implies

$$\|Vu\|_q \leqq C\delta^{-n/q}\|V\|_{r,t,\delta}\|u\|_{\sigma,\tau,\delta}. \tag{4.7}$$

provided

$$1/q = 1/r + 1/\sigma = 1/t + 1/\tau. \tag{4.8}$$

On the other hand Lemma 3.12 tells us that (3.29) holds if (3.28) holds and $\sigma \neq \infty$. Inequalities (3.28) and (4.8) are equivalent to

$$1/r + 1/p - s/n \leqq 1/q \leqq 1/t + 1/p \tag{4.9}$$

which is implied by (2.8). Moreover, in our case $\sigma \neq \infty$ since $q \neq r$. Hence (2.8) implies

$$\|Vu\|_q \leqq C\lambda^{(n-\alpha)/r}\|V\|_{r,t,\delta}\|(\lambda^2 - \Delta)^{s/2}u\|_r \tag{4.10}$$

where the constant does not depend on u, V or λ. Here we took $\delta = 1/\lambda$ in (4.7) and noted that

$$n(1/q + 1/p - 1/\sigma - 1/\tau - s/n) = (n-\alpha)/r.$$

In view of (2.4)–(2.6), inequality (4.10) implies (2.9) for the cases $\alpha < 0$ and $\alpha \leqq r$. The remaining cases follow from (4.6).

Proof of Theorem 2.2. Let W satisfy (2.10) for $\sigma > 1$. Let $r = \sigma q$. Then (2.7) and (2.8) are satisfied with α replaced by $\beta = \sigma\alpha$. Thus by Theorem 2.1

$$\|Vu\|_q \leqq CM_{\beta,r,t,1/\lambda}(V)\|(\lambda^2 - \Delta)^{s/2}u\|_p. \tag{4.11}$$

It is easily checked that in general

$$M_{\sigma\alpha,\sigma q,t,\delta}(V) \leqq CM_{\alpha,q,t,\delta}(V) \tag{4.12}$$

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when $W = |V|^q$ satisfies (2.10), the constant in (4.12) depending only on the constant in (2.10). In fact we have

$$M_{\sigma\alpha,\delta}\left(|V|^r\right)^{1/r} = \sup\left(\rho^{\sigma\alpha-n}\int_{B_\rho} W^\sigma\right)^{1/r} = \sup \rho^{\alpha/q}\left(\rho^{-n}\int_{B_\rho} W^\sigma\right)^{1/\sigma q}$$

$$\leqq C \sup \rho^{\alpha/q}\left(\rho^{-n}\int_{B_\rho} W\right)^{1/q} = CM_{\alpha,\delta}\left(|V|^q\right)^{1/q}.$$

Inequality (4.12) certainly holds if $\alpha > n$ or if $\alpha \leqq 0$. If we now combine (4.11) and (4.12) we obtain (2.13).

5. Relationship to other norms. The norms (3.10) give rise to mixed L^p spaces known as "amalgams" (cf. [**9**] for references to the literature). Lemma 3.7 and Corollary 3.8 give conditions under which they are equivalent to the $M_{\alpha,r,t,\delta}(V)$ norms.

Another set of related norms are given by the Lorentz spaces L_{pr} (for definitions and discussions we refer to [**5**]). We note the following

THEOREM 5.1. *If*

(5.1) $\quad 1 \leqq r < \sigma < t \leqq \infty$ *and* $0 < \alpha/nr \leqq 1/\sigma - 1/t < 1$

then

(5.2) $\quad M_{\alpha,r,t}(V) \leqq C\|V\|_{L_{\sigma,t}}.$

Proof. First we note that by Sobolev's inequality

(5.3) $\quad \|G_s * f\|_{t_i} \leqq C_i\|f\|_{\sigma_i}, \quad i = 1, 2, G_s = G_{s,1}$

provide $\sigma_i \neq 1$, $t_i \neq \infty$ and

$$0 < s/n = 1/\sigma_i - 1/t_i < 1.$$

Let Θ satisfy $0 < \Theta < 1$, and set

$$1/\sigma = (1-\Theta)/\sigma_1 + \Theta/\sigma_2, \qquad 1/t = (1-\Theta)t_1 + \Theta t_2.$$

If we apply the real method of interpolation to (5.3), we obtain

(5.4) $\quad \|G_s * f\|_t \leqq C\|f\|_{L_{\sigma t}}$

provided $\sigma \neq 1$, $t \neq \infty$ and

(5.5) $\quad 0 < s/n = 1/\sigma - 1/t < 1.$

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Inequality (5.4) implies by (3.24)

$$(5.6) \qquad \|M_{s,1}f\|_t \leqq \|f\|_{L_{\sigma t}}$$

provided (5.5) holds. Moreover (5.6) holds even when $t = \infty$. To see this note that

$$\int_{B_\rho(x)} |f(y)|dy \leqq \|f\|_{L\sigma,\infty}\|\chi_{B_\rho(x)}\|_{L_{\sigma',1}}.$$

It is easily checked that

$$\|\chi_{B_\rho(x)}\|_{L_{\sigma',1}} = C\rho^{n/\sigma'}.$$

Consequently,

$$(5.7) \qquad M_{n/\sigma,1}f(x) \leqq C\|f\|_{L_{\sigma,\infty}}$$

showing that (5.6) holds when $t = \infty$. Now let α, r, σ, t satisfy (5.1). Then by (5.6)

$$\|M_{\alpha,1}(V^r)\|_{t/r} \leqq C\|V^r\|_{L^{\sigma,r,t/r}} = C\|V\|^r_{L_{\sigma,t}}.$$

The last equality is a simple consequence of the definition of the space $L_{\sigma t}$.

6. A simple proof of the Fefferman-Phong estimate. We give a proof of (1.10) using only an elementary argument. By Holder's inequality

$$M_1\left(V^{1/2}u\right) \leqq M_q\left(V^{q/2}\right)^{1/q} M\left(|u|^{q'}\right)^{1/q'}$$

holds for any $q \geqq 1$. Let $p > 1$ be given and take $q = 2p > 2$. If

$$K_p = \|M_{2p}(V^p)\|_\infty^{1/p},$$

then

$$\begin{aligned}\|M_1(V^{1/2}u)\|_2 &\leqq K_p^{1/2}\|M(|u|^{q'})^{1/q'}\|_2 \\ &= K_p^{1/2}\|M(|u|^{q'})\|_{2/q'}^{1/q'} \leqq CK_p^{1/2}\||u|^{q'}\|_{2/q'}^{1/q'} \\ &= CK_p^{1/2}\|u\|_2\end{aligned}$$

since $q' < 2$. By a theorem of Muckenhoupt and Wheeden [**13**], this implies

$$\|I_1(V^{1/2}u)\|_2 \leqq C'K_p^{1/2}\|u\|_2.$$

This in turn is equivalent to

$$\|V^{1/2}I_1 v\|_2 \leqq C'K_p^{1/2}\|v\|_2.$$

If $u = I_1 v$, then $\|v\| = \|\nabla u\|$. Thus

$$(Vu, u) = \|V^{1/2}I_1 v\|^2 \leqq C'^2 K_p \|v\|^2 = C'^2 K_p \|\nabla u\|^2.$$

This shows that (1.10) holds.

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