CONGRUENCES ON ORTHODOX SEMIGROUPS II

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Introduction

If ρ is a congruence on a regular semigroup S, then the kernel of ρ is defined to be the set of ρ-classes which contain idempotents of S. Preston [7] has proved that two congruences on a regular semigroup coincide if and only if they have the same kernel: this naturally poses the problem of characterizing the kernel of a congruence on a regular semigroup and reconstructing the congruence from its kernel. In some sense this problem has been resolved by the author in [5]. Using the well-known theorem of M. Teissier (see for example, A. H. Clifford and G. B. Preston [1], Vol. II, Theorem 10.6), it is possible to characterize the kernel of a congruence on a regular semigroup S as a set $\mathscr{A} = \{A_i : i \in I\}$ of subsets of S which satisfy the Teissier-Vagner-Preston conditions:

(C1) each A_i contains an idempotent of S and each idempotent of S is contained in some A_i;

(C2) $xA_iy \cap A_j \neq \Box$ implies $xA_iy \subseteq A_j$ for each $x, y \in S^1$, $i, j \in I$.

Furthermore, the unique congruence $\rho_{\mathscr{A}}$ associated with such a set $\mathscr{A}$ of subsets of S may be defined as follows (see [5]):

(1) $\rho_{\mathscr{A}} = \{(a, b) \in S \times S$: there are inverses a' of a and b' of b such that ba', $aa' \in A_i$ and $b'a, b'b \in A_j$ for some $i, j \in I\}$.

The expression (1) for $\rho_{\mathscr{A}}$ provides us with a simple method of reconstructing the congruence from its kernel; in fact if we denote, for each idempotent e of S, the kernel class A_i to which e belongs by $A(e)$, then to decide if two elements a and b are related under $\rho_{\mathscr{A}}$ we merely have to check (for *any* inverses a' of a and b' of b that we like) that $ba' \in A(aa')$ and $b'a \in A(b'b)$. However, condition (C2) provides us with very little useful information about the kernel and one seeks a more meaningful and more informative characterization than that provided by (C1) and (C2). In [7], Preston was able to provide such a characterization for the kernel of a congruence on an inverse semigroup: an account of his 'kernel normal system' theory for congruences on inverse semigroups may be found in [1], Chapter 7. In [4], the author has provided one method of extending Preston's theory from inverse semigroups to orthodox semigroups (regular semigroups whose idempotents form a subsemigroup). The *regular kernel* of a congruence ρ on an orthodox semigroup was defined to be the set of maximal regular subsemigroups of the elements of the kernel of ρ: it was proved that two congruences on an orthodox semigroup coincide if and only if they have the same regular kernel; the regular kernel of a congruence on an orthodox semigroup S was characterized as a 'regular kernel normal system' of S and a construction of the unique congruence associated with such a regular kernel normal system of S was obtained.

In this paper we provide an alternative approach to the 'kernel normal system problem' for orthodox semigroups. We work directly with the kernel of a congruence on an orthodox semigroup and dispense with the notion of the regular kernel and its associated complications. The main result provides a set of conditions characterizing the kernel of a congruence on an orthodox semigroup, analagous to Preston's conditions characterizing the kernel of a congruence on an inverse semigroup. In addition, the final section is devoted to the determination of the minimum group congruence on an orthodox semigroup. The results obtained in this paper have appeared in slightly different form as a research announcement in the semigroup forum [5].

1. Preliminary results and notation

We shall adhere throughout to the notation and terminology of A. H. Clifford and G. B. Preston [1]. We denote the set of idempotents of a semigroup S by E_S and the set of inverses of an element x of a regular semigroup by $V(x)$. An *orthodox* semigroup is a regular semigroup S for which $E_S E_S \subseteq E_S$.

The following two results, due to N. R. Reilly and H. E. Scheiblich [8], are basic and will be used frequently throughout, normally without comment.

LEMMA 1.1. ([8], Lemma 1.3). *The following three conditions on a regular semigroup S are equivalent*:

(a) *S is orthodox*;

(b) *if $e \in E_S$ then $V(e) \subseteq E_S$*;

(c) *if a and b are arbitrary elements of S and if $a' \in V(a)$ and $b' \in V(b)$ then $a'b' \in V(ba)$*.

LEMMA 1.2. ([8], Lemma 1.4). *If a is an element of the orthodox semigroup S and if $a' \in V(a)$, then $aE_S a' \subseteq E_S$*.

We also have occasion to refer to the following basic lemma due to G. Lallement [3].

LEMMA 1.3. ([3], Proposition 3.5). *If ρ is a congruence on a regular semigroup S and if X is a ρ-class which is a sub-semigroup of S, then X contains an idempotent.*

2. Kernal normal systems of orthodox semigroups

The set $\mathscr{A} = \{A_i : i \in I\}$ of subsets of the orhodox semigroup S is define to be a *kernel normal system* of S if $\mathscr{A}$ satisfies:

(K1) $A_i \cap A_j = \square$ if $i \neq j$;

(K2) each A_i contains an idempotent of S and each idempotent of S is contained in some A_i;

(K3) if $a \in A = \bigcup_{i \in I} A_i$, then $V(a) \subseteq A$;

(K4) for each $a \in S$, $a' \in V(a)$ and $i \in I$, there exists $j \in I$ such that $aA_ia' \subseteq A_j$;

(K5) for each $i, j \in I$ there exists $k \in I$ such that $A_iA_j \subseteq A_k$;

(K6) if a, $ba \in A_i$ and $b'a$, $b'b \in A_j$ for some $b' \in V(b)$, $i, j \in I$, then $b \in A_i$.

LEMMA 2.1 *If ρ is a congruence on an orthodox semigroup S then the kernel $\mathscr{A} = \{A_i : i \in I\}$ of ρ is a kernel normal system of S.*

PROOF. Conditions K1, K2 and K5 are easily verified, and K3 follows from Lemma 2.2 and Lemma 2.3 of [4]. Condition K4 follows easily from Lemma 1.2. To prove K6, suppose that a, $ba \in A_i$ and $b'a$, $b'b \in A_j$, $b' \in V(b)$. Then $(a, ba) \in \rho$, so $(bb'a, bb'ba) \in \rho$, i.e. $(a, bb'a) \in \rho$. Hence $a\rho = (bb'a)\rho = b\rho(b'a)\rho = b\rho(b'b)\rho = (bb'b)\rho = b\rho$ and so $b \in A_i$. This completes the proof of the lemma.

We now introduce some notation. If $\mathscr{A} = \{A_i : i \in I\}$ is a kernel normal system of the orthodox semigroup S then we denote the set A_i which contains the idempotent e by $A(e)$, and by $a \sim b$ we mean that a and b are both elements of the same set A_i. Furthermore we define the relation $\rho_{\mathscr{A}}$ by:

(1) $\rho_{\mathscr{A}} = \{(a, b) \in S \times S$: there are inverses a' of a and b' of b such that $ba' \sim aa'$ and $b'a \sim b'b\}$.

We now state the main theorem of this paper.

THEOREM 2.2. *A set $\mathscr{A} = \{A_i : i \in I\}$ of subsets of an orthodox semigroup S is the kernel of a (necessarily unique) congruence on S if and only if $\mathscr{A}$ is a kernel normal system of S.*

We remark that Lemma 2.1 disposes of the 'only if' part of this theorem. To prove the converse we do not show the equivalence of K1-6 with C1-2 directly, but rather show that if $\mathscr{A}$ is a kernel normal system of S then $\rho_{\mathscr{A}}$ (as defined above) is a congruence on S with kernel $\mathscr{A}$. We break the proof into a sequence of lemmas.

LEMMA 2.3. *If $\mathscr{A}$ is a kernel normal system of the orthodox semigroup S and if $a, b \in S$, then a necessary and sufficient condition that $(a, b) \in \rho_{\mathscr{A}}$ is that $ba' \sim aa'$, $b'a \sim b'b$, $a'b \sim a'a$ and $ab' \sim bb'$ for any $a' \in V(a)$, $b' \in V(b)$.*

PROOF. The sufficiency is obvious. To prove the necessity, let $(a, b) \in \rho_{\mathscr{A}}$ and let $a' \in V(a)$ and $b' \in V(b)$ be such that

(2) $$ba' \sim aa' \text{ and } b'a \sim b'b.$$

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Then for any $a^* \in V(a)$ and $b^* \in V(b)$ we have $ab^* \in A_i$ for some $i \in I$ by K3, and so

$$\begin{aligned} ab^* &= (aa')(ab^*) \sim (ba')(ab^*) && \text{(by K5)} \\ &= b(b'b)(a'a)b^* \sim b(b'a)(a'a)b^* && \text{(by K4 and K5)} \\ &= b(b'a)b^* \sim b(b'b)b^* = bb^* && \text{(by K4),} \end{aligned}$$

and similarly $a^*b \sim a^*a$. A similar argument shows that if $a^*b \sim a^*a$ and $ab^* \sim bb^*$, then (2) is satisfied for any $a' \in V(a)$ and $b' \in V(b)$, and this establishes the truth of the lemma.

LEMMA 2.4. *If* $\mathscr{A}$ *is a kernel normal system of the orthodox semigroup* S *then the relation* $\rho_{\mathscr{A}}$ *defined by* (1) *is a congruence on* S.

PROOF. $\rho_{\mathscr{A}}$ is clearly a reflexive relation and the symmetry follows immediately from lemma 2.3. Suppose now that $(a, b) \in \rho_{\mathscr{A}}$ and $(b, c) \in \rho_{\mathscr{A}}$ and let $a' \in V(a)$, $b' \in V(b)$ and $c' \in V(c)$. We need only prove that $ca' \sim aa'$ and $c'a \sim c'c$. We first prove the preliminary results:

(3) $$aa' \sim ac'ca' \text{ and } cc' \sim ca'ac'.$$

Now

$$\begin{aligned} a(c'c)a' &= a(a'a)(c'c)a' \sim a(a'b)(c'c)a' && \text{(by K4 and K5)} \\ &= a[(a'b)(b'b)(c'c)]a' \sim a[(a'b)(b'c)(c'c)]a' \\ &= a[(a'b)(b'c)]a' \sim a[(a'b)(b'b)]a' \\ &= a(a'b)a' \sim a(a'a)a' = aa', \end{aligned}$$

and by symmetry the result $cc' \sim ca'ac'$ follows.

Now set

$$\begin{aligned} x = bc'cb'aa' \sim cc'cb'aa' = cb'aa' \sim bb'aa' \sim bb'ba' \\ = ba' \sim aa', \end{aligned}$$

and set $y = ca'$, $y' = ac'$. Then

$$\begin{aligned} yx &= (ca'bc')(cb')(aa') \sim (ca'ac')(cb')(aa') && \text{(by K4 and K5)} \\ &\sim (cc')(cb')(aa') \text{ (by (3) and K5)} \\ &= (cb')(aa') \sim (bb')(ba') = ba' \sim aa' \sim x, \end{aligned}$$

and

$$\begin{aligned} y'x &= a[(c'b)(c'c)(b'a)]a' \sim a[(c'c)(c'c)(b'a)]a' && \text{(by K4 and K5)} \\ &= a[(c'c)(b'a)]a' \sim a[(c'b)(b'b)]a' = a(c'b)a' \\ &\sim a(c'c)a' \sim aa', \end{aligned}$$

and also $y'y = ac'ca' \sim aa'$.

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Hence $x \sim yx \sim y'x \sim y'y \sim aa'$, and so $y \sim aa'$ by K6, i.e. $ca' \sim aa'$. By symmetry, $ac' \sim cc'$. Hence $c'a \in A_j$, some $j \in I$ by K3. It follows that

$$\begin{aligned} c'a &= (c'a)(a'a) \sim (c'a)(a'b) && \text{(by K5)} \\ &= (c'a)(a'b)(b'b) \sim c'[(aa')(bb')]c && \text{(by K5)} \\ &\sim c'[(aa')(ab')]c = c'(ab')c && \text{(by K4)} \\ &\sim (c'b)(b'c) \sim (c'b)(b'b) = c'b \sim c'c. \end{aligned}$$

Hence $\rho_{\mathscr{A}}$ is transitive.

To prove that $\rho_{\mathscr{A}}$ is left compatible we proceed as follows. Let $(a, b) \in \rho_{\mathscr{A}}$ and let c be any element of S, $a' \in V(a)$, $b' \in V(b)$ and $c' \in V(c)$. It suffices to prove that $cba'c' \sim caa'c'$ and $b'c'cb \sim b'c'ca$, since $a'c' \in V(ca)$ and $b'c' \in V(cb)$. That $cba'c' \sim caa'c'$ is trivial to prove: it follows instantly from K4 because $ba' \sim aa'$. Now set

$$x = b'c'cba'a, \; y = b'c'ca, \; y' = a'c'cb \in V(y).$$

Then

$$x = (b'c'cb)(b'b)(a'a) \sim (b'c'cb)(b'a)(a'a) = (b'c'cb)(b'a) \sim b'c'cb,$$

and

$$\begin{aligned} yx &= b'(c'cab'c'c)b(a'a) \\ &\sim b'(c'cbb'c'c)b(a'a) \quad \text{(by K4 and K5)} \\ &= (b'c'cb)(b'c'cb)(a'a) = (b'c'cb)(a'a) = x \sim b'c'cb. \end{aligned}$$

Furthermore,

$$\begin{aligned} y'x &= a'[(c'cbb'c'c)(ba')]a \sim a'[(c'cbb'c'c)(aa')]a \quad \text{(by K4)} \\ &= a'c'cbb'c'ca = y'y. \end{aligned}$$

Hence by K6 $y \sim x \sim b'c'cb$, i.e. $b'c'ca \sim b'c'cb$, and it follows that $\rho_{\mathscr{A}}$ is left compatible.

To prove that $\rho_{\mathscr{A}}$ is right compatible, we need to prove that $bcc'a' \sim acc'a'$ and $c'b'ac \sim c'b'bc$, and the second of these is trivial as it follows immediately from K4. Set

$$x = bb'acc'a', \; y = bcc'a', \; y' = acc'b' \in V(y).$$

Then $x = (bb')(aa')(acc'a') \sim acc'a'$ by the usual sort of argument, and

$$\begin{aligned} yx &= (bb')[b(cc'a'b)b'](acc'a') \\ &\sim (ab')[b(cc'a'b)b'](acc'a') = a[(b'b)(cc')(a'b)(b'a)(cc')]a' \\ &\sim a[(b'b)(cc')(a'b)(b'b)(cc')]a' = a[(b'b)(cc')(a'b)(cc')]a' \\ &\sim a[(b'a)(cc')(a'a)(cc')]a' = (ab')(acc'a')(acc'a') \\ &= (ab')(acc'a') \sim (bb')(acc'a') = x \sim acc'a'. \end{aligned}$$

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Also,

$$y'x = acc'b'bb'acc'a' = a(cc')(b'a)(cc')]a'$$
$$\sim a[(cc')(b'b)(cc')]a' = (acc'b')(bcc'a') = y'y.$$

Hence by K6 $y \sim x$, i.e. $bcc'a' \sim acc'a'$, and so $\rho_{\mathscr{A}}$ is a congruence. This completes the proof of the theorem.

REMARK. The apparent lack of symmetry in two places in the proof of the above theorem is due of course to the lack of symmetry of condition (K6). One may use *in place of* K6 its obvious dual, namely the condition: (K6′) if $a, ab \in A_i$ and $ab', bb' \in A_j$ for some $b' \in V(b)$, $i, j \in I$, then $b \in A_i$.

LEMMA 2.5. *If $\mathscr{A}$ is a kernel normal system of the orthodox semigroup S then $\mathscr{A}$ is the kernel of the congruence $\rho_{\mathscr{A}}$.*

PROOF. Let $\mathscr{K} = \{K_j : j \in J\}$ be the kernel of $\rho_{\mathscr{A}}$. Let $a \in K_j$ and choose $e \in K_j \cap E_S$. Suppose that $e \in A_i$. Now $e \in V(e)$ and $(a, e) \in \rho_{\mathscr{A}}$, so by lemma 2.3

$$ae \sim ee = e \text{ and } a'e \sim a'a,$$

where a' is any inverse of a. Hence $a \sim e$ by K6, i.e. $a \in A_i$. Thus each member K_j of $\mathscr{K}$ is a subset of some member A_i of $\mathscr{A}$.

Conversely, suppose that $a \in A_i$ and choose $e \in A_i$. Suppose that $e \in K_j$. Now A_i is a subsemigroup of S (by K5) so $ea \in A_i$, i.e. $ee \sim ea$. Furthermore, if a' is any inverse of a, then $a' \in A_j$, some $j \in J$, by K3, and so $aa' \sim ea'$. Hence $(a, e) \in \rho_{\mathscr{A}}$, and it follows that $a \in K_j$, and consequently that each member A_i of $\mathscr{A}$ is a subset of some member K_j of $\mathscr{K}$. Hence $\mathscr{A} = \mathscr{K}$ as required.

This completes the proof of theorem 3.2.

3. The minimum group congruence on an orthodox semigroup

A congruence ρ on a semigroup S is defined to be a *group congruence* if S/ρ is a group, and an *inverse semigroup congruence* if S/ρ is an inverse semigroup. From Lallement's result (lemma 1.3), we can see that if S is a regular semigroup then S has a minimum group congruence μ say, and a minimum inverse semigroup congruence ρ say, and of course $\rho \subseteq \mu$. Let μ' be the minimum group congruence on S/ρ: it is clear that $\mu = \{(x, y) \in S \times S : (x\rho, y\rho) \in \mu'\}$ (a full proof of this may be easily constructed using the methods of Reilly and Scheiblich [8]).

We now proceed to characterize the minimum group congruence on an orthodox semigroup.

THEOREM 3.1. *The minimum group congruence μ on an orthodox semigroup S is given by*

(4) $\quad \mu = \{(a, b) \in S \times S \colon V(fa) = V(fb) \textit{ for some idempotent } f \in S\}$

and also by

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(5) $\mu = \{(a, b) \in S \times S: eae = ebe \text{ for some idempotent } e \in S\}$.

PROOF.[1] We use Munn's determination of the minimum group congruence on an inverse semigroup [(6], corollary 2.9), the congruence $\mathscr{Y}$ of T. E. Hall ([2], theorem 3), and Lallement's lemma 1.3 to deduce that

$$\begin{aligned}\mu &= \{(a, b) \in S \times S: f\mathscr{Y}^{\natural} a\mathscr{Y}^{\natural} = f\mathscr{Y}^{\natural} b\mathscr{Y}^{\natural} \text{ for some } f \in E_S)\}\\ &= \{(a, b) \in S \times S: (fa)\mathscr{Y}^{\natural} = (fb)\mathscr{Y}^{\natural} \text{ for some } f \in E_S\}\\ &= \{(a, b) \in S \times S: V(fa) = V(fb) \text{ for some } f \in E_S\}.\end{aligned}$$

The proof will be complete when we prove the following:

LEMMA 3.2. *Let a, b be elements of an orthodox semigroup S. Then $V(fa) = V(fb)$ for some $f \in E_S$ if and only if $eae = ebe$ for some $e \in E_S$.*

PROOF. Suppose $V(fa) = V(fb)$ for some $f \in E_S$. Take any element $x \in V(fa) = V(fb)$. Then

$$fa = faxfa = fa(xfbx)fa = gfbh$$

where $g = fax$ and $h = xfa$ are idempotents. Then

$$(hgf)a(hgf) = (hgf)b(hgf)$$

and so $eae = ebe$ for $e = hgf$, an idempotent.

Conversely suppose $eae = ebe = x$ say, for some $e \in E_S$. Take any inverse x' of x and put $x^* = ex'e$, also an inverse of x. Then by routine calculations using $ex^* = x^* = x^*e$, we obtain $x^* \in V(xx^*a) \cap V(xx^*b)$. By theorem 2 of [2] we have $V(fa) = V(fb)$ for $f = xx^*$, an idempotent.

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