LOWER BOUNDS FOR THE ESSENTIAL SPECTRUM OF FOURTH-ORDER DIFFERENTIAL OPERATORS

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In this paper, we seek to determine the greatest lower bound of the essential spectrum of self-adjoint singular differential operators of the form

$$lu = \frac{d^2}{dx^2}\left(p(x)\frac{d^2u}{dx^2}\right) + q(x)u, \tag{1}$$

where $0 \leqq x < \infty$. In the event that this bound is $+\infty$, our results will yield criteria for the discreteness of the spectrum of (1).

Such bounds have been established by Friedrichs (**3**) for Sturm-Liouville operators of the form

$$-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

and our techniques will be closely related to those of (**3**). However, instead of studying the solutions of

$$lu = 0 \tag{2}$$

directly, we shall exploit the intimate connection between the infimum of the essential spectrum of (1) and the oscillation properties of (2). It will be shown (in terms to be made precise below) that if λ_0 is the infimum of the essential spectrum of (1), $-\infty \leqq \lambda_0 \leqq \infty$, and if $\lambda_0 < 0$, then (2) is oscillatory, whereas if $\lambda_0 > 0$, then (2) is not oscillatory. Estimates for λ_0 will then follow readily from known oscillation criteria (**5**; **7**) for (2).

It is assumed throughout that $p(x)$ is positive and of class C'' on $[0, \infty)$ and that $q(x)$ is real, continuous, and bounded above on $[0, \infty)$. (The case where $\lim_{x\to\infty} q(x) = +\infty$ will easily be seen to correspond to $\lambda_0 = +\infty$.) By allowing for a translation of the spectrum, it may be assumed without loss of generality that $q(x)$ is negative on $[0, \infty)$, and unless stated otherwise this assumption will be made throughout.

In order to justify speaking of "the essential spectrum of (1)" we consider the Hilbert space $\mathfrak{L}^2(0, \infty)$ and define L to be the symmetric operator obtained by applying l to $C_0^\infty(0, \infty)$. It is well known (**1**) that L has deficiency indices (m, m), where $2 \leqq m \leqq 4$, and from the finiteness of the deficiency indices it follows that all self-adjoint realizations of l will have the same essential spectrum (**1**, p. 108, Theorem 1).

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In determining λ_0, special use will be made of the Friedrichs extension for semi-bounded symmetric operators (**8**), which we review briefly. We shall consider Hilbert spaces $\mathfrak{L}^2(a, b)$, $0 \leqq a < b \leqq \infty$, and symmetric operators $L_{(a,b)}$ obtained by applying l to $C_0^\infty(a, b)$. If there exists a real constant γ such that

$$(L_{(a,b)}u, u) \geqq -\gamma(u, u)$$

for all $u \in C_0^\infty(a, b)$, then the Friedrichs extension $\tilde{L}_{(a,b)}$ is obtained as follows: complete $C_0^\infty(a, b)$ under the norm

$$|||u|||^2 = (lu, u) + (\gamma + 1)(u, u)$$

to construct a Hilbert space $\mathfrak{M}$ with inner product $((u, v)) = (lu, v) + (\gamma + 1)(u, v)$. Define the domain of $\tilde{L}_{(a,b)}$ to consist of those $u \in \mathfrak{L}^2(a, b)$ for which there exists a sequence $v_n \in C_0^\infty(a, b)$ such that

$$\lim_{n\to\infty} ||u - v_n|| = 0 \quad \text{and} \quad \lim_{m,n\to\infty} |||v_m - v_n||| = 0.$$

Then, according to Friedrichs' theorem, there exists a unique self-adjoint extension $\tilde{L}_{(a,b)}$ of $L_{(a,b)}$ whose domain is contained in $\mathfrak{M}$ and for which

$$((u, v)) = (\tilde{L}_{(a,b)}u, v)$$

for all $v \in \mathfrak{M}$ and u in the domain of $\tilde{L}_{(a,b)}$. In case $b < \infty$, $L_{(a,b)}$ is clearly bounded below so that $\tilde{L}_{(a,b)}$ exists. In fact, for this non-singular case, $\tilde{L}_{(a,b)}$ is just the self-adjoint extension obtained by imposing the boundary conditions

$$u(a) = u'(a) = 0 = u(b) = u'(b) \tag{3}$$

on the domain of $L_{(a,b)}^*$. Finally, $\tilde{L}_{(a,b)}$ has the greatest lower bound of all self-adjoint extensions of L, in the sense that if $\tilde{\tilde{L}}_{(a,b)}$ is any other self-adjoint extension of L, then

$$\inf(\tilde{L}_{(a,b)}u, u) \geqq \inf(\tilde{\tilde{L}}_{(a,b)}v, v),$$

where the infima are taken over all normalized u and v in the domains of $\tilde{L}_{(a,b)}$ and $\tilde{\tilde{L}}_{(a,b)}$, respectively.

Definition. We say that (2) is oscillatory if any solution of (2) has an infinite number of zeros on $[0, \infty)$.

THEOREM 1. *If $lu = 0$ is oscillatory and λ_0 is the greatest lower bound of the essential spectrum of* (1), *then either $\lambda_0 \leqq 0$ or else $L_{(0,\infty)}$ is not bounded below.*

Remark. If $L_{(0,\infty)}$ is not bounded below, we shall set $\lambda_0 = -\infty$.

Proof. To simplify notation we write L for $L_{(0,\infty)}$ and $\tilde{L}$ for $\tilde{L}_{(0,\infty)}$ if the latter exists. Suppose, to the contrary, that L is bounded below and that the spectrum of $\tilde{L}$ is discrete below $\lambda' > 0$. Then for any u in the domain of $\tilde{L}$ we have that

$$u = \sum_{i=1}^{n} (u, \phi_i)\phi_i + \int_{\lambda'}^{\infty} dE_\lambda u,$$

where $\phi_1, \ldots, \phi_n$ are normalized eigenfunctions of $\tilde{L}$ corresponding to eigenvalues $-\infty < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n < \lambda'$. To obtain a contradiction, it is sufficient (see **3**) to show the existence of $n + 1$ linearly independent functions $w_1, \ldots, w_{n+1}$ in $C_0^\infty(0, \infty)$ such that

$$(\tilde{L}w_i, w_i) < \lambda'(w_i, w_i), \qquad i = 1, \ldots, n + 1.$$

To construct such a set of $\{w_i\}$ we make use of the assumption $q(x) < 0$ to apply an oscillation theorem of Leighton and Nehari (**7**, Part I) to the equation (2). Since (2) is assumed oscillatory, (**7**, Theorem 3.6) implies the existence of points $0 = x_0 < x_1 < \ldots < x_{n+1}$ such that $lu = 0$ has non-trivial solutions u_i with double zeros at x_{i-1} and x_i. In each of the Hilbert spaces $\mathfrak{H}_i = \mathfrak{L}^2(x_{i-1}, x_i)$ the non-singular operator l generates a Friedrichs extension $\tilde{L}_i$ as described above, where 0 is the first eigenvalue of each $\tilde{L}_i$ corresponding to the eigenfunction u_i, $i = 1, \ldots, n + 1$. Furthermore, to each u_i there exists a sequence $v_{ij} \in C_0^\infty(x_{i-1}, x_i)$ with the properties

$$\text{(3)} \qquad \begin{aligned} &\lim_{j\to\infty} ||v_{ij} - u_i||_i = 0, \qquad i = 1, \ldots, n + 1, \\ &\lim_{j\to\infty} |||v_{ij} - u_i|||_i = 0, \qquad i = 1, \ldots, n + 1, \end{aligned}$$

where the subscript i outside the norm symbol indicates a norm taken in $\mathfrak{H}_i$. By the triangle inequality,

$$|||v_{ij}|||_i \leqq |||v_{ij} - u_i|||_i + |||u_i|||_i.$$

Furthermore,

$$|||u_i|||_i = (\gamma_i + 1)^{1/2}||u_i||_i \leqq (\gamma_i + 1)^{1/2}(||u_i - v_{ij}||_i + ||v_{ij}||_i),$$

where γ_i may be taken non-negative since zero is the greatest lower bound of $\tilde{L}_i$ on $C_0^\infty(x_{i-1}, x_i)$. Combining these inequalities, we have that

$$|||v_{ij}|||_i \leqq |||v_{ij} - u_i|||_i + (\gamma_i + 1)^{1/2}(||u_i - v_{ij}||_i + ||v_{ij}||_i).$$

Using (3) we have that

$$\lim_{j\to\infty} |||v_{ij}|||_i^2 \leqq \lim_{j\to\infty} (\gamma_i + 1)||v_{ij}||_i^2, \qquad i = 1, \ldots, n + 1.$$

From the fact that

$$|||v_{ij}|||_i^2 = (\tilde{L}_i v_{ij}, v_{ij})_i + (\gamma_i + 1)||v_{ij}||_i^2$$

we obtain, finally,

$$\lim_{j\to\infty} (\tilde{L}_i v_{ij}, v_{ij})_i \leqq 0, \qquad i = 1, \ldots, n + 1.$$

Since $\lim_{j\to\infty} ||v_{ij}||_i^2 = ||u_i||_i^2 > 0$, for every $\epsilon > 0$ there exists a j_0 such that

$$(\tilde{L}v_{ij_0}, v_{ij_0})_i \leqq \epsilon||v_{ij_0}||_i^2, \qquad i = 1, \ldots, n + 1.$$

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Choosing $\epsilon = \frac{1}{2}\lambda'$ and defining

$$w_i = v_{ij_0} \quad \text{for } x_{i-1} \leqq x \leqq x_i,$$
$$= 0 \quad \text{for } x < x_{i-1} \text{ or } x > x_i,$$

we have $n + 1$ orthogonal functions $w_1, \ldots, w_{n+1}$ in $C_0^\infty(0, \infty)$ with the property that

$$(\tilde{L}w_i, w_i) < \lambda'(w_i, w_i), \qquad i = 1, \ldots, n + 1.$$

This completes the proof.

THEOREM 2. *If $lu = 0$ is non-oscillatory, then $\lambda_0 \geqq 0$.*

Proof. Here, fundamental use will be made of the fact (see **2**, Chapter XIII, Theorem 7.4) that for any $a > 0$, the essential spectrum of l is the same on $[a, \infty)$ as on $[0, \infty)$. Thus, it will be sufficient to establish the existence of a positive number a such that

$$(lu, u) \geqq 0 \tag{4}$$

for all $u \in C_0^\infty(a, \infty)$.

To that end we note (**7**, Theorem 3.9) that if $lu = 0$ is non-oscillatory, then there exists a positive a such that no solution of

$$lu = \lambda u, \qquad u(a) = u'(a) = 0$$

has a double zero on (a, ∞). According to Theorem 1 of Hinton (**4**), this implies that for any $b > a$

$$\int_a^b [pu''^2 + qu^2]\, dx \geqq 0$$

for all $u \in C_0^\infty(a, b)$. Integrating by parts twice we obtain $(lu, u) \geqq 0$ for all $u \in C_0^\infty(a, b)$. Since b is arbitrary, (4) follows.

Specific estimates for λ_0 can now be obtained from the following oscillation theorem due to Leighton and Nehari (**7**, Theorem 6.2).

THEOREM 3. *If α is an arbitrary real constant, then $lu = 0$ is oscillatory if*

$$\limsup_{x\to\infty} x^{-2-\alpha}p(x) < 1, \qquad \limsup_{x\to\infty} x^{2-\alpha}q(x) < -\frac{(1-\alpha^2)^2}{16},$$

and it is non-oscillatory if

$$\liminf_{x\to\infty} x^{-2-\alpha}p(x) > 1, \qquad \liminf_{x\to\infty} x^{2-\alpha}q(x) > -\frac{(1-\alpha^2)^2}{16}.$$

By Theorems 1 and 2, the above are just criteria for $\lambda_0 \leqq 0$ and $\lambda_0 \geqq 0$, respectively. To simplify the formulation of subsequent results we shall assume that $p(x)$ "behaves like $p_0x^{2+\alpha}$ at $+\infty$" for some positive number p_0.

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With this assumption on the behaviour of $p(x)$ we immediately obtain the following.

COROLLARY 1. *Suppose that* $\lim_{x\to\infty} x^{-2-\alpha}p(x) = p_0 > 0$ *and consider the quantity*

$$Z(x) = q(x) + p_0 \frac{(1-\alpha^2)^2}{16} x^{\alpha-2}.$$

If $\limsup_{x\to\infty} Z(x) < 0$, *then* $\lambda_0 \leqq 0$, *whereas if* $\liminf_{x\to\infty} Z(x) > 0$, *then* $\lambda_0 \geqq 0$.

In order to derive estimates for λ_0, suppose first that $\tilde{q}(x)$ is strictly increasing and that

$$\lim_{x\to\infty} \tilde{q}(x) = q_0, \qquad \lim_{x\to\infty} Z(x) = M.$$

Then $\tilde{q}(x) - M$ is negative and we can apply Corollary 1 to $l_M u = (pu'')'' + (\tilde{q} - M)u$ to conclude that $\lambda_0 - M = 0$. If now $\tilde{q}(x)$ is replaced by a function $q(x)$ which is not necessarily strictly increasing but which still satisfies $\lim_{x\to\infty} q(x) = q_0$, then it follows from the previously cited theorem (**2**, Theorem 7.4) that this change does not affect the essential spectrum of l_M. Thus, we obtain the following result.

COROLLARY 2. *If* $\lim_{x\to\infty} x^{-2-\alpha}p(x) = p_0 > 0$ *and if* $\lim_{x\to\infty} q(x)$ *exists, then* $\lambda_0 = \lim_{x\to\infty} Z(x)$.

If $\lim_{x\to\infty} q(x)$ does not exist, then we define $\underline{q}(x) = \inf_{t\geqq x} q(t)$ and $\bar{q}(x) = \sup_{t\geqq x} q(t)$ and apply Corollary 2 to the operators $\underline{l}u = (pu'')'' + \underline{q}u$ and $\bar{l}u = (pu'')'' + \bar{q}u$ whose essential spectra are bounded below by $\underline{\lambda}_0$ and $\bar{\lambda}_0$, respectively. Since $\underline{q}(x) \leqq q(x) \leqq \bar{q}(x)$, it follows that $\underline{\lambda}_0 \leqq \lambda_0 \leqq \bar{\lambda}_0$, from which we obtain our principal result.

THEOREM 4. *If there exists a real* α *such that* $\lim_{x\to\infty} x^{-2-\alpha}p(x) = p_0 > 0$ *and if* $q(x)$ *is bounded above on* $[0, \infty)$, *then*

$$\liminf_{x\to\infty} Z(x) \leqq \lambda_0 \leqq \limsup_{x\to\infty} Z(x).$$

If $\liminf_{x\to\infty} Z(x) = +\infty$, *then the spectrum of* (1) *is discrete.*

If $\limsup_{x\to\infty} Z(x) = -\infty$, *then* L *is not bounded below.*

Remarks. (1) In case $q(x)$ is not bounded above, the preceding arguments are not valid. However, if $\lim_{x\to\infty} q(x) = \infty$, then $q(x)$ is eventually positive, and for sufficiently large values of x, solutions of $lu = 0$ can have at most one double zero (**7**, Lemma 8.2). Thus, the technique of Theorem 2 readily shows that $\lambda_0 = \infty$ whenever $\lim_{x\to\infty} q(x) = \infty$.

(2) The previously cited theorem (**2**, Theorem 7.4) implies that the essential spectrum of l on $(-\infty, \infty)$ consists of the union of the essential spectra of l on $(-\infty, 0)$ and on $(0, \infty)$. Therefore, the above techniques can also be used to determine λ_0 when l is defined for $-\infty < x < \infty$.

(3) If under the hypotheses of Theorem 4 $\limsup_{x\to\infty} Z(x) = -\infty$, then $lu = \lambda u$ is oscillatory for all real λ.

Finally, we derive a criterion under which λ_0 is the same as the lower bound of $\tilde{L}$.

THEOREM 5. *If* $p(x)$ *and* $q(x)$ *are monotonically decreasing on* $[0, \infty)$, *then*

$$\lambda_0 = \inf(Lu, u),$$

where the infimum is taken over all normalized elements of $C_0^\infty(0, \infty)$.

Proof. If $\tilde{L}$ is not bounded below, the result is trivial. Suppose that $\tilde{L}$ is bounded below and has (not necessarily simple) eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_0$. In order to obtain a contradiction, we first show that λ_1 is a "mobile" eigenvalue with respect to small changes in the non-singular end point $x = 0$. Choose $x_1 > 0$ sufficiently small so that $u_i(x) \neq 0$ in $(0, x_1]$ for all eigenfunctions u_i corresponding to λ_1 and let L_1 denote the symmetric operator obtained by applying l to $C_0^\infty(x_1, \infty)$. Since every element in $C_0^\infty(x_1, \infty)$ can be extended to an element in $C_0^\infty(0, \infty)$ by defining it identically zero in $(0, x_1]$, we have from the classical theory that

$$(Lu_i, u_i) = \lambda_0 \leqq \inf_{\substack{v \in \mathfrak{D}L_1; \\ \|v\|=1}} (L_1 v, v).$$

If we had equality in the above, then it would follow that some $u_i \equiv 0$ in $[0, x_1)$, which contradicts our assumption $u_i(x_1) \neq 0$ (as well as the uniqueness theorem). Therefore, the lower bound of $\tilde{L}_1$ is greater than λ_0.

On the other hand, $\tilde{L}_1$ has the same lower bound as the Friedrichs extension generated by the operator

$$ku \equiv \frac{d^2}{dx^2}\left(p(x - x_1)\frac{d^2u}{dx^2}\right) + q(x - x_1)u$$

in the Hilbert space $\mathfrak{L}^2(0, \infty)$. By hypothesis, $p(x - x_1) \leqq p(x)$ and $q(x - x_1) \leqq q(x)$ for all x in $[0, \infty)$ so that

$$(ku, u) \leqq (lu, u) \quad \text{for all } u \in C_0^\infty(0, \infty).$$

Thus, the lower bound of $\tilde{L}_1$ is no greater than λ_0 and we have the desired contradiction.

REFERENCES

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Vol. II (Ungar, New York, 1963).
2. N. Dunford and J. Schwartz, *Linear operators*, Part II (Interscience, New York, 1963).
3. K. O. Friedrichs, *Criteria for discrete spectra*, Comm. Pure Appl. Math. *3* (1950), 439–449.
4. D. Hinton, *Clamped end boundary conditions for fourth-order self-adjoint differential equations*, Duke Math. J. *34* (1967), 131–138.
5. H. Howard, *Oscillation criteria for fourth-order linear differential equations*, Trans. Amer. Math. Soc. *96* (1960), 296–311.
6. K. Kreith, *An abstract oscillation theorem*, Proc. Amer. Math. Soc. *19* (1968), 17–20.
7. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of fourth order*, Trans. Amer. Math. Soc. *89* (1958), 325–377.
8. F. Riesz and B. Sz.-Nagy, *Functional analysis* (Ungar, New York, 1955).

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