Discussion and History of Certain Geometrical Problems of Heraclitus and Apollonius.

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Not the least interesting portions of the wonderful "Mathematical Collections" of Pappus are those which reproduce parts of the νεύσεις, or the two lost books of Apollonius (247–205 B.C.). Pappus (*c.* 300 A.D.) writes[1]:—"A line is said to verge (using Heath's translation[2]) toward a point if, being produced, it reach the point," and among other particular cases of the general problem he gives the following as treated by Apollonius:

Problem A: Between two lines, given in position, to place a straight line given in length and verging toward a given point.

Problem B: If there be given in position a semi-circle and a straight line at right angles to the base, to place between the two lines a straight line of given length and verging to a corner of the semi-circle.

Problem C: Between the side of a given rhombus and its adjacent side produced, to insert a straight line of given length and verging to the opposite corner.

Problem *C* is evidently the particular case of problem *A* when the "corner" is on the bisector of an angle between the given lines. For the purposes of this paper we shall refer to the rhombus case of Problem *A* as the *Problem of Apollonius*. This problem may be thought of in another way, for it is, roughly, equivalent to the following:

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[1] *Collectio*, Ed. by Hultsch, *Liber* VII., p. 670. In what follows all references will be to this edition.

[2] T. L. Heath, *The Works of Archimedes*, Cambridge, 1897, p.c. Chapter V. (p. c.–cxxii.) of this work is entitled "On the Problems known as ΝΕΥΣΕΙΣ." Hereafter when we quote Dr. Heath it will be with reference to this chapter.

Problem D: Given the base of a triangle, the vertical angle and either the internal or external bisector of the angle, to construct the triangle.[3]

The particular case of the *Problem of Apollonius* when the rhombus is a square was called by Pappus the *Problem of Heraclitus*. These are the two problems which are considered in detail in this paper. Among others, Ghetaldi, Girard, Descartes, Van Schooten, Huygens, L'Hospital, Newton, Gergonne, and Steiner have treated them from various points of view. Algebraic solutions lead to interesting discussion of quadratic and biquadratic equations; geometric solutions are of great variety and elegance. A number of Huygens' results are taken from hitherto unpublished manuscripts.[4]

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[3] The case of this problem for the internal bisector with the further (unnecessary) supposition of given perimeter was proposed for solution in Leybourn's *Mathematical Repository*, O.S. No. 12, Dec. 20, 1801, III., 69, and solved geometrically in No. 14, May 1, 1804, III., 188–9. Francoeur, *Cours Complet de Mathématiques Pures*, Paris, 1809, Tome I., p. 309–10 (4th Éd. 1837, I., 356–7), discussed the problem algebraically under the following form:—(*Problem B*): *From the point S at the extremity of the diameter of a circle perpendicular to a chord, E_2F_2, draw a line SBD such that the part BD between the chord and arc be of given length. Cf.* F.G.M., *Exercices de Géométrie*, 4th Ed., Tours and Paris, 1907, p. 170–1, also p. 163, 701. See also Notes 34, 41.

[4] To economise space when indicating references in what follows, I give here a complete list of the portions of Huygens' writings which deal with our problems. Nine pieces are to be found in the Library of the University of Leyden in the manuscripts "Codex Hugeniorum, No. 12." Only two of these have been published, but all the others have been placed at my disposal through the courtesy of the librarian, Dr S. G. de Vries, and Professor D. J. Korteweg of the Huygens' Commission at Amsterdam. The other references are to three letters to Van Schooten and four problems in *De Circuli magnitudine*, etc.

"Travaux Mathématiques Divers de 1650—*Oeuvres Complètes*, La Haye, Tome XI., 1908.

1. "No. IV.," p. 226–7.

2. "No. VIII.." p. 239–42.

"Travaux Mathématiques Divers de 1652 et 1653" (MSS. unpublished at present, although later to appear in *Oeuvres Complètes*, Tome XII.

3. "No. II.," dated 30th Jan. 1652.

4. "No. IV.," dated 1652.

5. "No. VI.," dated 9th Feb. 1652 and 19th Oct. 1653.

6. "No. VII.," dated Feb. 1652.

The discussion is divided into four parts:—I. Related famous problems and solutions by curves other than circles. II. Algebraic solutions. III. Ruler and compass solutions. IV. Concluding remarks.

I.

1. The ancients, as Pappus tells us, tried at first to solve the general problems of the Trisection of an Angle and the Duplication of the Cube (which is equivalent to the finding of two mean proportionals between two given unequal straight lines) by means of the straight line and circle only—means which defined a problem (possible of solution by them) as "plane." Their efforts being futile, they were led to consider many other lines, such as the Conchoide of Nicomedes, Cissoide, and Conic Sections, and by means of some of these curves they resolved the two problems which were accordingly called "solid."

2. It was Nicomedes (*c.* 100 B.C.) who made use of the conchoide to reduce the problem of the Duplication of a Cube[5] to a "solid" νεῦσις. The construction (after Pappus) is as follows:—Let GD, GH (GD<GH, Fig. 1*), the two lines between which we wish to find two mean proportionals, be the sides of a rectangle GHKD. Bisect HG in L. Join KL and produce it to A, the

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7. "No. VIII.," dated 14th Feb. 1652.

8. "No. IX.," dated 17th Feb. 1652.

9. "No. XIII.," dated 16th Aug. 1652.

"Correspondance" *Oeuvres Complètes*, La Haye, Tome I., 1888.

10. "No. 164," dated Oct. 1653, p. 244–5.

11. "No. 166," dated 23rd Oct. 1653, p. 247–51. Modification of *H. 4, 5.*

12. "No. 168," dated 10th Dec. 1653, p. 256–7. Modification of *H. 9.*

De Circuli magnitudine inventa accedunt ejusdem problematum quorundum illustrium constructiones, Amsterdam, 1654. [Another edition, Fensburg, 1668. This work was included in Huygens' *Opera Varia*, 1724, Tome 2.]

13. "Prob. IV.," p. 56–7. Slightly modified from *H. 11.*

14. "Prob. V.," p. 57–8 *idem.*

15. "Prob. VI.," p. 59–61. Slightly modified from *H. 5, 9, 11, 12.* *Cf.* Note 39.

16. "Prob. VII.," p. 62–69. Slight modification of *H. 9, 11, 12.*

References to this list will be by such an abbreviation as "*H. 9*," which indicates the piece of 16th Aug. 1652.

[5] Pappus, III., 58–63; IV., p. 242 *et seq.*

* See folding-out plates.

point of intersection with DG produced. At M, the middle point of DG, erect a perpendicular MB such that BD equals HL. Join AB and through D draw DC parallel to AB and meeting BC parallel to AD in C. If we now consider the conchoide generated with respect to the pole B, the base line DC, and the constant length HL, it will meet AGD in a point F. If FK be drawn to meet GH produced in N, HN and DF are the required mean proportionals. The proof of this construction[6] does not concern us here. But it is to be noted that if E be the point of intersection of BF, CD, we have a parallelogram ABCD in connection with which FE (equal to a given length HL) verges to the point B.

As particular cases of the parallelogram, we may therefore apply the Conchoide of Nicomedes to the solution of the Problems of Apollonius and Heraclitus.

3. For reducing the problem of the Trisection of an Angle to a "solid" νεῦσις, suppose that D (Fig. 2) is any point on the side BD of a given angle DBC (there is evidently no lack of generality in taking this angle acute). Complete the rectangle ABCD and suppose a point F on AD produced determine that EF, which verges to B, is double of BD. Then BF trisects the angle CBD. For, bisect EF in O and join D to O. Then $EO = OF = DO = BD$, and thence $\angle OBD = \angle DOB = 2 \angle DFO = 2 \angle CBE$. Therefore $\angle CBE = \frac{1}{3} \angle CBD$. The question then is, how to determine F.

According to Pappus[7] the solution of this νεῦσις was known to the ancient Greeks. Proclus suggests that Nicomedes used the conchoide[8] for the purpose. Pappus, however, indicates a solution (p. 298) known about the same time, where the hyperbola was employed.

4. Suppose the problem done and the rectangle BF (Fig. 2) completed. Draw CH parallel to EF, and HL, EK to AF. Since $CH = EF = k$, the determination of H would evidently solve the

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[6] For geometrical proof Pappus or Heath may be consulted. Conti gives an analytic proof in ENRIQUES' *Fragen der Elementargeometrie*, II., 207-08, Liepzig, 1907.

[7] P. 272, *et seq.*

[8] ZEUTHEN, *Die Lehre von dem Kegelschnitten im Altertum*, Kopenhagen, 1886, p. 267. The topics of this paper are touched on in various places, p. 267–283.

problem. Now rectangle LF equals rectangle BK, equals rectangle BD. That is $HL \cdot HF = \text{const.}$ Therefore H is on a rectangular hyperbola passing through C, and with asymptotes AF, AB. A circle described with centre C and radius k will therefore determine H. Although Pappus considers the rectangle, the construction applies equally to any parallelogram.

Professor G. Russo's way of stating the proof is elegant[9]. Let CD, AF (Fig. 4) be any two intersecting lines, and B the given point. Through B draw BC parallel to AF and form a parallelogram CEFH. As BEF turns about B the lines CH, FH form homographic involutions (of which the vertices are C and the point at infinity on DC), and therefore their intersection H describes a hyperbola. To resolve the problem one seeks the point of intersection of this hyperbola and the circle with centre C and radius k.

In particular, then, solutions of the problems of Apollonius and Heraclitus may be derived by means of the hyperbola.

5. The hyperbola was introduced for the solution of Problem A in yet another way[10], by Huygens in 1652 (*H.* 7, *8*). The intimate connection of what we are about to give in proof, with the solutions of our more special problems will appear later (§§9, 25). Let B be the point between the lines DA, DC where ABCD is a parallelogram (Fig. 10). Suppose the problem done, and E_2BF_2 a line drawn such that $E_2F_2 = k$. Produce BF_2 and make $BH_2 = k$. Cut off $AT = AD$ and through T draw $TP \parallel AB$. Also draw $H_2U \parallel PT$, $H_2P \parallel TD$.

Since the triangles H_2UF_2, BE_2C are similar, and since $H_2F_2 = BE_2$, we have $H_2U = CE_2$, $UF_2 = BC = AT$. $\therefore UT = AF_2 = H_2P$. Again $F_2A : AB = BC : CE_2$.

$$\therefore \quad H_2P : AB = BC : H_2U, \text{ or } H_2P \cdot H_2U = BA \cdot BC.$$

Hence H_2 is on a hyperbola at its intersection with a circle, with centre at B and radius k. The proof we have given for a general parallelogram is in particular true for a rhombus.

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[9] *Mathesis*, March 1889. IX., 96–7. Analytic treatment is added by J. Neuberg.

[10] Newton found a solution with a hyperbola, in still a third way—*Arithmetica Universalis*, 1707; Second Edition, London, 1722, p. 290–92.

II.

6. As the conchoide of Nicomedes is a curve of the fourth degree, which is also the degree of the equation for determining the intersection of a circle and a hyperbola, an algebraic equation of the same order will certainly, in general, result from the attempted algebraic solution of these problems.

The first discussion along this line was made by Albert Girard in his *Invention Nouvelle en Algèbre* (Amsterdam, 1629),[11] under the heading "Problème d'Inclinaison." He considers two lines at right angles, and forms the square ABCD (Fig. 3) with side of length 4 and $EF = k = \sqrt{153}$. Setting $CE = x$ he finds

$$x^4 = 8x^3 + 121x^2 + 128x - 256.$$

Whence the four values of x are found to be

$$1,\ 16,\ -4\tfrac{1}{2} + \sqrt{4\tfrac{1}{4}},\ -4\tfrac{1}{2} - \sqrt{4\tfrac{1}{4}}.$$

From the figure it is apparent that there are four lines EF, E_1F_1, E_2F_2, E_3F_3, which can satisfy the conditions of the problem; the four values of x therefore correspond respectively to the lengths CE, CE_1, CE_2, CE_3. Such was the reasoning of Girard, the geometrical interpretation of the negative quantities, at his time, being especially notable.

7. The next discussion is by Descartes in Livre III. of his *Géométrie* (1637).[12] In considering the question of when a biquadratic may be broken up into quadratic expressions, he discovers in the Problem of Heraclitus an excellent example of "plane" geometry to illustrate his process. Setting $AB = a$ (Fig. 3), $CE = x$ and $EF = k$, he arrives at the equation

$$x^4 - 2ax^3 + (2a^2 - k^2)x^2 - 2a^3x + a^4 = 0.$$

After transforming this into another with the term involving the third power of the unknown wanting, and expressing the

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[11] This rare little book, so important in the history of Algebra and Spherical Triangles, was reprinted at Leyden in 1884, with an introduction by Bierens de Haan. The pages are unnumbered.

[12] Nouv. Édition, Paris, 1887, p. 70–71; also *Oeuvres*, ed. by Adam et Tannery, VI., 461–463, Paris, 1902.

condition (by means of a cubic in y^2) that the biquadratic is resolvable into quadratic factors, he finds

$$x = \frac{a}{2} + \sqrt{\tfrac{1}{4}a^2 + \tfrac{1}{4}k^2} - \sqrt{\tfrac{1}{4}c^2 - \tfrac{1}{2}a^2 + \tfrac{1}{2}a\sqrt{a^2+k^2}}$$

but omits consideration of the other roots. This omission was observed by Van Schooten in his Commentary of 1649.[13] He not only pointed out that the four solutions

$$x = \frac{a}{2} \underset{=}{\overset{+}{+}} \sqrt{\tfrac{1}{4}a^2 + \tfrac{1}{4}k^2} \overset{\mp}{\pm} \sqrt{\tfrac{1}{4}k^2 - \tfrac{1}{2}a^2 \underset{=}{\overset{+}{+}} \tfrac{1}{2}a\sqrt{a^2+k^2}}$$

correspond to the lengths CE, CE_1, CE_2, CE_3, but also that two of these values will be imaginary when $k < 2AC$. This is evident geometrically, but Van Schooten made it clear by numerical examples.

8. We have seen that the problems of Heraclitus and Apollonius had a great fascination for Christian Huygens. He discovered no less than fourteen solutions during the years 1650–53—that is between his twenty-first and twenty-fourth years. His earlier algebraic solutions (1650, *H. 1, 2*) were prefaced with geometrical proofs of the possibility of solutions by means of intersecting circles and hyperbolas as indicated in §4. From the point H (Fig. 4) thus determined a perpendicular HL was dropped on AC produced. Huygens set $CL = x$, $BD = b$, $AC = c$. In the case of the square he was led to the equations

$$(1) \qquad \begin{cases} 2x^2 + 2cx - k^2 = 0 \\ x = \sqrt{\tfrac{1}{4}c^2 + \tfrac{1}{2}k^2} - \tfrac{1}{2}c, \end{cases}$$

and in the case of the rhombus to

$$(2) \qquad \begin{cases} x^2 + 2px - q^2 = 0 \\ x = \sqrt{p^2+q^2} - p \end{cases}$$

where $$p = \frac{b^2c}{c^2+b^2}, \quad q = \frac{c^2k^2}{c^2+b^2}.$$

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[13] *Geometria à Renato des Cartes anno 1637, gallicè edita; nunc autem cum Notis Florimondi de Beavne . . . in linguam Latinam versa & Commentariis illustra operâ atque studio Francisci à Schooten.*

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The geometrical constructions which may be deduced from these equations we shall consider later (§§20, 22), but the striking simplicity of these quadratic equations with rational coefficients instead of the earlier biquadratic are so noticeable, it may be well to seek the reason. Huygens considers but one solution in each case, and a single line EF corresponding. To this single solution, however, correspond *two* lines EF, E_1F_1 (Fig. 4), constructed from the points of intersection, H, H_1, of circle and hyperbola. It is to be noted that one vertex of this hyperbola is at C[14], and its centre at A. In order therefore that this curve be cut in four real points by the circle it is necessary that $k \geqq 2AC$. The other two points, H_2, H_3 (which may be coincident) thus derived are those which correspond to the root of (2) when the negative sign before the radical is taken. Whence E_2F_2, E_3F_3 (Fig. 4).

9. From the hyperbola which we derived in §5, Huygens deduced another quadratic equation (1652, *H. 7, 8*). Produce BT (Fig. 10) till KT = TB. Draw H_2H_3 perpendicular to TK produced, and make $KJ \parallel TF_2$, to which KR and JQ_2 are perpendicular. Setting $RQ_2 = y$, $AC = c$, $BD = b$, he found

$$(1) \quad y^2 + 2py = k^2 - 4b^2, \text{ where } p = \frac{c^2 + 2b^2}{\sqrt{c^2 + b^2}}; \text{ whence}$$

$$(2) \quad y_1 = -p + \sqrt{p^2 + c^2 - 4b^2}, \quad y_2 = -p - \sqrt{p^2 + c^2 - 4b^2}.$$

Here again, Huygens chose the single solution y_1, and from that deduced the point H_2. But the four points (H_2, H_3, M, H_1) where circle and hyperbola meet, give solutions, as we shall see when we come to represent (2) geometrically (§25).

Finally, Huygens also derived two equations of the fourth degree. Taking as unknown BF (Fig. 4) he derived a complete equation of the fourth degree. On eliminating the second term he found the fourth term also disappeared, whence he saw that if he had chosen $BO = y$ as unknown in the first place, he would have got

$$(3) \quad y^4 - (2l^2 + b^2)y^2 + l^2(l^2 + b^2 - 4a^2) = 0,$$

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[14] This is a distinguishing characteristic of the "plane" case.

where $2l = k$. The same equation was found on taking as unknown BO_2 where O_2 is the centre of E_2F_2.

10. Abbé[15] de Vaumesle of "Basse Normandie" wrestled unsuccessfully with the biquadratic resulting from an attempted solution of the Apollonian Problem.[16] This fact of itself would hardly be worth recording were it not that up to the present few have been more successful, and that his name has some interest from the fact that he was the inspirer of Huygens' papers on epicycloidal curves,[16] and the discoverer of the cardoid and some of its properties as early as 1675.[16]

11. At the time of his death in 1704 L'Hospital had practically completed his *Traité analytique des sections Coniques.*[17] On pages 366–70 of this work our two problems are treated both analytically and geomctrically. For the problem of Heraclitus, considering $AF = x$ (Fig. 3) he found

$$(1) \quad x^4 - 2ax^3 + (2a^2 - k^2)x^2 - 2a^3x + a^4 = 0,$$

which may be written

$$(x^2 + a^2 - ax)^2 - (a^2 + k^2)x^2 = 0, \quad \text{whence}$$

$$(2) \quad x^2 + a^2 - ax + \sqrt{a^2 + k^2}x = 0,$$

$$(3) \quad x^2 + a^2 - ax - \sqrt{a^2 + k^2}x = 0.$$

Concerning the roots of (2) L'Hospital remarks "parce que $\sqrt{a^2 + k^2}$ surpassant a, la disposition des signes me fait connoître qu'elles sont tout deux fausses." Whether this is intended to

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[15] *Cf.* Reference by De la Hire in the preface to his *Traité des Epicycloïdes*, 1694.

[16] Letters, dated Nov. 19 and Dec. 3, 1678, from De Vaumesle to Huygens. They were published by P. J. Uylenbroek in his *Christiani Hugenii aliorumque seculi* XVII., *virorum celebrium exercitationes mathematicae*, etc., La Haye, 1833; Fasc. II., p. 42–51. See also *Oeuvres Complètes* VIII. 115–7; 125–7.

[17] This work was first published at Paris in 1720. The problems are treated as "Ex. 2," Book X.

convey the idea that the roots are negative or imaginary is not quite clear; but they are evidently real for $k \geqq 2\sqrt{2}a$. The roots of (3) are always real, and just as before we could indicate two points to the right of A, and, in general, two points to the left of A, the measures of whose distances from A geometrically indicate the magnitude of the four roots of (1). (*Cf.* § 18). It may be remarked that equation (1) is identical with Pascal's equation (§ 7).

L'Hospital also found a quadratic equation for the solution of the Problem of Apollonius by taking as unknown quantity $BG = z$ (Fig. 5) where G is a point in BD produced such that $BG \cdot GD = GE^2$. This leads to the equation

$$(4) \qquad b^2z^2 - b^3z = a^2k^2.$$

A similar equation would have been reached by taking $GD = z$. The geometric construction for determining G and E will be given later (§ 17).

12. Newton made use of the Heraclitus Problem for exemplifying the rule that in order to reduce an equation to its lowest degree the unknown quantity must be so chosen that it cannot with equal correctness be changed for another.[18]

Let O (Fig. 3) be the middle point of EF, $BO = x$, $EF = k = 2l$, then

$$(1) \quad x^4 = 2(a^2 + l^2)x^2 + 2a^2l^2 - l^4,$$

$$\text{or} \quad x = \pm\sqrt{a^2 + l^2 \pm \sqrt{a^4 + 4a^2l^2}},$$

the particular case of Huygens' Equation (§ 9 (3)) when $b = \sqrt{2}a$. If, however, $BE = x$ we get the more complicated form

$$x^4 + 2kx^3 + (k^2 - 2a^2)x^2 - 2a^2kx - a^2k^2 = 0.$$

But on setting $a^2k^2 + a^4$ on the right side we have on the left a complete square, whence

$$x^2 + kx - a^2 = \pm a\sqrt{a^2 + k^2}.$$

This complication has been introduced through bringing in the point E; the same would result if BF had been chosen (§ 9) for the

[18] *Arithmetica Universalis*, 1707. Second edition, Lond., 1722, Sec. IV., Problem XXIV., p. 148–151. See also the edition of Noel Beaudeux, Paris, 1802, Tome II., p. 141–2, 56–59.

unknown. According to the above rule, the reason of this is that if we imagine F_1E_1 as drawn in the angle ADE_1 ($E_1F_1 = k$), the points E and F will have been interchanged with respect to the point B. The same would further result in choosing as unknown CE (§ 7), or AF (§ 11), DE or DF (§§15, 13). Similar uncertainty would arise if we chose as unknown the lengths of the perpendiculars from O on DF or DC, on AB or BR. If, however, we drop the perpendicular OP on BD produced and set $DP = x$, $BD = b$, we arrive at a still simpler form

$$(2) \quad 2x^2 = bx + l^2.$$

A geometric determination of the point O is given in § 30, and as before (§§ 8, 9) to each root of (2) correspond, in general, two lines, EF, E_1F_1, or E_2F_2, E_3F_3.

We get still another equation of the second degree if we drop FR (Fig. 3) perpendicular to EF to meet BC produced in R. If $CR = x$,[19]

$$(3) \quad (x + a)^2 = k^2 + a^2.$$

The geometrical construction which follows from this was that given by Heraclitus (§ 18).

13. We now come to Kästner's treatment of the problem of Apollonius in 1799[20]. After enthusiastic reference to Ghetaldi's geometric solution (§ 19) he deduces a biquadratic involving a function of the angle $BCD = \alpha$ (Fig. 5). Cantor[21] sets $DF = x$ and derives a similar equation

$$(1) \quad (x^2 - k^2)(x + a)^2 = \{2(a + x)\cos\alpha - a\}ax^2.$$

It seems very evident that Kästner never solved his biquadratic, although his geometric interpretation of the four roots was, in effect, the following. As BF (Fig. 4) rotates about B there is one position in the angle CDF and another in the vertically opposite

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[19] Guisnée, *Application de l'Algebre a la Geometrie*, Paris, 1733, p. 54.

[20] Kästner, *Geschichte der Kunste und Wissenschaften*, etc. Bd. III., Göttingen, 1799.

[21] M. Cantor, *Vorlesungen über die Geschichte der Mathematik*, II., 809–11, 2te Auf., Leipzig, 1899. As Cantor only refers to Ghetaldi in connection with the rhombus problem, it is evident that he did not know its interesting history. G. Ritt (*Problèmes de géométrie et de Trigonometrie*, Neuvième Edition, Paris, 1894, p. 311–16) chose $AF = x$, and was led to a reciprocal equation.

angle where lengths equal to k are cut off. The argument then continues, the other solutions are indicated by a line, rotating about C, from which lengths, k, are cut off by the lines DA, BA produced, or BA and DA produced! This appears to be nothing but a wild guess. That Gelcich[22] should reproduce this interpretation, and Cantor give it his approval,[21] is somewhat astonishing. I find the solutions of (1) to be given by

$$x = -a\sin^2\frac{a}{2} \pm \sqrt{a^2\sin^4\frac{a}{2} + l^2} \pm \sqrt{\left(a\cos^2\frac{a}{2} \pm \sqrt{a^2\sin^4\frac{a}{2} + l^2}\right)^2 - a^2}$$

which are to be interpreted as the results of §§ 6, 7, 8, 9, 12.

14. It is apparent that an infinite number of other equations could be found. For the square (Fig. 3): (1) $DE + DF = x$ leads to an equation of the fourth degree which can be at once solved, since the terms involving x, and x^3 are lacking; (2) $DF - DE = x$ gives an equation of the second degree, and the same is true when the unknown quantity is (3) the perpendicular from D on EF; (4) the radius of the inscribed circle of the triangle DEF; (5) $y = \sin 2 \angle DFE$.

If, however, we chose for unknown quantities (6) $\tan DFE = x$, or (7)[23] $DF : DE = x$, we should be led to complete biquadratic equations *which* (in common with all other complete biquadratics which we have found in connection with the Problem of Heraclitus, §§ 6, 7, 12) *are reciprocal equations.* These can be readily reduced to quadratics. For example (7) leads to $k^2x^2 - a^2(x^2+1)(x-1)^2 = 0$, or $a^2y^2 + 2ay - m^2 = 0$, where $y = x + \frac{1}{x}$.

So also Pascal's equation (§ 7) becomes[24] $z^2 - 2az - k^2 = 0$, where $x + \frac{a^2}{x} = z$. Another statement which can be made concerning

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[22] E. Gelcich, *Zeitschrift für Mathematik und Physik*, XXVII., Suppl., p. 213–4, 1882. "Eine studie ueber die Entdeckung der analytischen Geometrie, mit Berüchsichtigung eines werkes des Marino Ghetaldi . . . aus dem Jahre 1630."

[23] (1), (2), (3), (5), (6), (7) are given by Momenheim et Franck, *Examens et Compositions de Mathématiques*, Paris, 1862, p. 6–28; (4) was given by Transon, *Nouvelles Annales de Mathématiques*, 1847, VI., 458–61.

[24] *Cf.* E. Combette, *Cours d'Algèbre Élémentaire*, Paris, 1882, p. 504–519; also *Cours d'Algèbre*, Paris, 1891, p. 396–411.

these biquadratic equations is that the squares of their roots are in arithmetic progression. For example,[25] if $DF = x$

$$DF^2 + DE^2 = DF^2 + DF_1^2 = k^2 = DF_2^2 + DF_3^2.$$

15. In a similar way we can get rhombus equations.

If we take as unknown quantity: (1)[26] $BR = z$ (where $\angle BFR = \angle CDF$, Fig. 8), corresponding to § 12 (3), we get $z^2 + 2a(1 - \cos\alpha)z - k^2 = 0$; (2)[27] $B'G'' = z$ (where G'' is the point of intersection of B'D' produced, with the circumference of the circumcircle of the triangle D'E'F', Fig. 7), we simply get L'Hospital's equation in slightly different form, $z(z - c) = k^2 \frac{(1 - \cos\alpha)}{2\sin^2\alpha}$; (3)[28] the length of the perpendicular, r, from D' on E'F' we get the quadratic

$$r\left\{r - \frac{2a^2\sin\alpha}{k}(1 - \cos\alpha)\right\} = a^2\sin^2\alpha.$$

Interesting geometrical constructions corresponding to these will be given later (§§ 24, 19, 29). It may be remarked here that the absolute values of r in (3) are the radii of the concentric circles touching the four lines corresponding to EF, E_1F_1, E_2F_2, E_3F_3.

Gergonne considered the general problem of any two lines intersecting at an angle α, point $B'(p, q)$, and $\tan D'F'B' = M$, and found[28]

$$(4)\quad p^2M^4 - 2p(q + p\cos\alpha)M^3 + (p^2 + q^2 - k^2 + 4pq\cos\alpha)M^2 - 2q(p + q\cos\alpha)M + q^2 = 0.$$

If $p = q = -a$ we have the rhombus case, and this equation becomes reciprocal (*Cf.* Ritt, Note 21) and can therefore be solved at once. Further, since the equation of E'F' is $y - q = M(x - p)$, $D'F' = p - \frac{q}{M} = x$, and hence from (4)

$$(5)\quad x^4 + 2(p - q\cos\alpha)x^3 + (p^2 + q^2 - k^2 - 2pq\cos\alpha)x^2 - 2pk^2x - p^2k^2 = 0,$$

of which §13 (1) is a particular case. For $\alpha = \frac{\pi}{2}$ this becomes

$$x^4 + 2px^3 + (p^2 + q^2 - k^2)x^2 - 2pk^2x - p^2k^2 = 0.$$

On comparing this with the general biquadratic (6) $z^4 + rz^2 + sz + t = 0$,

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[25] Momenheim et Franck. *Cf.* Note 23.

[26] Fontés, *Nouvelles Annales des Mathématiques*, 1847, VI., 180–5.

[27] Transon. *Cf.* Note 23.

[28] J. D. Gergonne, *Annales de Mathématiques Pures et Appliquées* (Gergonne), Jan., 1820, X., 204–16.

we find

$$p=\frac{1}{3}\sqrt{-6(r\mp\sqrt{r^2+12t})}\quad q=\frac{1}{4p}\sqrt{-2(4sp+p^4-16t)}$$

$$k=\frac{1}{4p}\sqrt{2(-4sp+p^4-16t)}.$$

Hence when (6) is given p, q, and k may be found, and we have a *geometrical method for graphing the roots of a biquadratic* when there are at least two real. If $p=q$ we have the condition that the construction can be performed with ruler and compass only, as we shall find in the paragraphs which now follow.

III.

16. In what has gone before we have seen that the solution of certain cases of "solid" νεύσεις was reducible to the solution of equations whose roots involved quadratic irrationalities only. It is well known that this is a sufficient condition that such problems are "plane." The sections that immediately follow will display some methods for demonstrating this. No attempt will be made to consider all cases arising and, for the most part, we shall confine ourselves to determining EF (Figs. 5, 6). Those interested in the further geometrical determination of such lines as E_2F_2 (Figs. 3, 4) should also consult the numerous references we have given, but in particular, those to the works of Pappus, Ghetaldi, and Huygens.

17. *Apollonius' Solution of his Problem:* The "8th problem" of the first book of the νεύσεις is enunciated in the following form[29]: *Given a rhombus, AC* (Fig, 5), *with diagonal BD produced to G, if GK be a mean proportional between BG, GD, and if a circle be described with centre G and radius GK cutting DC in E and AD produced in F, BEF shall be a straight line.*

Proof: Join GF_1, GF. Since the angles F_1DK, KDE are equal, the angles F_1DG, GDE are equal. But $GF_1=GE$ and GD is common to the triangles F_1GD, GDE, which are therefore equal, and $F_1D=DE$, $\angle DF_1G=\angle DEG$. But $\angle DF_1G=\angle GFD$. $\therefore$ $\angle DEG=\angle DFG$ and the points D, E, F, G are concyclic. But from the given proportion the triangles BGE, DGE are similar. Therefore $\angle DBE=\angle DEG=\angle DFG$. But $\angle BDE=\angle FDG$ (each

[29] Pappus, VII., p. 778 *et seq.*

half of $\angle$ ADC). Therefore the remaining angles DGF, DEB are equal. But the sum of the angles DGF, DEF is two right angles. Therefore the sum of the angles BED, DEF is two right angles, which was to be proved.

This ends Pappus' reference to this problem, but from this construction the triangles BDF, BEG are similar. Hence $BE : GE(=GK) = BD : DF$. But since AB, DC are parallel $BE : EF = AD : DF$.

$$\therefore \quad GK : EF(=k) = AD : BD$$

a proportion with only one unknown quantity. Hence Apollonius' construction for the solution of his problem may be stated:

Find (1) *PQ* (*GK*) *such that* $PQ : k = AD : BD$, (2) *a point G in BD produced such that* $BG : PQ = PQ : GD$; *then a circle described with centre G and radius PQ determines* $EF = k$ *verging towards the point B.*[30]

18. *Heraclitus' Solution of his Problem*: It seems clear that Pappus must have recognised that the preceding construction was

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[30] Contrary to what Zeuthen states (*l.c.*, p. 281, *cf.* Note 8) Pappus mentions (p. 670) the rhombus as one of the "plane" *νεύσεις* which the Greeks had solved; not only this, but he spoke of "two cases," which evidently cover all possible solutions of the problem of Apollonius. What must have been the solution of Apollonius for the case of the lines E_2F_2, E_3F_3, does not seem to have been explicitly pointed out before. If a point G′ be taken on DB produced such that G′W = GW (where W is the centre of BD), exactly the same construction may be employed on substituting G′ for G. For further comment on this construction see §30. The method employed by the Greeks is, then, now evident.

This same result, attributed to Apollonius, was arrived at by SAMUEL HORSLEY in his restoration of *Apollonii Pergaei Inclinationum libri duo* (Oxford, 1770), and if doubts of the result were still held, Heath's independent research and discussion would certainly dissipate them. FLAUTI in his "Su due libri di Apollonio Pergeo detti delle inclinazioni e sulle diverse restituzioni di essi disquisizione" [1850] (*Memorie di matematica e di Fisica della Società Italiana della Scienze*, Modena, 1852, XXV., P. I., p. 223–36) gives what practically amounts to Apollonius' construction. The same is true of L'Hospital, 1704, and this is his geometrical construction referred to in connection with the algebraic equation § 11 (4). Apparently independent of others D'OMERIQUE discovered this same solution, Prop. XXXII., p. 216 *et seq.* of his *Analysis geometrica sive nova, et vera methodus resolvendi tam problemata quam Arithmeticas Quaestiones* (1698).

immediately applicable to a square[31], and only gave as alternate, Heraclitus' simple and elegant construction (II., p. 782, *et seq.*). Pappus first gave "A Lemma useful for Heraclitus' solution." Let AC (Fig. 3) be the given square and suppose BEF drawn, and let FR be perpendicular to EF and meet BC produced in R; to prove $CR^2 = BC^2 + EF^2$.

Draw FS ‖ DC. Then since BFR is a right angle, the angles EBC, RFS are equal. Therefore the triangles BEC, FRS are equal, and FR = BE. Now $BR^2 = BF^2 + FR^2$, or

$$BC \cdot BR + BR \cdot RC = BF \cdot BE + BF \cdot EF + FR^2.$$

But the angles ECR, RFE being right angles, the points E, C, R, F are concyclic, and therefore $BC \cdot BR = BE \cdot BF$.

$\therefore BR \cdot RC = BF \cdot EF + FR^2 = BF \cdot EF + BE^2$

or $BR \cdot RC = BE \cdot EF + EF^2 + BE^2 = BF \cdot BE + EF^2 = BC \cdot BR + EF^2$.

Take away the common part BC . CR, and $CR^2 = BC^2 + EF^2$. Q.E.D.

Heraclitus' analysis and construction then followed:

Suppose BEF drawn so that EF has a given length k. Since $CR^2 = BC^2 + EF^2 = BC^2 + k^2$, and BC and k are both given, CR is given, and therefore BR is given. Thus the semicircle on BR as diameter is given, and therefore also F, its intersection with the given line ADF; hence BF is given.

To effect the construction, we first find a square equal to the sum of the given square and the square on k. We then produce BC to R so that CR is equal to the side of the square so found. If a semi-circle be now described on BR as diameter, it will pass above D (since CR > CD and therefore $BC \cdot CR > CD^2$), and will therefore meet AD produced in some point F. Join BF meeting CD in E. Then $EF = k$, and the problem is solved.[32]

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[31] This method was indicated by D'Omerique (1698, *cf.* Note 30), and also by "Tyco Oxoniensis" in *The Mathematician*, No. 2, p. 105 (Lond. 1746).

[32] This proof has been given *in extenso* and almost verbatim (*cf.* Heath) in order to illustrate the ancient mode of discussion, which we would now greatly abbreviate. Huygens gave three other geemetrical proofs with practically the same initial construction as Heraclitus. Two of these which differ little in essentials from the above are given here out of chronological order. In both it is supposed that R has been determined such that $CB^2 + k^2 = CR^2$; the semi-circle is described and F determined. *I.* (*H. 11*, 1653). Join ER. Add

Besides giving Heraclitus' construction for the points F, F_1 (Fig. 3), L'Hospital (1704) extended it for the determination of F_2, F_3 (in spite of the solutions being "*fausses*") § 11 (2). Huygens (1653, *H. 11*) gave the same construction.

Produce CB to R′ such that $CR'^2 = CB^2 + k^2$. On BR′ describe a semi-circle. For $k > 2\sqrt{2}a$ this circle will cut DA produced in the two points F_2, F_3. Whence the geometrical construction for the four solutions. These constructions also follow from the algebraic equation, § 12 (3).

19. *Ghetaldi's Solution of the Apollonian Problem.* This solution was first published in *Patritii Ragusini Apollonius Redivivus seu Restituta a Apollonii Pergaei Inclinationum Geometria* (Venice, 1607)[33], and is made to depend on the solution of the "plane" νεῦσις[34], Problem B, of our introductory paragraph.

Suppose (Fig. 5) ABCD be the given rhombus with the side AD produced to F. Join BD. On $E'F' = k$ describe a segment of a circle (Fig. 7) containing an angle equal to the angle EDF.

CE^2 to the equals $CB^2 + k^2$ and CR^2, then $BC^2 + k^2 + CE^2 = CE^2 + CR^2$, or $BE^2 + k^2 = ER^2 = EF^2 + FR^2$. But since FG, BE are between equally distant parallel lines at right angles to one another and equally inclined to these lines, FR = BE, $\therefore EF^2 = k^2$. Q.E.D. *II.* (*H. 16*, 1654). Join ER and draw FS ∥ DC. Since triangles BEC, FSR are similar and the sides BC, FR are equal, the side BE = FR and EC = SR. But $EF^2 + FR^2 = EF^2 + FS^2 + SR^2 = ER^2 = EC^2 + CR^2$. But $EC^2 = SR^2$. $\therefore EF^2 = CR^2 - FS^2 = k^2$ (by constr.).

[33] It also appeared in a posthumous work (which has an important bearing on the history of analytical geometry – Gelcich, note 22) entitled *Marini Ghetaldi Patritii Ragusini Mathematici praestantissimi de Resolutione & Compositione Mathematica libri quinque* (Rome, 1630; another edition, 1640) p. 330–2. The same solution in somewhat abbreviated form was given by Pierre Herigone in Tome IV., p. 912–3 of his *Cursus Mathematicus* (Paris, 1634; another edition, 1644).

[34] Prob. II., case 5 of Ghetaldi's restitution of Apollonius: Suppose SE′G″ the semi-circle (Fig. 7). Produce SE′ to J such that JE′ equals half the given length BD (Fig. 5). With centre J (Fig. 7) and radius JE′ describe a circle which cuts G″J in D″ and X. In the semi-circle place a chord G″D′ = G″D″, and produce it to meet O′E′ produced in B′, then D′B′ is equal to required length DB. For $G''D'' \cdot G''X = G''E'^2 = G''O' \cdot G''S = G''D' \cdot G''B'$. But G″D″ = G″D′. $\therefore$ G″X = G″B′, or D″X = D′B′ = DB. This is also a solution of Problem D for the external bisector D′B′ of the triangle D′E′F′. *Cf.* Note 3. The naturalness of Ghetaldi's proof of this problem of Apollonius is the more striking if Fig. 7 be thought of as applied to Fig. 5, the singly-primed letters correspond to those unprimed, and G″ with G. This comparison also suggests that the algebraic equations of the solutions of Apollonius and Ghetaldi might be the same.

Draw the diameter G″S of this circle perpendicular to E′F′. From G″ draw a line G″D′B′, meeting the circumference of the circle again in D′ and the line F′E′ produced in B′, and such that D′B′ = BD (Problem B). Join D′F′, D′E′, F′S, SE′. If we now make (Fig. 5) DF = D′F′, EF = k as required.

Proof: For the angles E′SF′ and E′D′F′ are supplementary. But $\angle$ B′D′E′ = $\angle$ E′SG″, which is half the supplement of the angle E′D′F′, which is equal to the angle EDF. Therefore $\angle$ B′D′E′ = $\angle$ BDE or $\angle$ B′D′F′ = $\angle$ BDF. Hence the triangles B′D′F′, BDF are equal in all respects, and BF = B′F′; also since EDF = E′D′F′, EF = E′F′ = k.

20. *Huygens' Second*[35] *Solution, 1650, (H. 1)*: Huygens deduced this solution from the equation, § 8 (1). Let ABCD (Fig. 6) be the given square. In BC cut off BM = k, and with centre A and radius AM describe a circle which cuts AB produced in N. With centre on BD describe a semi-circle BMR, and in this semi-circle place the chord RS = BN. BS produced will cut DC in E, and AD produced in F, such that EF = k.

The Editors of Huygens' works state that they fail to find the connection between this construction and earlier discussion. Nevertheless this connection is easily deduced as follows:

Produce the diagonal DB to meet the line through N parallel to CB in L. It may be shown that LB = $2x = \sqrt{\frac{1}{4}b^2 + \frac{1}{2}k^2} - \frac{1}{2}b$, the result Huygens gives in § 8 (1). Erect LT perpendicular to LB, T being determined by the distance TB = k. If we now prove that BD and TB make the same angle with BC or BC produced, we have exactly Huygens' earlier construction (with the hyperbola), except that the diagonal DB has been used instead of AC. [Through an oversight the line LVT of Fig. 6 is drawn incorrectly. It should bisect LB at right angles, at, say, L′.] This follows readily, for on joining RM we have an isosceles right-angled triangle RMB. Then $RS : RB = BN : RB = \frac{BN}{\sqrt{2}} : \frac{RB}{\sqrt{2}} = \frac{LB}{2} : BM = BL' : BT.$

Therefore the angle BRS is equal to the angle TBL′. But $\angle$ BRM = $\angle$ MBR = $\angle$ L′BV, and $\angle$ MRS = $\angle$ MBS. Therefore $\angle$ MBS = $\angle$ TBV.

[35] We have considered the algebraic solutions given in § 8 as the "first" and "fourth" in chronological order. *Cf.* Note 38.

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21. *Huygens' Third Solution*, 1650, (*H. 1*)[36]: The construction for this case is practically the same as that of Heraclitus, but the proof is radically different. Produce AD to R″ (Fig. 3) such that DR″ = k. In BC produced cut off CR = CR″. Bisect BR in M and describe a circle with radius MR. This cuts AR″ in F.

Proof: Draw the diameter GMN perpendicular to BR. BD produced will evidently go through G. Join GF, GE. Since GB is a quadrant, the angle GFE is half a right angle. But the angle GDE is a right angle and a half. Therefore the sum of the angles GDE, GFE is equal to two right angles, and the semi-circle described on EF as diameter will pass through D and G. The angle GFE being half a right angle and the angle FGE a whole right angle, the angle GEF is also half a right angle and GF = GE.

Take a point Z in CR such that CZ = BC. Then ZR = CR″ − CD. Again BM = ½BR, BC = ½BZ. ∴ CM = ½ZR.

∴ 2CM = 2DL = 2GL = CR″ − CD.

But BR = GN = CR″ + CD.

∴ 2GL . GN = CR″² − CD² = DR″².

But GL . GN = GF².

∴ 2GF² = DR″² = EF² (since GF = GE). ∴ EF = DR″ = k.

22. *Huygens' Fifth*[35] *Solution*, 1650, (*H. 2*): This solution of the Apollonian problem is deduced from equation 2, § 8.

Erect a perpendicular CN to BC (Fig. 5) meeting AD produced in N. From N drop a perpendicular NR on AC. It is then not difficult to show that CR = p. Again, in AF cut off AT = k and drop a perpendicular TQ on AC; then AQ = q. Hence to get $x = \sqrt{p^2 + q^2} - p$, prolong NR to Y such that RY = AQ. With centre C and radius CY describe an arc of a circle to cut AC in S. Then SR = x. The point L of our earlier discussion is therefore found by producing AC till CL = SR, etc.

23. *Huygens' Sixth Solution*, 1650, (*H. 2*): Draw DG (Fig. 8) perpendicular to BC and make BK = BG. Then determine L in BC such that KL² = k^2 + BK². Draw LM parallel to BD cutting DC in N, and make BM = k. Then DEMF is the line required.

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[36] This construction and proof were discovered independently by John Turner, and given in *The Mathematician*, No. 2 (Lond. 1746), p. 104–5. *Cf.* Note 31. *N.B.*—GD of this solution is the same length as L′B (or LC) in hyperbola solution, §§20, 8.

There certainly seems, at first glance, to be no connection between this solution and the one just before; nevertheless it was from the fifth that Huygens derived the sixth. He must have reasoned somewhat as follows: suppose the sides CH, CV of the triangle HCV (Fig. 5) to be in coincidence with BE, BC respectively, then H will be at a distance k from B along BF, and HV will be parallel to BD. If then the position of V could be determined in terms of known quantities the problem would be solved.

Draw CU ∥ BD; then from similar triangles

$$\frac{CV}{CL} = \frac{NU}{RC} = \frac{k}{AQ} = \frac{k}{RY} = \lambda, \text{ say.}$$

Hence $k = \lambda . RY$, $NU = \lambda . RC$. $\therefore \sqrt{k^2 + NU^2} = \lambda . CY = \lambda . CS$.
Therefore $\sqrt{k^2 + NU^2} - NU = \lambda . SR = \lambda . CL = CV$.
Interpreting this in terms of Fig. 8, $NU = BG$, $CV = BL$. The relation which must subsist between BG, BL is then
$\sqrt{k^2 + BG^2} - BG = BL$, or $KL^2 = k^2 + KB^2$, with which we started.[37]

24. *Huygens' Ninth*[38] *Solution*, 1652–3 (*H. 5, 15*): This may be regarded as a kind of generalization of Heraclitus' solution. Produce BC (Fig. 8) and make $DR^2 = BD^2 + k^2$. On BR describe a segment of a circle containing an angle equal to the angle BAD. The circumference of this circle will cut AD produced in F.

The proof of this construction is long and complicated, and since it may be found in Sir John Leslie's *Geometrical Analysis and*

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[37] We shall return to this a little later (§ 26). Huygens gave another construction (*H. 2*) practically the same as the above, where instead of drawing LM parallel to BD he made CN = CL.

[38] In order to shorten a somewhat lengthy paper three solutions (beside the proof of the ninth) have been omitted; the seventh for the case of a square, two sides produced; the tenth and fourteenth for the rhombus case, two sides produced. These may be considered as other cases of the third, ninth, and thirteenth respectively. One of the most interesting features of the various published and unpublished solutions of Huygens is that we can trace the manner of their evolution to the refined forms. Only those constructions or proofs radically different have been counted as new solutions. The two algebraic equations, indicated in connection with § 9 (3) are classed as the eighth solution. The eleventh solution is § 9 (1). *Cf.* Note 35.

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Geometry of Curve Lines (Edinburgh, 1821)[39], I will not take up space for it here.

It is to be observed, however, that this construction and the consequent proof were developed, by Huygens, from equation (2) §9. The details of this derivation, though interesting, are too lengthy to reproduce. The same geometrical construction was deduced by Fontés (1847)[26] from equation (1) §15. No reference was made to Huygens.

Just as for the square case, the points R, R′ are determined by the intersection of a circle with centre C and (radius)$^2 = k^2 + BC^2$ (§18), so in the rhombus case the points R, R′ (and hence the segment of a circle on BR′ as chord, containing an angle equal to ADC—Huygens' tenth construction) are determined by the intersection of a circle with centre G (Fig. 8) and (radius)$^2 = k^2 + BG^2$. [Implied in *H. 5, 6.* Compare DIESTERWEG (p. 33), Note 41].

25. *Huygens' Twelfth*[38] *Solution*, 1652 (*H. 8*): This is derived from the hyperbola and algebraic solution of §5 and §9 (1) (2).

Make AT (Fig. 10) equal to AB, join BT and produce it to K where TK = TB. Drop $KR \perp DF_2$, then $DR = p$. Find a point Q_2 such that $DQ_2^2 = p^2 + c^2 - 4b^2$, then $Q_2R = y_1$. Erect $Q_2J \parallel KR$ to meet $KJ \parallel RQ_2$ in J. Through J draw the perpendicular to BK, meeting the circle with centre B and radius k in H_2, H_3. The line H_2B will meet DC, DA produced in E_2, F_2. This is the end of Huygens' solution. The other three cases readily follow, however. The line H_3B gives E_3F_3. Cut off $DQ_1 = DQ_2$, then $RQ_1 = y_2$. Erect $QQ_1 \perp TQ_1$ to meet JK produced in Q, then $QS \perp KB$ will cut the circle with centre B and radius k in M, H_1. The lines BM, BH_1 give EF, E_1F_1. As we have seen (§9), the points H_1, H_2, H_3, M are on a hyperbola. Its asymptotes are TP, TD, and its major axis BK.

26. *Huygens' Thirteenth Solution*, 1652, (*H. 9, 12, 15*): Make $BI^2 = BD^2 + k^2$ (*Fig.* 8). Through I draw MX parallel to BD, where $BM = k$. Then EF is also equal to k.

Proof: Make AT = AD and draw $TX \parallel AB$, meeting DB, IB, IM, produced, in the points W, V, X.

[39] P. 102–4. *Cf.* also pp. 44–51, 101, 443. I am indebted to J. S. Mackay, Esq., LL.D., for this reference. Leslie reproduced *H. 15.* The modifications in *H. 5* are considerable.

Since DA = AT, and TV, AB, DE are parallel, VB = BE and WB = BD. The angle TBD is right, therefore $\angle$ TSI is right. Therefore IS = SX. Now $BI^2 = BM^2 + IM \cdot MX = k^2 + BD^2$ (by constr.). $\therefore$ XM : BD = BD : IM (since BM = k). But BD = WB, therefore MV : VB = BD : MI = BF : FM.

$\therefore$ BM(MV − VB) : VB = BM : FM. But VB = BE. $\therefore$ BE = FM, or BM = k = EF.

Had we considered M′, the second point of intersection of the circle with centre B and radius k with IX, we would have had the line BM′, which determines E_1, F_1.

Were it not recognised that in both the sixth and thirteenth solutions of Huygens we were led to the same line parallel to BD we might proceed as follows to derive the latter from the former.

$$BI^2 = BD^2 + DI^2 + 2BD \cdot DI \cos BDG \text{ (Fig. 8)},$$
$$= BD^2 + BL^2 + 2BL \cdot BG = BD^2 + (BL + BG)^2 - BG^2$$
$$= BD^2 + KL^2 - KB^2$$
or $$BI^2 = BD^2 + k^2.$$

This connection Huygens does not seem to have remarked. On the other hand, his writings point almost conclusively to the view that the thirteenth solution was derived from the twelfth. A comparison of Figs. 8 and 10 tends but to confirm it.

27. *Gergonne's Solution*, 1820[28]: In DC produced determine K′ (Fig. 9) such that DK′ = k, join AK′ and draw CL $\parallel$ AK′. With centre L and radius LC describe a circle to cut AD in M_1, M_2. With centre D and radii DM_1, DM_2 describe two circles. The tangents drawn to these circles from B give EF, E_1F_1, E_2F_2, E_3F_3.

Proof: $DM_1 \cdot DM_2 = DC^2$; if $M_1D = r$, we have therefore $r\left(r - \frac{2a^2}{k}\right) = a^2$, which is simply the particular case of Gergonne's equation when $\alpha = \frac{\pi}{2}$, §15 (3).

28. *Cirodde's Solution*, 1843[40]: In AB produced, make BK″ = k and join K″C. Erect CI perpendicular to CK″ and meeting AB in I. With centre I and radius IB describe a circle to cut CI in M_1', M_2'. If $CM_2' = r$ we have again the equation of the last paragraph, and the circles with centre D and radii CM_1', CM_2'.

[40] P. L. Cirodde, *Leçons de Géométrie Analytique*. Paris, 1843 p. 104–109.

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29. *Fontes' Solution*, 1847[26]: This may be considered as a generalization of Gergonne's. Erect $GDK'(=k$, Fig. 8$)\perp BC$. Join BK′ and draw $DM_1D'M_2 \parallel BK'$. The circle with centre D′ and radius D′G cuts DD′ in M_1, M_2. Then the circles with centre D and radii DM_1, DM_2 are the circles required. For, on substituting in the geometrical relation $DM_1 . DM_2 = DG^2$, we get at once §15 (3). In our figure tangents can be drawn from B to the smaller circle only, which indicates, what is otherwise evident, that only two solutions are possible for the value of k chosen.

30. From Newton's equation, §12 (2), I deduce the following construction: Produce CD (Fig. 6) to X such that $DX = l$. Draw XH perpendicular to XD to meet the bisector of the angle XDA in H. With centre W, the middle point of BD, and radius WH, describe a circle which cuts BD produced in G. The perpendicular bisector of DG meets the circle described with centre D and radius l at two points O, O′. The lines from B through these points give two solutions of the problem.

It is easily verified that the point G, as just determined, is the same as that found before in more than one solution (*Cf.* Note 31 and §21). If then we describe a circle with centre G and radius HD, it will cut CD in E and E_1. Moreover, when the other two solutions of the problem are possible, the geometrical construction is almost the same; determine a point G′ on DB, symmetrical with respect to W, and the circle with G′ as centre and HD as radius will cut DC produced in E_2, E_3.[41]

The same is true for the case of the rhombus; the circle with centre G (Fig. 5) and radius $\frac{ak}{b}$ (§15 (2)), cuts the line CD in E, E_1.

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[41] This is practically the construction of MOMENHEIM ET FRANÇOIS-FRANCK (p. 25, *cf.* Note 23), who consider Heraclitus' problem as a particular case of Problem D (introductory paragraph). See also F. GIANNATTASIO, *Atti della reale Accademia della Scienze, sezione della Societa Reale Barbonica*, Naples, II. Parte I., p. 49–50, 1825, and Note 3. *Cf. Bulletin des Sciences Mathématiques et Physiques*, 13e année, Mai, Juillet, 1908, p. 237–239, 297–300. Problem D, explicitly stated for the internal bisector, was proved geometrically by W. A. DIESTERWEG, *Die Bücher des Apollonius von Perga De Inclinationibus wiederhergestellt von Sam. Horsley nach dem Lateinischen frey bearbeitet*, Berlin, 1823, p. 41–44.

Reflect G about the line AC, and we have G′, the centre of a circle with the same radius determining E_2, E_3, F_2, F_3 (Horsley, 1770, Note 30). The points G, G′ may be found as follows (Diesterweg)[41]: In DB or DB produced cut off $DK' = k$. Through K′ draw K′I ∥ BA and erect $DH' = DI (= DE)$ perpendicular to BD. Then the circle with centre W and radius WH′ cuts BD produced in G and G′; and the circles with centres G and G′ have radii equal to DH′.

IV.

31. Some forty solutions of the Problems of Heraclitus and Apollonius are indicated in the foregoing pages. The geometric solutions offer peculiarly striking illustrations of the value of algebraic analysis in the matter of their derivation. All of Huygens' solutions, with a single possible exception (seventh) were found, more or less directly, in this manner. Geometrical solutions might thus be greatly multiplied, but sufficient has been given to indicate the wealth of possibilities when attacking a single problem in the "Fairy Land of Geometry." For this reason, apart from the Huygens' solutions,[38] four others have been omitted: (1) Diesterweg, 1823 (p. 41–2)[41]; (2) G. Sangro, 1825[42]; (3) Momenheim et Franck, 1862 [p. 11; *cf.* Note 23; deduced from § 14 (1)]; (4) B. Niewenglowski, 1908.[43] Otherwise, I have intended to make the history of the problem as complete as my notes would allow. To this end I add at the close of this paper a *selection* from a list of references to other writers who have treated our problems (in some cases from different points of view), but whose results were not new. In conclusion, the connection between our discussion and the problems of drawing tangents to certain sextic curves may be given, along with some miscellaneous comment.

It was John Bernoulli who first imagined the fourth part of the Astroid[44] (so called four-cusped hypocycloid) as the envelope of a

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[42] "Nuovo Soluzione Geometrica di un Problema del 1° libro della Inclinazione di Apollonio Pergeo." *Atti della reale Accademia della Scienze sezione della Societa Reale Barbonica*, Naples, II., Parte I., 45–49, 1825.

[43] *L'Intermédiaire des Mathématiciens*, Mars, 1908, XV., 71–2; in answer to Question No. 3309 (XIV., 266–7).

[44] *Acta Eruditorum Lipsiensis*, Jan. 1692, p. 33. *Opera Omnia*, 1742, III., 447. Bernoulli found the equation in its expanded form. The form $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ was first given by Hermann in a letter to Leibnitz, dated Nov. 22, 1715. *Cf.* Leibnitz-Gerhardt, *Mathematische Schriften*, IV., 407–408, 1859.

line of constant length sliding between two lines at right angles to one another. From what has gone before we see then that the problem of drawing tangents to an Astroid from any point on the bisectors of the angles between the cuspidal tangents is a "plane" νεῦσις.

A similar result may be stated for the tetracuspide of Bellavitis,[45] which is enveloped by a line of constant length sliding between intersecting lines not at right angles. (*Cf.* F. Joachimsthal, *Nouv. Ann. de Math.*, 1847, VI., 260). The fixed lines in this case are not common cuspidal tangents, although the bisectors of their angles are still lines of symmetry. It is well known that the locus of the centres of the sliding line of fixed length is an ellipse. Moreover, a given point B on a bisector having been chosen we may state, with Steiner,[46] that *the centres of the four resulting segments lie on a circle whose centre is independent of k* (Fig. 4). Hence we have another *solution of the Problem of Apollonius by means of the intersections of a circle and an ellipse*, whose equations are easily found.[47] (They become coincident for the Heraclitus Problem). But as the perpendiculars from G on EF, E_1F_1, and from G′ on E_2F_2, E_3F_3, bisect these segments, we have the further solution: *the points of intersection of the Steiner circle*[48] *with the circles on BG, BG′ as diameters are the middle points of the lines EF, E_1F_1,*

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[45] *Spozione de metodo delle equipollenze*, Modena, 1854, p. 189–191; translation, *Nouvelles Annales de Mathématiques*, 1874 (2), T. XIII., p. 229–230. This curve is parallel to a hypocycloid.

[46] *Crelle's Journal*, Bd. LV., s. 362, 1858, or *Gesammelte Werke*, II., 668–9. This theorem seems to have been rediscovered by J. MIDZUHARA, and published in *Journal of the Mathematico-Physical Society of Tokyo*, IV., 1899 or 1900. *Cf. Mathematical Papers from the Far East.* Ed. by Y. Mikami, Leipzig, 1910, p. 135.

[47] *Cf.* Note by J. NEUBERG, *Mathesis*, Aug. 1889, IX., 183-4. Midzuhara gives as solution (*Cf.* reference Note 46) the intersections of this ellipse and the hyperbola which is the locus of all points O got by varying *k*. That the locus of O is a hyperbola was indicated by C. SMITH in his *Elementary Treatise on Conic Sections*, London, 1892—*Cf.* Ex. 20, p. 85, Ex. 2, p. 162, Ex. 12, p. 163. It is also implied in Newton's discussion (*Cf.* Note 10).

A solution by K. Tsuruta was also given in the *Journal of the Mathematico-Physical Society in Tokyo*, Vol. IV. (*Cf.* Note 46). It is dated May 1889, and is made to depend upon the theorems: (1) the locus of the middle point of a line of fixed length sliding between two fixed lines is an ellipse; (2) "The

E_2F_2, E_3F_3. It does not seem to have been observed before that *the centre S of the Steiner circle is the orthocentre of the triangle ADC* (Fig. 4). Since the power of the point D is l, the radius may be at once deduced. For the square, the orthocentre is D and the radius l. *The circle with centre S and radius SD is cut orthogonally by the circles on EF, E_1F_1, E_2F_2, E_3F_3* (Fig. 4) *as diameters*—since the ends of these diameters are conjugate points with respect to it. Again, *the centres of the circles circumscribing the triangles EDF, E_1DF_1, E_2DF_2, E_3DF_3, are determined by the intersection of a fifth equal circle* $\left(\text{centre } D, \text{ radius } \frac{l}{\sin\alpha}\right)$ *with the perpendicular bisectors of DG, DG'; whence EF, E_1F_1, E_2F_2, E_3F_3.*

It is further easy to see that the problem of drawing tangents to an Astroid from any point on the inscribed circle leads to the exact construction for the trisection of an angle which we have given in § 3—a "solid" νεῦσις, in general.

If BEF always trisected ∠ ABC while α changed from 0° to 180°, E and F would lie on a Trisectrix of Maclaurin (Aubry, *Journal de Math. Spéc*, 1895, p. 83).

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locus of the centre of gravity of the triangle formed by a straight line through a given point with two given straight lines coplanar with the point is a hyperbola." The construction is then given as follows:

"Describe an ellipse (1) taking the constant length $=\frac{2}{3}$ of the given length [k]; then describe a hyperbola (2), the given point being taken as the fixed point.

"Next describe another ellipse similar to and concentric with the ellipse already described, the former bearing to the latter the ratio of similitude of 3 : 2. The minor ellipse will in general intersect with the hyperbola in four points.

"Again draw from the intersection of the given straight lines the four radii vectores through these points, and let them intersect the ellipse in O, O_1, O_2, O_3. Then the straight lines BO, BO_1, BO_2, BO_3 have their segments included between the given straight lines equal to the given length."

Finally, it may be remarked that T. HAYASHI, in *Journal of Physics School in Tokyo* for December 1900, X., 1–4 (*Cf.* Note 46), discusses the general problem and arrives at the equation of Gergonne (§15 (5)). He then considers $p=q$ and $\alpha=\frac{\pi}{2}$ and, apparently, because of guess work arrives at the erroneous conclusion that "our problem is geometrically insoluble."

[48] This term has been chosen simply as a convenient one for this paper.