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PARALLEL SUBMANIFOLDS OF COMPLEX SPACE FORMS II

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Introduction

This is a continuation of Part I, which appeared in this journal.

In the previous paper I we have defined the following notions: orthogonal Jordan triple system (OJTS), orthogonal symmetric graded Lie algebra (OSGLA), orthogonal Jordan algebra (OJA), Hermitian symmetric graded Lie algebra (HSGLA). And we have shown that equivalent classes of OJTS naturally correspond to equivalent classes of OSGLA and through this correspondence we have naturally constructed HSGLA's from the OJTS's associated with OJA's with unity.

In the present paper we will give OJA's with unity and thus HSGLA's associated with r-dimensional complete totally real parallel submanifolds of an r-dimensional complex space form of nonzero constant holomorphic sectional curvature. And we will show that rigid classes of these submanifolds correspond to equivalent classes of the associated OJA's with unity and thus the associated HSGLA's (Theorem 6.3).

Moreover, in the section 7, we will decompose these OJA's and HSGLA's into indecomposable ones (Theorem 7.10). The indecomposable OJA's and HSGLA's are devided into the following types: almost nilpotent type or simple type. In the section 8 we will determine almost nilpotent HSGLA's by using results of Cahen-Parker [2], and in the section 9 simple HSGLA's by using results of Berger [1] and Kobayashi-Nagano [6].

We retain the definitions and notations in Part I.

§ 6. The OJA and HSGLA associated with a complete inverse

Let L^ℓ be an ℓ-dimensional pseudo-Euclidean space and N^n an n-dimensional connected parallel pseudo-riemannian submanifold of L^ℓ. Denote by $\tilde{\sigma}$, $\tilde{A}$ the second fundamental form, the shape operator of the

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inclusion $N \hookrightarrow L^\ell$ respectively. The pseudo-riemannian metrics on L^ℓ, N will be denoted by the same notation $\langle\ \ \rangle_L$. Denote by $\hat{R}$ the curvature tensor for the Levi-Civita connection $\hat{\nabla}$ of N and by $\tilde{R}^\perp$ the curvature tensor for the normal connection $\tilde{D}$ of the inclusion $N \hookrightarrow L^\ell$ respectively. Then we have the following identities:

$$\hat{R}(X, Y)Z = \tilde{A}_{\tilde{\sigma}(Y,Z)}X - \tilde{A}_{\tilde{\sigma}(X,Z)}Y, \tag{6.1}$$

$$\hat{R}(X, Y)\cdot\hat{R} = 0, \tag{6.2}$$

$$\tilde{R}^\perp(X, Y)\tilde{\sigma}(Z, W) = \tilde{\sigma}(\hat{R}(X, Y)Z, W) + \tilde{\sigma}(Z, \hat{R}(X, Y)W) \tag{6.3}$$

for vector fields X, Y, Z, W of N. In fact, the identity (6.1) is the result of Gauss equation. The identities (6.2), (6.3) are attained from the parallelity of $\hat{R}$, $\tilde{\sigma}$ respectively.

Fix a point $p \in N$ and put $V = T_p(N)$. Define a V-valued trilinear form $\{\ \ \}$ on V by

$$\{X, Y, Z\} = \hat{R}(X, Y)Z + \tilde{A}_{\tilde{\sigma}(X,Y)}Z$$

for $X, Y, Z \in V$. Then the object $(V, \{\ \ \})$ is a JTS. In fact, the condition (JT 1) is attained by (6.1) and the condition (JT 2) by (6.2), (6.3) respectively (See Ferus [3], Lemma 2 for the detailed proof). Denote also by $\langle\ \ \rangle_L$ the restriction of $\langle\ \ \rangle_L$ into $V \times V$. Then we have

$$L(X, Y)^t = -\hat{R}(X, Y) + \tilde{A}_{\tilde{\sigma}(X,Y)} = \hat{R}(Y, X) + \tilde{A}_{\tilde{\sigma}(Y,X)} = L(Y,X)$$

for $X, Y \in V$. Hence the object $(V, \{\ \ \}, \langle\ \ \rangle_L)$ is an OJTS. Note that the tensor $\tilde{A}_{\tilde{\sigma}(*,*)}(*)$ on N is parallel. Then the OJTS is independent of the choice of a point $p \in N$ by the parallelity of $\hat{R}$, $\tilde{A}_{\tilde{\sigma}(*,*)}(*)$.

Let H^n be a $2n$-dimensional pseudo-Hermitian space with the complex structure i_H and the pseudo-riemannian metric $\langle\ \ \rangle_H$, and N^n an n-dimensional totally real pseudo-riemannian submanifold of H^n. Then we have the following identities:

$$\tilde{D}_X i_H Z = i_H \hat{\nabla}_X Z \quad \text{and thus} \quad \tilde{R}^\perp(X, Y)i_H Z = i_H\hat{R}(X, Y)Z, \tag{6.4}$$

$$\tilde{A}_{i_H X}Y = -i_H\tilde{\sigma}(X, Y), \tag{6.5}$$

$$\langle\tilde{\sigma}(X, Y), i_H Z\rangle_H = \langle\tilde{\sigma}(X, Z), i_H Y\rangle_H \tag{6.6}$$

for vector fields X, Y, Z of N^n. In fact, the identities (6.4), (6.5) are attained by Gauss-Weingarten formulas and the identity (6.6) is attained by (6.5).

Let N^n be an n-dimensional connected totally real parallel pseudo-riemannian submanifold of $\boldsymbol{H}^n$. Fix a point $p \in N^n$ and define a product on $V = T_p(N)$ by

$$X \cdot Y = i_H \tilde{\sigma}(X, Y)$$

for $X, Y \in V$. Denote by A this algebra $(V, \cdot)$. Note that

$$(6.7) \qquad \hat{\nabla}(i_H \tilde{\sigma}) = 0 \quad \text{and thus} \quad \hat{R}(X, Y) \cdot (i_H \tilde{\sigma}) = 0$$

for $X, Y \in V$ by the parallelity of $\tilde{\sigma}$ and (6.4).

LEMMA 6.1. *The object* $(A, \langle\ \rangle_H)$ *is an OJA. Moreover the OJTS associated with* $(A, \langle\ \rangle_H)$ *is the object* $(V, \{\ \}, \langle\ \rangle_H)$.

Proof. We show that A is a JA. The condition (J 1) is obvious. Now we have

$$\begin{aligned} X^2 \cdot (X \cdot Y) - X \cdot (X^2 \cdot Y) & \\ &= i_H \tilde{\sigma}(i_H \tilde{\sigma}(X, X), i_H \tilde{\sigma}(X, Y)) - i_H \tilde{\sigma}(X, i_H \tilde{\sigma}(i_H \tilde{\sigma}(X, X), Y)) \\ &= -\tilde{A}_{\tilde{\sigma}(X, X)} \tilde{A}_{i_H X} Y + \tilde{A}_{i_H X} \tilde{A}_{\tilde{\sigma}(X, X)} Y = [\tilde{A}_{i_H X}, \tilde{A}_{\tilde{\sigma}(X, X)}](Y) \end{aligned}$$

for $X, Y \in V$ by (6.5), and moreover

$$\begin{aligned} \langle [\tilde{A}_{i_H X}, \tilde{A}_{\tilde{\sigma}(X, X)}](Y), Z \rangle_H &= \langle \tilde{R}^{\perp}(Y, Z) i_H X, \tilde{\sigma}(X, X) \rangle_H \\ &= -\langle i_H \hat{R}(Y, Z) X, \tilde{\sigma}(X, X) \rangle_H = \langle \hat{R}(Y, Z) X, i_H \tilde{\sigma}(X, X) \rangle_H \\ &= (1/3)\{-\langle X, \hat{R}(Y, Z) i_H \tilde{\sigma}(X, X) \rangle_H + 2\langle X, i_H \tilde{\sigma}(\hat{R}(Y, Z) X, X) \rangle_H\} \\ &= -(1/3)\langle X, \{\hat{R}(Y, Z) \cdot (i_H \tilde{\sigma})\}(X, X) \rangle_H = 0 \end{aligned}$$

for $Z \in V$ by Ricci equation and (6.6), (6.7). This implies the condition (J 2). Hence the algebra A is a JA. Since

$$\begin{aligned} \langle T_X(Y), Z \rangle_H = \langle X \cdot Y, Z \rangle_H &= \langle i_H \tilde{\sigma}(X, Y), Z \rangle_H = \langle i_H \tilde{\sigma}(X, Z), Y \rangle_H \\ &= \langle X \cdot Z, Y \rangle_H = \langle T_X(Z), Y \rangle_H \end{aligned}$$

for $X, Y, Z \in A$ by (6.6), endomorphisms T_X, $X \in A$, are symmetric for $\langle\ \rangle_H$. Hence the object $(A, \langle\ \rangle_H)$ is an OJA.

Let $(V_A, \{\ \}_A, \langle\ \rangle_H)$ be the OJTS associated with $(A, \langle\ \rangle_H)$. Then we have

$$\begin{aligned} \{X, Y, Z\}_A &= (X \cdot Y) \cdot Z + X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) \\ &= i_H \tilde{\sigma}(i_H \tilde{\sigma}(X, Y), Z) + i_H \tilde{\sigma}(X, i_H \tilde{\sigma}(Y, Z)) - i_H \tilde{\sigma}(Y, i_H \tilde{\sigma}(X, Z)) \\ &= \tilde{A}_{\tilde{\sigma}(X, Y)} Z + \tilde{A}_{\tilde{\sigma}(Y, Z)} X - \tilde{A}_{\tilde{\sigma}(X, Z)} Y \\ &= \tilde{A}_{\tilde{\sigma}(X, Y)} Z + \hat{R}(X, Y) Z = \{X, Y, Z\} \end{aligned}$$

for $X, Y, Z \in A$ by (6.1), (6.5), and thus $(V_A, \{\ \}_A, \langle\ \rangle_H) = (V, \{\ \}, \langle\ \rangle_H)$.
q.e.d.

Note that the OJA $(A, \langle\ \rangle_H)$ is independent of the choice of $p \in N^n$ by the parallelity of $i_H\tilde{\sigma}$.

Let M^r be an r-dimensional connected complete totally real parallel submanifold of $\bar{M}^r(c)$, $c \neq 0$. The complete inverse $\hat{M}^{r+1}$ of M^r is a connected complete totally real parallel submanifold of E^{r+1} by Proposition 4.1, Lemma 1.1, (3), (4.5). Then we have the following

LEMMA 6.2. *Let $(A, \langle\ \rangle_E)$ be the OJA constructed as above from the complete inverse $\hat{M}^{r+1} \hookrightarrow E^{r+1}$. Then $(A, \langle\ \rangle_E)$ has the unity E and satisfies that*

(E_c 1) *the signature of $\langle\ \rangle_E$ is $(1, r)$, $(0, r+1)$ according as $c < 0$, $c > 0$ respectively,*

(E_c 2) $\langle E, E\rangle_E = 4/c$.

Proof. Note that

$$\tilde{A}_\nu = -(\sqrt{|c|}/2)\ \mathrm{id}_A \tag{6.8}$$

by (4.1), (4.6). This implies that

$$i\tilde{\sigma}(i\nu, X) = -(\sqrt{|c|}/2)X \tag{6.9}$$

for $X \in A$ by (6.5). Put $E = -(2/\sqrt{|c|})i\nu$. Then E is the unity of A.
The properties (E_c 1), (E_c 2) are obvious. q.e.d

This object $(A, \langle\ \rangle_E)$ (resp. $(\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$ coming from $(A, \langle\ \rangle_E)$) is called the OJA (resp. HSGLA) *associated with* a complete inverse $\hat{M}^{r+1}$ and will be denoted by $\mathscr{A}_{\hat{M}}$ (resp. $\mathscr{G}_{\hat{M}}$).

Fix a real number $c \neq 0$. Let $\mathscr{A} = (A, \langle\ \rangle)$ be an OJA with unity E satisfying (E_c 1), (E_c 2) and $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$ the HSGLA coming from $\mathscr{A}$. Denote by $E_{\mathscr{G}}^{r+1}$ the pseudo-Hermitian space $(\mathfrak{p}_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$. Put $N_{\mathscr{G}}^{2r+1}(c/4) = \{X \in \mathfrak{p}_A; \langle X, X\rangle_{\mathfrak{p}_A} = c/4\}$ and denote by $\bar{M}_{\mathscr{G}}^r(c)$ the complex space form of constant holomorphic sectional curvature c, defined by the set of orbits in $N_{\mathscr{G}}^{2r+1}(c/4)$ by the S^1-action: $\theta \to \exp \theta J_{\mathfrak{p}_A}$. Then $K(\boldsymbol{\nu})$ is an $(r+1)$-dimensional connected complete totally real parallel submanifold of $E_{\mathscr{G}}^{r+1}$ and is left invariant by the S^1-action (Theorem 5.7). Since K acts isometrically on $E_{\mathscr{G}}^{r+1}$ and $\langle \boldsymbol{\nu}, \boldsymbol{\nu}\rangle_{\mathfrak{p}_A} = 4/c$, the submanifold $K(\boldsymbol{\nu})$ is contained in $N_{\mathscr{G}}^{2r+1}(c/4)$. Denote by $\pi_{\mathscr{G}}$ the projection of $N_{\mathscr{G}}^{2r+1}(c/4)$

onto $\overline{M}^r_{\mathscr{G}}(c)$. Then $M^r_{\mathscr{G}} = \pi_{\mathscr{G}}(K(\nu))$ is an r-dimensional connected complete totally real parallel submanifold of $\overline{M}^r_{\mathscr{G}}(c)$ and the complete inverse $\hat{M}^{r+1}_{\mathscr{G}}$ of $M^r_{\mathscr{G}}$ is the submanifold $K(\nu)$ (Lemma 1.1, (3), Proposition 4.1).

THEOREM 6.3. (1) *Let M^r be an r-dimensional connected complete totally real parallel submanifold of $\overline{M}^r(c)$. Then $M^r \hookrightarrow \overline{M}^r(c)$ is holomorphically congruent to $M^r_{\mathscr{G}_{\hat{M}}} \hookrightarrow \overline{M}^r_{\mathscr{G}_{\hat{M}}}(c)$, i.e., there exists a holomorphic isometry $\bar{\delta}$ of $\overline{M}^r(c)$ onto $\overline{M}^r_{\mathscr{G}_{\hat{M}}}(c)$ such that $\bar{\delta}(M^r) = M^r_{\mathscr{G}_{\hat{M}}}$.*

(2) *Let $\mathscr{A}$ be an OJA with unity E satisfying* (E_c 1), (E_c 2) *and $\mathscr{G}$ the HSGLA coming from $\mathscr{A}$. Then $\mathscr{A}_{\hat{M}_{\mathscr{G}}}$ is equivalent to $\mathscr{A}$ and thus $\mathscr{G}_{\hat{M}_{\mathscr{G}}}$ is equivalent to $\mathscr{G}$.*

(3) *Let $\mathscr{A}$, $\mathscr{A}'$ be OJA's with unities E, E' satisfying* (E_c 1), (E_c 2) *respectively and $\mathscr{G}$, $\mathscr{G}'$ the HSGLA's coming from $\mathscr{A}$, $\mathscr{A}'$ respectively. Then $\mathscr{A}$ is equivalent to $\mathscr{A}'$ if and only if $\mathscr{G}$ is equivalent to $\mathscr{G}'$ if and only if $M^r_{\mathscr{G}} \hookrightarrow \overline{M}^r_{\mathscr{G}}(c)$ is holomorphically congruent to $M^r_{\mathscr{G}'} \hookrightarrow \overline{M}^r_{\mathscr{G}'}(c)$.*

Proof. (1) Let $\hat{M}^{r+1}$ be the complete inverse of M^r and fix a point $z \in \hat{M}$. Denote by $\mathscr{A}_{\hat{M}} = (A, \langle\ \rangle_E)$ (resp. $\mathscr{G}_{\hat{M}} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$) the OJA (resp. HSGLA) associated with $\hat{M}$. Identify $A \oplus iA = T_z(\hat{M}) \oplus iT_z(\hat{M})$ with $\boldsymbol{E}^{r+1}$.

Define a linear isomorphism δ of $\boldsymbol{E}^{r+1}$ onto $\mathfrak{p}_A$ by

$$\delta(X + iY) = (-X, -T_Y, X) \qquad \text{for } X, Y \in A.$$

Then we have

$$\begin{aligned}
\langle \delta(X + iY), \delta(X' + iY')\rangle_{\mathfrak{p}_A} &= \langle X, X'\rangle_E + \langle (0, T_Y, 0), (0, T_{Y'}, 0)\rangle_{\mathfrak{p}_A} \\
&= \langle X, X'\rangle_E + \langle (0, (1/2)\{L_A(Y, E) + L_A(E, Y)\}, 0), (0, T_{Y'}, 0)\rangle_{\mathfrak{p}_A} \\
&= \langle X, X'\rangle_E + \langle T_{Y'}(E), Y\rangle_E = \langle X, X'\rangle_E + \langle Y, Y'\rangle_E \\
&= \langle X + iY, X' + iY'\rangle_E
\end{aligned}$$

for $X + iY$, $X' + iY' \in \boldsymbol{E}^{r+1}$ by (5.12), and moreover

$$\delta i(X + iY) = (Y, -T_X, -Y) = J_{\mathfrak{p}_A}(-X, -T_Y, X) = J_{\mathfrak{p}_A}\delta(X + iY)$$

for $X + iY \in \boldsymbol{E}^{r+1}$. Hence δ is a holomorphic isometry of the pseudo-Hermitian space $\boldsymbol{E}^{r+1}$ onto the pseudo-Hermitian space $\boldsymbol{E}^{r+1}_{\mathscr{G}_{\hat{M}}}$. This implies that $\delta(\hat{M}^{r+1})$ is a connected complete totally real parallel submanifold of $\boldsymbol{E}^{r+1}_{\mathscr{G}_{\hat{M}}}$ which is left invariant by the S^1-action: $\theta \to \exp \theta J_{\mathfrak{p}_A}$. Denote by j the imbedding: $K/K_0 \ni kK_0 \to k(\nu) \in \mathfrak{p}_A$. Then the second fundamental form $(\tilde{\sigma}_j)_o$ at $o = K_0$ is given by

$$(\tilde{\sigma}_j)_o(\boldsymbol{A}, \boldsymbol{B}) = \{\mathrm{ad}\,(\boldsymbol{A})\,\mathrm{ad}\,(\boldsymbol{B})\boldsymbol{\nu}\}_{\mathfrak{p}_A \cap (\mathfrak{g}_A)_0}$$

for $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{m} = \{(X, 0, X); X \in A\}$ (See Ferus [4], Lemma 1 for the proof). Note that $\delta(z) = \boldsymbol{\nu}$ by (6.9) and that $T_\nu(K(\boldsymbol{\nu})) = T_\nu(\delta(\hat{M})) = [\mathfrak{m}, \boldsymbol{\nu}]$. Denote by $(\tilde{\sigma}_{K(\nu)})_\nu$ (resp. $(\tilde{\sigma}_{\delta(\hat{M})})_\nu$) the second fundamental form at $\boldsymbol{\nu}$ of $K(\boldsymbol{\nu}) \subset \boldsymbol{E}^{r+1}_{\mathscr{G}_{\hat{M}}}$ (resp. $\delta(\hat{M}) \subset \boldsymbol{E}^{r+1}_{\mathscr{G}_{\hat{M}}}$). Then we have

$$\begin{aligned}(\tilde{\sigma}_{K(\nu)})_\nu((X, 0, -X), (Y, 0, -Y)) &= (\tilde{\sigma}_j)_o((X, 0, X), (Y, 0, Y)) \\ &= (0, (1/2)\{L_A(X, Y) + L_A(Y, X)\}, 0) = (0, T_{X\cdot Y}, 0) = -\delta(iX\cdot Y) \\ &= \delta(\tilde{\sigma}_z(X, Y)) = (\tilde{\sigma}_{\delta(M)})_\nu((X, 0, -X), (Y, 0, -Y))\end{aligned}$$

for $(X, 0, -X), (Y, 0, -Y) \in T_\nu(K(\boldsymbol{\nu})) = T_\nu(\delta(\hat{M}))$ and thus

$$(6.10) \qquad (\tilde{\sigma}_{K(\nu)})_\nu = (\tilde{\sigma}_{\delta(\hat{M})})_\nu\,.$$

Let $\bar{\delta}$ be a holomorphic isometry of $\bar{M}^r(c)$ onto $\bar{M}^r_{\mathscr{G}_{\hat{M}}}(c)$, induced by δ. Then we note that $\bar{\delta}(M^r) = \pi_{\mathscr{G}_{\hat{M}}}(\delta(\hat{M}^{r+1}))$. Put $p = \pi_{\mathscr{G}_{\hat{M}}}(\boldsymbol{\nu})$ and denote by $(\sigma_{M_{\mathscr{G}_{\hat{M}}}})_p$ (resp. $(\sigma_{\bar{\delta}(M)})_p$) the second fundamental form at p of $M_{\mathscr{G}_M} \hookrightarrow \bar{M}^r_{\mathscr{G}_{\hat{M}}}(c)$ (resp. $\bar{\delta}(M) \hookrightarrow \bar{M}^r_{\mathscr{G}_{\hat{M}}}(c)$). Then we have

$$(\sigma_{M_{\mathscr{G}_{\hat{M}}}})_p = (\sigma_{\bar{\delta}(M)})_p$$

by (6.10), (4.1), (4.6), Lemma 1.1, (1). Hence we have $\bar{\delta}(M) = M^r_{\mathscr{G}_{\hat{M}}}$ (Naitoh [8], Lemma 3.2).

(2) Let $\mathscr{A} = (A, \langle\ \ \rangle)$ and $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \ \rangle_{\mathfrak{p}_A})$. Then the second fundamental form $(\tilde{\sigma}_{\hat{M}_{\mathscr{G}}})_\nu$ at $\boldsymbol{\nu}$ of $\hat{M}_{\mathscr{G}} = K(\boldsymbol{\nu}) \subset \boldsymbol{E}^{r+1}_{\mathscr{G}}$ is given by

$$(6.11) \qquad (\tilde{\sigma}_{\hat{M}_{\mathscr{G}}})_\nu(\boldsymbol{A}, \boldsymbol{B}) = (0, T_{X\cdot Y}, 0)$$

for $\boldsymbol{A} = (X, 0, -X)$, $\boldsymbol{B} = (Y, 0, -Y)$ and thus

$$J_{\mathfrak{p}_A}(\tilde{\sigma}_{\hat{M}_{\mathscr{G}}})_\nu(\boldsymbol{A}, \boldsymbol{B}) = (-X\cdot Y, 0, X\cdot Y)\,.$$

Define a linear isomorphism g of A onto $[\mathfrak{m}, \boldsymbol{\nu}]$ by $g(X) = (-X, 0, X)$ for $X \in A$. Then we have

$$\begin{aligned}g(X\cdot Y) = (-X\cdot Y, 0, X\cdot Y) &= J_{\mathfrak{p}_A}(\tilde{\sigma}_{\hat{M}_{\mathscr{G}}})_\nu(\boldsymbol{A}, \boldsymbol{B}) \\ &= J_{\mathfrak{p}_A}(\tilde{\sigma}_{\hat{M}_{\mathscr{G}}})_\nu(g(X), g(Y))\end{aligned}$$

and moreover

$$\langle g(X), g(Y)\rangle_{\mathfrak{p}_A} = \langle X, Y\rangle\,.$$

This implies that g is an isomorphism of OJA $\mathscr{A}$ onto OJA $\mathscr{A}_{\hat{M}_{\mathscr{G}}}$ and thus

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$\mathscr{A}$ is equivalent to $\mathscr{A}_{\hat{\mathfrak{M}}_{\mathscr{G}}}$. Hence $\mathscr{G}$ is equivalent to $\mathscr{G}_{\hat{\mathfrak{M}}_{\mathscr{G}}}$ by Theorem 5.5, (2).

(3) It is obvious by Theorem 5.5, (2) that $\mathscr{A}$ is equivalent to $\mathscr{A}'$ if and only if $\mathscr{G}$ is equivalent to $\mathscr{G}'$.

Assume that $\mathscr{A} = (A, \langle\ \rangle)$ is equivalent to $\mathscr{A}' = (A', \langle\ \rangle')$, i.e., there exists an algebra isomorphism α of A onto A' such that $\langle \alpha(X), \alpha(Y)\rangle' = \langle X, Y\rangle$ for $X, Y \in A$. The isomorphism α induces the isomorphism τ_{g_α} of $\mathscr{G}$ onto $\mathscr{G}'$. Then we have

$$\tau_{g_\alpha}(\boldsymbol{\nu}) = \tau_{g_\alpha}(0, -\mathrm{id}_A, 0) = (0, -g_\alpha \circ \mathrm{id}_A \circ g_\alpha^{-1}, 0) = (0, -\mathrm{id}_{A'}, 0) = \boldsymbol{\nu}' .$$

The restriction δ of τ_{g_α} into $\mathfrak{p}_A$ is a holomorphic isometry of $\boldsymbol{E}_{\mathscr{G}}^{r+1}$ onto $\boldsymbol{E}_{\mathscr{G}'}^{r+1}$ and $\delta(K(\boldsymbol{\nu})) = K'(\boldsymbol{\nu}')$. Hence we have $\bar{\delta}(M_{\mathscr{G}}^r) = M_{\mathscr{G}'}^r$, where $\bar{\delta}$ denotes a holomorphic isometry of $\overline{M}_{\mathscr{G}}^r(c)$ onto $\overline{M}_{\mathscr{G}'}^r(c)$, induced by δ.

Conversely, assume that $M_{\mathscr{G}}^r \hookrightarrow \overline{M}_{\mathscr{G}}^r(c)$ is holomorphically congruent to $M_{\mathscr{G}'}^r \hookrightarrow \overline{M}_{\mathscr{G}'}^r(c)$, i.e., there exists a holomorphic isometry $\bar{\delta}$ of $\overline{M}_{\mathscr{G}}^r(c)$ onto $\overline{M}_{\mathscr{G}'}^r(c)$ such that $\bar{\delta}(M_{\mathscr{G}}^r) = M_{\mathscr{G}'}^r$. Then $\bar{\delta}$ induces a holomorphic isometry δ of $\boldsymbol{E}_{\mathscr{G}}^{r+1}$ onto $\boldsymbol{E}_{\mathscr{G}'}^{r+1}$ such that $\delta(\hat{M}_{\mathscr{G}}^{r+1}) = \hat{M}_{\mathscr{G}'}^{r+1}$. Denote by $\tilde{\sigma}_{\mathscr{G}}$ (resp. $\tilde{\sigma}_{\mathscr{G}'}$) the second fundamental form of $\hat{M}_{\mathscr{G}}^{r+1} \hookrightarrow \boldsymbol{E}_{\mathscr{G}}^{r+1}$ (resp. $\hat{M}_{\mathscr{G}'}^{r+1} \hookrightarrow \boldsymbol{E}_{\mathscr{G}'}^{r+1}$). Then we have

$$\delta(J_{\mathfrak{p}_A}\tilde{\sigma}_{\mathscr{G}}(X, Y)) = J_{\mathfrak{p}_{A'}}\delta(\tilde{\sigma}_{\mathscr{G}}(X, Y)) = J_{\mathfrak{p}_{A'}}\tilde{\sigma}_{\mathscr{G}'}(\delta X, \delta Y)$$

for $X, Y \in T_z(\hat{M}_{\mathscr{G}}^{r+1})$. Hence $\mathscr{A}_{\hat{\mathfrak{M}}_{\mathscr{G}}}$ is equivalent to $\mathscr{A}_{\hat{\mathfrak{M}}_{\mathscr{G}'}}$. This implies that $\mathscr{A}$ is equivalent to $\mathscr{A}'$ by (2). q.e.d.

Remark 6.4. Theorem 6.3 implies that the classification of r-dimensional connected complete totally real parallel submanifold of $\overline{M}^r(c)$, $c \neq 0$, reduces to that of HSGLA's associated with OJA's with unities satisfying (E_c 1), (E_c 2).

Remark 6.5. The proof of Theorem 6.3, (2) doesn't need the conditions (E_c 1), (E_c 2) for $\mathscr{A}$. Hence the claim (2) is true for any OJA with unity.

Let $\mathscr{A} = (A, \langle\ \rangle)$ be an OJA with unity and $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$ the HSGLA coming from $\mathscr{A}$. Denote by $\boldsymbol{H}_{\mathscr{G}}^{r+1}$ the pseudo-Hermitian space $(\mathfrak{p}_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$ and put $\hat{M}_{\mathscr{G}}^{r+1} = K(\boldsymbol{\nu})$. Then the proof of Theorem 6.3, (3) implies that the claim (3) can be generalized as follows: Let $\mathscr{A}$, $\mathscr{A}'$ be OJA's with unity and $\mathscr{G}$, $\mathscr{G}'$ the HSGLA's coming from $\mathscr{A}$, $\mathscr{A}'$ respectively. Then $\mathscr{A}$ is equivalent to $\mathscr{A}'$ if and only if $\mathscr{G}$ is equivalent to $\mathscr{G}'$ if and only if $\hat{M}_{\mathscr{G}}^{r+1} \hookrightarrow \boldsymbol{H}_{\mathscr{G}}^{r+1}$ is holomorphically and linearly congruent to $\hat{M}^{r+1}_{\mathfrak{g}'} \hookrightarrow H^{r+1}_{\mathfrak{g}'}$, i.e., there exists a holomorphic and linear isometry δ of $H^{r+1}_{\mathfrak{g}}$ onto $H^{r+1}_{\mathfrak{g}'}$ such that $\delta(\hat{M}^{r+1}_{\mathfrak{g}}) = \hat{M}^{r+1}_{\mathfrak{g}'}$.

§ 7. A decomposition of the HSGLA coming from an OJA with unity

Firstly we define the following two notions for each category given in this series: one is "the sum of objects" and the other is "a decomposition of an object".

Let $(V_i, \{\ \ \}_i)$, $1 \leqq i \leqq s$, be JTS's. Put $V = \oplus_{i=1}^s V_i$ and define a V-valued trilinear form $\{\ \ \}$ on V by

$$\{\sum X_i, \sum Y_i, \sum Z_i\} = \sum \{X_i, Y_i, Z_i\}_i$$

for $X_i, Y_i, Z_i \in V_i$, $1 \leqq i \leqq s$. Then the object $(V, \{\ \ \})$ is a JTS. This JTS is called the *sum* of JTS's $(V_i, \{\ \ \}_i)$ and is denoted by $(V, \{\ \ \}) = \oplus (V_i, \{\ \ \}_i)$. Conversely, let $(V, \{\ \ \})$ be a JTS. Let $V = \oplus_{i=1}^s V_i$ be the direct sum of linear subspaces V_i satisfying that $\{V_i, V_j, V_k\} \subset V_i \cap V_j \cap V_k$ for $1 \leqq i, j, k \leqq s$. Denote by $\{\ \ \}_i$, $1 \leqq i \leqq s$, the restrictions of $\{\ \ \}$ into subspaces V_i respectively. Then the objects $(V_i, \{\ \ \}_i)$, $1 \leqq i \leqq s$, are JTS's and the JTS $(V, \{\ \ \})$ is equivalent to $\oplus (V_i, \{\ \ \}_i)$. The sum $\oplus (V_i, \{\ \ \}_i)$ is called a *decomposition* of the JTS $(V, \{\ \ \})$.

Let $\mathscr{V}_i = (V_i, \{\ \ \}_i, \langle\ \ \rangle_i)$, $1 \leqq i \leqq s$, be OJTS's. Let $(V, \{\ \ \})$ be the sum of JTS's $(V_i, \{\ \ \}_i)$ and define a non-degenerate symmetric bilinear form $\langle\ \ \rangle$ on V by

$$\langle \sum X_i, \sum Y_i \rangle = \sum \langle X_i, Y_i \rangle_i$$

for $X_i, Y_i \in V_i$, $1 \leqq i \leqq s$. Then the object $\mathscr{V} = (V, \{\ \ \}, \langle\ \ \rangle)$ is an OJTS. This OJTS is called the *sum* of OJTS's $\mathscr{V}_i$ and is denoted by $\mathscr{V} = \oplus \mathscr{V}_i$. Conversely, let $\mathscr{V} = (V, \{\ \ \}, \langle\ \ \rangle)$ be an OJTS. Let $\oplus (V_i, \{\ \ \}_i)$ be a decomposition of the JTS $(V, \{\ \ \})$ such that $\langle V_i, V_j \rangle = \{0\}$ for $i \neq j$. Denote by $\langle\ \ \rangle_i$, $1 \leqq i \leqq s$, the restrictions of $\langle\ \ \rangle$ into subspaces V_i respectively. Then the objects $\mathscr{V}_i = (V_i, \{\ \ \}_i, \langle\ \ \rangle_i)$, $1 \leqq i \leqq s$, are OJTS's and $\mathscr{V}$ is equivalent to $\oplus \mathscr{V}_i$. The sum $\oplus \mathscr{V}_i$ is called a *decomposition* of the OJTS $\mathscr{V}$.

Let $\mathscr{A}_i = (A_i, \langle\ \ \rangle_i)$, $1 \leqq i \leqq s$, be OJA's. Let $A = \oplus A_i$ be the sum of JA's A_i and define a non-degenerate symmetric bilinear form $\langle\ \ \rangle$ on A by

$$\langle \sum X_i, \sum Y_i \rangle = \sum \langle X_i, Y_i \rangle_i$$

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for $X_i, Y_i \in A_i$, $1 \leqq i \leqq s$. Then the object $\mathscr{A} = (A, \langle\ \rangle)$ is an OJA. This OJA is called the *sum* of OJA's $(A_i, \langle\ \rangle_i)$ and is denoted by $\mathscr{A} = \oplus \mathscr{A}_i$. Conversely, let $\mathscr{A} = (A, \langle\ \rangle)$ be an OJA. Let $\oplus A_i$ be a decomposition of A into the sum of ideals A_i satisfying that $\langle A_i, A_j \rangle = \{0\}$ for $i \neq j$. Denote by $\langle\ \rangle_i$, $1 \leqq i \leqq s$, the restrictions of $\langle\ \rangle$ into ideals A_i respectively. Then the objects $\mathscr{A}_i = (A_i, \langle\ \rangle_i)$, $1 \leqq i \leqq s$, are OJA's and $\mathscr{A}$ is equivalent to $\oplus \mathscr{A}_i$. The sum $\oplus \mathscr{A}_i$ is called a *decomposition* of the OJA $\mathscr{A}$.

Moreover, two notions of "sum" and "decomposition" can be defined naturally for other categories: OSLA, HSLA, OSGLA, HSGLA, etc. Since the definitions are clear, they are not discribed here. But the notions are often used in this paper.

PROPOSITION 7.1. (1) *Let $\mathscr{V}, \mathscr{V}_1, \mathscr{V}_2$ be OJTS's and $\mathscr{G}, \mathscr{G}_1, \mathscr{G}_2$ the OSGLA's associated with $\mathscr{V}, \mathscr{V}_1, \mathscr{V}_2$ respectively. Then $\mathscr{V}$ is equivalent to $\mathscr{V}_1 \oplus \mathscr{V}_2$ if and only if $\mathscr{G}$ is equivalent to $\mathscr{G}_1 \oplus \mathscr{G}_2$.*

(2) (*a*) *Let $\mathscr{A}_1, \mathscr{A}_2$ be OJA's. Then $\mathscr{A}_1 \oplus \mathscr{A}_2$ has the unity if and only if each $\mathscr{A}_i$ has the unity.*

(*b*) *Let $\mathscr{G}_1, \mathscr{G}_2$ be HSGLA's. Then $\mathscr{G}_1 \oplus \mathscr{G}_2$ is equivalent to an HSGLA coming from an OJA with unity if and only if each $\mathscr{G}_i$ is equivalent to an HSGLA coming from an OJA with unity.*

(3) *Let $\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2$ be OJA's with unity and $\mathscr{G}, \mathscr{G}_1, \mathscr{G}_2$ the HSGLA's coming from $\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2$ respectively. Then $\mathscr{A}$ is equivalent to $\mathscr{A}_1 \oplus \mathscr{A}_2$ if and only if $\mathscr{G}$ is equivalent to $\mathscr{G}_1 \oplus \mathscr{G}_2$.*

Proof. (1) Note that the OSGLA associated with $\mathscr{V}_1 \oplus \mathscr{V}_2$ is equivalent to $\mathscr{G}_1 \oplus \mathscr{G}_2$. Then our claim is clear by Theorem 5.4, (2).

(2) The claim (a) is obvious. We show the claim (b). Assume that $\mathscr{G}_1 \oplus \mathscr{G}_2$ is equivalent to $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \rangle_{\mathfrak{p}_A})$ coming from $\mathscr{A} = (A, \langle\ \rangle)$ with the unity E by an isomorphism $\tau\colon \mathscr{G}_1 \oplus \mathscr{G}_2 \rightrightarrows \mathscr{G}$. Let $\mathscr{G}_i = (\mathfrak{g}_i = \sum (\mathfrak{g}_i)_\mu, \rho_i, J_{\mathfrak{p}_i}, \langle\ \rangle_{\mathfrak{p}_i})$, $i = 1, 2$. Put $A_i = \{X \in A; (X, 0, 0) \in \tau(\mathfrak{g}_i)\}$, $i = 1, 2$. Then A is decomposed into the sum of linear subspaces A_i. We show that A_i, $i = 1, 2$, are orthogonal ideals of the JA A for $\langle\ \rangle$. Suppose that $X \in A_i$, i.e., $(X, 0, 0) \in \tau(\mathfrak{g}_i)$. Since $\tau(\mathfrak{g}_i)$ is an ideal of $\mathfrak{g}_A$, we have

$$(X \cdot Y, 0, 0) = (L_A(X, Y)E, 0, 0) = -2[[(X, 0, 0), (0, 0, Y)], (E, 0, 0)] \in \tau(\mathfrak{g}_i)$$

for $Y \in A$ and thus $X \cdot Y \in A_i$. This implies that A_i is an ideal of A.

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Moreover, noting that $(X_i, 0, -X_i) \in \tau(\mathfrak{p}_i)$ for $X_i \in A_i$, $i = 1, 2$, we have

$$\langle X_1, X_2\rangle = \langle (X_1, 0, -X_1), (X_2, 0, -X_2)\rangle_{\mathfrak{p}} = 0$$

and thus $\langle A_1, A_2\rangle = \{0\}$. Denote by $\langle\ \ \rangle_i$, $i = 1, 2$, the restrictions of $\langle\ \ \rangle$ into the ideals A_i respectively and put $\mathscr{A}_i = (A_i, \langle\ \ \rangle_i)$ for $i = 1, 2$. Then OJA's $\mathscr{A}_i$ have the unities by (a) and the OJA $\mathscr{A}$ is decomposed into the sum $\mathscr{A}_1 \oplus \mathscr{A}_2$.

Let $\mathscr{G}'_i = (\mathfrak{g}'_i = \sum (\mathfrak{g}'_i)_\mu, \rho'_i, J_{\mathfrak{p}'_i}, \langle\ \ \rangle_{\mathfrak{p}'_i})$, $i = 1, 2$, be the HSGLA's coming from $\mathscr{A}_i$ respectively. Denote by F_i, $i = 1, 2$, the restrictions of $F \in L_A$ into A_i respectively. Then we have

$$(\mathfrak{g}'_i)_0 = \{(0, (L_A(X, Y))_i, 0);\ X, Y \in A_i\}$$

Define mappings λ_i, $i = 1, 2$, of $\mathfrak{g}'_i$ into $\tau(\mathfrak{g}_i)$ by

$$\lambda_i((X_i, (L_A(Z_i, W_i))_i, Y_i)) = (X_i, L_A(Z_i, W_i), Y_i)$$

for X_i, Y_i, Z_i, $W_i \in A_i$, $i = 1, 2$. Then we can easily see that λ_i are well-defined isomorphisms of the HSGLA's $\mathscr{G}'_i$ onto the HSGLA's $\tau(\mathscr{G}_i)$ respectively. Hence $\mathscr{G}_i$ are equivalent to the HSGLA's $\mathscr{G}'_i$ coming from $\mathscr{A}_i$ respectively.

The converse is obvious.

(3) Note that the HSGLA coming from $\mathscr{A}_1 \oplus \mathscr{A}_2$ is equivalent to $\mathscr{G}_1 \oplus \mathscr{G}_2$. Then our claim is clear by Theorem 5.5, (2). q.e.d.

Let $\mathscr{V} = (V, \{\ \ \}, \langle\ \ \rangle)$ be an OJTS and denote by β the trace form of the JTS $(V, \{\ \ \})$. Then β is a symmetric bilinear form on V. Define a symmetric endomorphism $L_{\mathscr{V}}$ of V by

$$\beta(X, Y) = \langle L_{\mathscr{V}}(X), Y\rangle$$

for X, $Y \in V$. Let $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ be an orthonormal basis of V, i.e., $\langle e_i, e_i\rangle = -1$ for $1 \leqq i \leqq k$, $\langle e_j, e_j\rangle = 1$ for $k+1 \leqq j \leqq n$, and $\langle e_i, e_j\rangle = 0$ for $1 \leqq i \neq j \leqq n$. Then the symmetric endomorphism $L_{\mathscr{V}}$ is given by

$$L_{\mathscr{V}} = -\textstyle\sum_{i=1}^k L(e_i, e_i) + \sum_{j=k+1}^n L(e_j, e_j)\,. \tag{7.1}$$

In fact, note that $\langle L(X, Y)Z, W\rangle = \langle L(Z, W)X, Y\rangle$ for X, Y, Z, $W \in V$ by (5.3), (JT 1). Then we have

$$\begin{aligned}\beta(X, Y) = \operatorname{Tr} L(X, Y) &= -\textstyle\sum_{i=1}^k \langle L(X, Y)e_i, e_i\rangle + \sum_{j=k+1}^n \langle L(X, Y)e_j, e_j\rangle \\ &= -\textstyle\sum_{i=1}^k \langle L(e_i, e_i)X, Y\rangle + \sum_{j=k+1}^n \langle L(e_j, e_j)X, Y\rangle \\ &= \langle(-\textstyle\sum_{i=1}^k L(e_i, e_i) + \sum_{j=k+1}^n L(e_j, e_j))X, Y\rangle\end{aligned}$$

for $X, Y \in V$. This implies (7.1).

Moreover the symmetric endomorphism $L_{\mathscr{V}}$ is contained in the center of L, i.e.,

$$(7.2) \qquad [L_{\mathscr{V}}, L(X, Y)] = 0$$

for $X, Y \in V$. In fact, note that $\beta(L(X, Y)Z, W) = \beta(Z, L(Y, X)W)$ for $X, Y, Z, W \in V$ by (JT 2). Then we have

$$\begin{aligned} \langle [L_{\mathscr{V}}, L(X, Y)]\,(Z), W\rangle &= \langle L(X, Y)Z, L_{\mathscr{V}}(W)\rangle - \langle L_{\mathscr{V}}(Z), L(Y, X)W\rangle \\ &= \beta(L(X, Y)Z, W) - \beta(Z, L(Y, X)W) = 0 \end{aligned}$$

for $X, Y, Z, W \in V$. This implies (7.2).

Let $\mathscr{V} = (V, \{\ \ \}, \langle\ \ \rangle)$ be an OJTS and $(V_1, \{\ \ \}_1) \oplus (V_2, \{\ \ \}_2)$ a decomposition of the JTS $(V, \{\ \ \})$. Denote by $\langle\ \ \rangle_i$, $i = 1, 2$, the restrictions of $\langle\ \ \rangle$ into subspaces V_i respectively.

LEMMA 7.2. *Assume that the symmetric endomorphism $L_{\mathscr{V}}$ is non-degenerate on V_1. Then the objects $\mathscr{V}_i = (V_i, \{\ \ \}_i, \langle\ \ \rangle_i)$, $i = 1, 2$, are OJTS's and the OJTS $\mathscr{V}$ is equivalent to the sum $\mathscr{V}_1 \oplus \mathscr{V}_2$.*

Proof. It is sufficient to show that $\langle V_1, V_2\rangle = \{0\}$. Note that $\beta(V_1, V_2) = \{0\}$ and thus $\langle L_{\mathscr{V}}(V_1), V_2\rangle = \{0\}$. Since $L_{\mathscr{V}}$ is non-degenerate on V_1, we have $L_{\mathscr{V}}(V_1) = V_1$ and thus $\langle V_1, V_2\rangle = \{0\}$. q.e.d.

Let $\mathscr{V} = (V, \{\ \ \}, \langle\ \ \rangle)$ be an OJTS. Put $V_i = (L_{\mathscr{V}})^i(V)$ for non-negative integers i. Then we have a decreasing sequence:

$$V = V_0 \supseteqq V_1 \supseteqq \cdots \supseteqq V_i \supseteqq \cdots .$$

Let ℓ be the least number of i's such that $V_i = V_{i+1}$, and put $V_{\text{non}} = V_\ell$, $V_{\text{deg}} = \operatorname{Ker}(L_{\mathscr{V}})^\ell$. Note that $L_{\mathscr{V}}(V_{\text{non}}) = V_{\text{non}}$. Then we have the direct sum $V = V_{\text{deg}} \oplus V_{\text{non}}$. Denote by $\{\ \ \}_{\text{deg}}$, $\{\ \ \}_{\text{non}}$ (resp. $\langle\ \ \rangle_{\text{deg}}$, $\langle\ \ \rangle_{\text{non}}$) the restrictions of $\{\ \ \}$ (resp. $\langle\ \ \rangle$) into V_{deg}, V_{non} respectively.

PROPOSITION 7.3. (1) *The objects $\mathscr{V}_{\text{deg}} = (V_{\text{deg}}, \{\ \ \}_{\text{deg}}, \langle\ \ \rangle_{\text{deg}})$, $\mathscr{V}_{\text{non}} = (V_{\text{non}}, \{\ \ \}_{\text{non}}, \langle\ \ \rangle_{\text{non}})$ are OJTS's and the OJTS $\mathscr{V}$ is equivalent to the sum $\mathscr{V}_{\text{deg}} \oplus \mathscr{V}_{\text{non}}$.*

(2) *The symmetric endomorphisms $L_{\mathscr{V}_{\text{deg}}}$, $L_{\mathscr{V}_{\text{non}}}$ are the restrictions of $L_{\mathscr{V}}$ into subspaces V_{deg}, V_{non} respectively. Hence,*

(*a*) $(L_{\mathscr{V}_{\text{deg}}})^\ell = 0$,

(*b*) *$L_{\mathscr{V}_{\text{non}}}$ is non-degenerate, and thus the JTS underlying $\mathscr{V}_{\text{non}}$ is non-degenerate.*

(3) *Let* $\mathscr{G}$, $\mathscr{G}_{\text{deg}}$, $\mathscr{G}_{\text{non}}$ *be the OSGLA's associated* with $\mathscr{V}$, $\mathscr{V}_{\text{deg}}$, $\mathscr{V}_{\text{non}}$ *respectively. Then* $\mathscr{G}$ *is equivalent to the sum* $\mathscr{G}_{\text{deg}} \oplus \mathscr{G}_{\text{non}}$. *Moreover the Lie algebra* $\mathfrak{g}_{\text{deg}}$ *underlying* $\mathscr{G}_{\text{deg}}$ *is not semi-simple and the Lie algebra* $\mathfrak{g}_{\text{non}}$ *underlying* $\mathscr{G}_{\text{non}}$ *is semi-simple.*

(4) *Let* $\mathscr{V}$, $\mathscr{V}'$ *be OJTS's and* $\mathscr{G}$, $\mathscr{G}'$ *the OSGLA's associated with* $\mathscr{V}$, $\mathscr{V}'$ *respectively. Then the following four statements are equivalent to one another*:

(a) $\mathscr{V}$ *is equivalent to* $\mathscr{V}'$.

(b) $\mathscr{V}_{\text{deg}}$, $\mathscr{V}_{\text{non}}$ *are equivalent to* $\mathscr{V}'_{\text{deg}}$, $\mathscr{V}'_{\text{non}}$ *respectively.*

(c) $\mathscr{G}$ *is equivalent to* $\mathscr{G}'$.

(d) $\mathscr{G}_{\text{deg}}$, $\mathscr{G}_{\text{non}}$ *are equivalent to* $\mathscr{G}'_{\text{deg}}$, $\mathscr{G}'_{\text{non}}$ *respectively.*

Proof. (1) Note that

$$L(V, V)V_{\text{deg}} \subset V_{\text{deg}}, \; L(V, V)V_{\text{non}} \subset V_{\text{non}} \tag{7.3}$$

by (7.2), and moreover

$$L(V_{\text{deg}}, V)V_{\text{non}} = L(V_{\text{non}}, V)V_{\text{deg}} = \{0\} \tag{7.4}$$

by (7.3), (JT 1). Since $L(L_{\gamma}(X), Y) = L(X, L_{\gamma}(Y))$ for $X, Y \in V$ by (7.2), (JT 2), we have

$$L(V_{\text{deg}}, V_{\text{non}}) = L(V_{\text{non}}, V_{\text{deg}}) = \{0\}\,. \tag{7.5}$$

Hence, by (7.3) ~ (7.5), the objects $(V_{\text{deg}}, \{\;\}_{\text{deg}})$, $(V_{\text{non}}, \{\;\}_{\text{non}})$ are JTS's and the JTS $(V, \{\;\})$ is decomposed into the sum $(V_{\text{deg}}, \{\;\}_{\text{deg}}) \oplus (V_{\text{non}}, \{\;\}_{\text{non}})$. Note that L_{γ} is non-degenerate on V_{non}. Then, by Lemma 7.2, the objects $\mathscr{V}_{\text{deg}}$, $\mathscr{V}_{\text{non}}$ are OJTS's and the OJTS $\mathscr{V}$ is equivalent to the sum $\mathscr{V}_{\text{deg}} \oplus \mathscr{V}_{\text{non}}$.

The claim (2) is clear by (7.1), (7.4) and the claim (3) by Proposition 7.1, (1) and the claims (1), (2).

(4) We show that (a) $\Rightarrow$ (b). Assume that $\mathscr{V}$ is equivalent to $\mathscr{V}'$ by an isomorphism $g: \mathscr{V} \rightleftarrows \mathscr{V}'$. Then we have $L_{\gamma'} = g \circ L_{\gamma} \circ g^{-1}$ by (7.1). Hence the restriction of g into V_{deg} (resp. V_{non}) is an isomorphism of $\mathscr{V}_{\text{deg}}$ (resp. $\mathscr{V}_{\text{non}}$) onto $\mathscr{V}'_{\text{deg}}$ (resp. $\mathscr{V}'_{\text{non}}$).

The claims that (b) $\Rightarrow$ (d), (c) $\Rightarrow$ (a) are obvious by Theorem 5.4, (2) and the claim that (d) $\Rightarrow$ (c) by the claim (3). q.e.d.

Let $\mathscr{A} = (A, \langle\;\rangle)$ be an OJA with the unity E and $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu$, ρ_A, $J_{\mathfrak{p}_A}$, $\langle\;\rangle_{\mathfrak{p}_A})$ the HSGLA coming from $\mathscr{A}$. Denote by $\mathscr{V}(\mathscr{A})$ the OJTS $(V_A, \{\;\}_A, \langle\;\rangle)$ associated with $\mathscr{A}$ and by $\mathscr{O}(\mathscr{G})$ the OSGLA $(\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu$, ρ_A, $\langle\;\rangle_{\mathfrak{p}_A})$ underlying $\mathscr{G}$. Then the OSGLA $\mathscr{O}(\mathscr{G})$ is associated with the

OJTS $\mathscr{V}(\mathscr{A})$. Regard $(V_A)_{\rm deg}$, $(V_A)_{\rm non}$ as subspaces of A and denote them by $A_{\rm deg}$, $A_{\rm non}$ respectively.

PROPOSITION 7.4. (1) *The objects* $\mathscr{A}_{\rm deg} = (A_{\rm deg}, \langle\ \rangle_{\rm deg})$, $\mathscr{A}_{\rm non} = (A_{\rm non}, \langle\ \rangle_{\rm non})$ *are OJA's with unity and the OJA* $\mathscr{A}$ *is decomposed into the sum* $\mathscr{A}_{\rm deg} \oplus \mathscr{A}_{\rm non}$. *Hence the HSGLA* $\mathscr{G}$ *is equivalent to the sum* $\mathscr{G}_{\rm deg} \oplus \mathscr{G}_{\rm non}$ *of the HSGLA's* $\mathscr{G}_{\rm deg}$, $\mathscr{G}_{\rm non}$ *coming from* $\mathscr{A}_{\rm deg}$, $\mathscr{A}_{\rm non}$ *respectively.*

(2) *The OJTS's* $\mathscr{V}(\mathscr{A}_{\rm deg})$, $\mathscr{V}(\mathscr{A}_{\rm non})$ *are equivalent to the OJTS's* $\mathscr{V}(\mathscr{A})_{\rm deg}$, $\mathscr{V}(\mathscr{A})_{\rm non}$ *respectively. Hence the OSGLA's* $\mathscr{O}(\mathscr{G}_{\rm deg})$, $\mathscr{O}(\mathscr{G}_{\rm non})$ *are equivalent to the OSGLA's* $\mathscr{O}(\mathscr{G})_{\rm deg}$, $\mathscr{O}(\mathscr{G})_{\rm non}$ *respectively.*

(3) *Let* $\mathscr{A}$, $\mathscr{A}'$ *be OJA's with unity and* $\mathscr{G}$, $\mathscr{G}'$ *the HSGLA's coming from* $\mathscr{A}$, $\mathscr{A}'$ *respectively. Then the following four statements are equivalent to one another:*

(a) $\mathscr{A}$ *is equivalent to* $\mathscr{A}'$.

(b) $\mathscr{A}_{\rm deg}$, $\mathscr{A}_{\rm non}$ *are equivalent to* $\mathscr{A}'_{\rm deg}$, $\mathscr{A}'_{\rm non}$ *respectively.*

(c) $\mathscr{G}$ *is equivalent to* $\mathscr{G}'$.

(d) $\mathscr{G}_{\rm deg}$, $\mathscr{G}_{\rm non}$ *are equivalent to* $\mathscr{G}'_{\rm deg}$, $\mathscr{G}'_{\rm non}$ *respectively.*

Proof. (1) The OJTS $\mathscr{V}(\mathscr{A})$ is decomposed into the sum $\mathscr{V}(\mathscr{A})_{\rm deg} \oplus \mathscr{V}(\mathscr{A})_{\rm non}$ by Proposition 7.3, (1). Note that the Jordan product on A is given by $X\cdot Y = \{X, Y, E\}_A$. Then the objects $\mathscr{A}_{\rm deg}$, $\mathscr{A}_{\rm non}$ are OJA's and the OJA $\mathscr{A}$ is decomposed into the sum $\mathscr{A}_{\rm deg} \oplus \mathscr{A}_{\rm non}$. Since $\mathscr{A}$ has the unity, $\mathscr{A}_{\rm deg}$, $\mathscr{A}_{\rm non}$ also have the unities by Proposition 7.1, (2), (a).

(2) Note that $\{X, Y, Z\}_A = T_{X\cdot Y}(Z) + [T_X, T_Y](Z)$ for $X, Y, Z \in A$. Then our claim is obvious by a routine way.

(3) We show that (a) $\Rightarrow$ (b). Assume that $\mathscr{A} = (A, \langle\ \rangle)$ is equivalent to $\mathscr{A}' = (A', \langle\ \rangle')$ by an isomorphism α: $\mathscr{A} \rightleftarrows \mathscr{A}'$. The isomorphism α induces the isomorphism g_α of the OJTS $\mathscr{V}(\mathscr{A})$ onto the OJTS $\mathscr{V}(\mathscr{A}')$. Then g_α translates subspaces $(V_A)_{\rm deg}$, $(V_A)_{\rm non}$ to subspaces $(V_{A'})_{\rm deg}$, $(V_{A'})_{\rm non}$ respectively. Hence α translates JA's $A_{\rm deg}$, $A_{\rm non}$ to JA's $A'_{\rm deg}$, $A'_{\rm non}$ respectively. This implies that $\mathscr{A}_{\rm deg}$, $\mathscr{A}_{\rm non}$ are equivalent to $\mathscr{A}'_{\rm deg}$, $\mathscr{A}'_{\rm non}$ respectively.

The claims that (b) $\Rightarrow$ (d), (c) $\Rightarrow$ (a) are obvious by Theorem 5.5, (2) and the claim that (d) $\Rightarrow$ (c) by the claim (1). q.e.d.

Let $\mathscr{A}$ be an OJA with unity and $\mathscr{G}$ the HSGLA coming from $\mathscr{A}$. Denote the OJA $\mathscr{A}_{\rm non}$, the HSGLA $\mathscr{G}_{\rm non}$ by $\mathscr{A}_{\rm non} = (A_{\rm non}, \langle\ \rangle_{\rm non})$, $\mathscr{G}_{\rm non} = (\mathfrak{g}_{\rm non} = \sum (\mathfrak{g}_{\rm non})_\mu, \rho_{\rm non}, J_{\mathfrak{p}_{\rm non}}, \langle\ \rangle_{\mathfrak{p}_{\rm non}})$. Then the Lie algebra $\mathfrak{g}_{\rm non}$ is semi-simple by Propositions 7.3, (3) and 7.4, (2). Let $\mathfrak{g}_{\rm non} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of the Lie algebra $\mathfrak{g}_{\text{non}}$ into simple Lie algebras $\mathfrak{g}_i$. Define subspaces A_i, $1 \leqq i \leqq s$, of A_{non} by $A_i = \{X \in A_{\text{non}}; (X, 0, 0) \in \mathfrak{g}_i\}$ and denote by $\langle\ \rangle_i$, $1 \leqq i \leqq s$, the restrictions of $\langle\ \rangle_{\text{non}}$ into subspaces A_i respectively.

THEOREM 7.5. (1) *The objects $\mathscr{A}_i = (A_i, \langle\ \rangle_i)$, $1 \leqq i \leqq s$, are OJA's with unity and the OJA $\mathscr{A}_{\text{non}}$ is decomposed into the sum $\oplus \mathscr{A}_i$. Hence the OJA $\mathscr{A}$ is decomposed into the sum $\mathscr{A}_{\text{deg}} \oplus (\oplus \mathscr{A}_i)$.*

(2) *Let $\mathscr{G}_i$, $1 \leqq i \leqq s$, be the HSGLA's coming from $\mathscr{A}_i$ respectively. Then the Lie algebras underlying $\mathscr{G}_i$, $1 \leqq i \leqq s$, are isomorphic to $\mathfrak{g}_i$ respectively and the HSGLA $\mathscr{G}_{\text{non}}$ is equivalent to the sum $\oplus \mathscr{G}_i$. Hence the HSGLA $\mathscr{G}$ is equivalent to the sum $\mathscr{G}_{\text{deg}} \oplus (\oplus \mathscr{G}_i)$.*

(3) *Let $\mathscr{A}$, $\mathscr{A}'$ be OJA's with unity and $\mathscr{G}$, $\mathscr{G}'$ the HSGLA's coming from $\mathscr{A}$, $\mathscr{A}'$ respectively. Let $\mathscr{A}_{\text{deg}} \oplus (\bigoplus_{i=1}^s \mathscr{A}_i)$, $\mathscr{A}'_{\text{deg}} \oplus (\bigoplus_{i=1}^t \mathscr{A}'_i)$ be the decompositions of $\mathscr{A}$, $\mathscr{A}'$ respectively given in* (1) *and let $\mathscr{G}_{\text{deg}} \oplus (\bigoplus_{i=1}^s \mathscr{G}_i)$, $\mathscr{G}'_{\text{deg}} \oplus (\bigoplus_{i=1}^t \mathscr{G}'_i)$ be the decompositions of $\mathscr{G}$, $\mathscr{G}'$ respectively given in* (2). *Then the following four statements are equivalent to one another*:

(a) *$\mathscr{A}$ is equivalent to $\mathscr{A}'$.*

(b) *The object $(\mathscr{A}_{\text{deg}}, \mathscr{A}_1, \cdots, \mathscr{A}_s)$ is equivalent to the object $(\mathscr{A}'_{\text{deg}}, \mathscr{A}'_1, \cdots, \mathscr{A}'_t)$, i.e.,* (i) *$\mathscr{A}_{\text{deg}}$ is equivalent to $\mathscr{A}'_{\text{deg}}$, and* (ii) *$s = t$ and there exists a permutation Σ such that $\mathscr{A}'_{\Sigma(i)}$ is equivalent to $\mathscr{A}_i$ for any i.*

(c) *$\mathscr{G}$ is equivalent to $\mathscr{G}'$.*

(d) *The object $(\mathscr{G}_{\text{deg}}, \mathscr{G}_1, \cdots, \mathscr{G}_s)$ is equivalent to the object $(\mathscr{G}'_{\text{deg}}, \mathscr{G}'_1, \cdots, \mathscr{G}'_t)$, i.e.,* (i) *$\mathscr{G}_{\text{deg}}$ is equivalent to $\mathscr{G}'_{\text{deg}}$, and* (ii) *$s = t$ and there exists a permutation Σ such that $\mathscr{G}'_{\Sigma(i)}$ is equivalent to $\mathscr{G}_i$ for any i.*

Proof. (1) Let $\boldsymbol{\nu} = (0, -\mathrm{id}_{A_{\text{non}}}, 0) \in (\mathfrak{g}_{\text{non}})_0$. Then $(\mathfrak{g}_{\text{non}})_\mu$, $\mu = 0, \pm 1$, are characterized as eigen spaces of $\mathrm{ad}(\boldsymbol{\nu})$ for eigen values μ respectively. Hence, putting $(\mathfrak{g}_i)_\mu = \mathfrak{g}_i \cap (\mathfrak{g}_{\text{non}})_\mu$, $\mu = 0, \pm 1$, $1 \leqq i \leqq s$, we have

$$\mathfrak{g}_i = \textstyle\sum_\mu (\mathfrak{g}_i)_\mu \tag{7.6}$$

for $1 \leqq i \leqq s$ and

$$(\mathfrak{g}_{\text{non}})_\mu = \textstyle\bigoplus_{i=1}^s (\mathfrak{g}_i)_\mu \tag{7.7}$$

for $\mu = 0, \pm 1$. Since $(\mathfrak{g}_{\text{non}})_{-1} = (A_{\text{non}}, 0, 0)$ and $(\mathfrak{g}_i)_{-1} = (A_i, 0, 0)$, $1 \leqq i \leqq s$, the JA A_{non} is decomposed into the sum $\oplus A_i$ of linear subspaces A_i by (7.7). Let $X \in A_i$, $Y \in A_{\text{non}}$. Since $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}_{\text{non}}$, we have

$$(X \cdot Y, 0, 0) = [(0, T_Y, 0), (X, 0, 0)] \in \mathfrak{g}_i$$

and thus $X \cdot Y \in A_i$. This implies that A_i is an ideal of A. Hence the

JA A_{non} is decomposed into the sum $\oplus A_i$ of JA's A_i.

Let $\mathscr{V}(\mathscr{A}_{\mathrm{non}}) = (V_{\mathrm{non}}, \{\ \}_{\mathrm{non}}, \langle\ \rangle_{\mathrm{non}})$. Regard A_i, $1 \leqq i \leqq s$, as subspaces V_i of V_{non} respectively and denote by $\{\ \}_i$, $1 \leqq i \leqq s$, the restrictions of $\{\ \}_{\mathrm{non}}$ into subspaces V_i respectively. Then the JTS $(V_{\mathrm{non}}, \{\ \}_{\mathrm{non}})$ is decomposed into the sum $\oplus(V_i, \{\ \}_i)$ of JTS's $(V_i, \{\ \}_i)$ and moreover the symmetric endomorphism $L_{\mathscr{V}(\mathscr{A}_{\mathrm{non}})}$ is non-degenerate on V_{non} and thus on V_i, $1 \leqq i \leqq s$, by Proposition 7.3, (2), (b). Hence the restrictions $\langle\ \rangle_i$, $1 \leqq i \leqq s$, of $\langle\ \rangle_{\mathrm{non}}$ into subspaces V_i are non-degenerate by Lemma 7.2. This implies that the OJA $\mathscr{A}_{\mathrm{non}}$ is decomposed into the sum $\oplus \mathscr{A}_i$ of OJA's $\mathscr{A}_i$. Since $\mathscr{A}_{\mathrm{non}}$ has the unity, $\mathscr{A}_i$, $1 \leqq i \leqq s$, also have the unities.

(2) Let E_{non} be the unity of $\mathscr{A}_{\mathrm{non}}$ and put $\boldsymbol{\varepsilon}_{-1} = (E_{\mathrm{non}}, 0, 0) \in (\mathfrak{g}_{\mathrm{non}})_{-1}$, $\boldsymbol{\varepsilon}_1 = (0, 0, E_{\mathrm{non}}) \in (\mathfrak{g}_{\mathrm{non}})_1$. Then we have

$$\mathrm{ad}(\boldsymbol{\varepsilon}_{-1})^2(0, 0, X) = (1/2)(X, 0, 0)\,,$$
$$\mathrm{ad}(\boldsymbol{\varepsilon}_1)^2(X, 0, 0) = -(1/2)(0, 0, X)$$

for $X \in A_{\mathrm{non}}$. This implies that $(X, 0, 0) \in (\mathfrak{g}_i)_{-1}$ if and only if $(0, 0, X) \in (\mathfrak{g}_i)_1$. Since $(\mathfrak{g}_i)_{-1} = (A_i, 0, 0)$ by definition, we have $(\mathfrak{g}_i)_1 = (0, 0, A_i)$, and thus $(\mathfrak{g}_i)_0 = \{(0, L_{\mathrm{non}}(Z, W), 0); Z, W \in A_i\}$ by (7.6). Note that $L_i(Z, W)$, $Z, W \in A_i$, are the restrictions of $L_{\mathrm{non}}(Z, W)$ into A_i. Define a mapping λ_i of the Lie algebra underlying $\mathscr{G}_i$ onto the Lie algebra $\mathfrak{g}_i$ by

$$\lambda_i(X, L_i(Z, W), Y) = (X, L_{\mathrm{non}}(Z, W), Y)$$

for $X, Y, Z, W \in A_i$. Then we can easily see that λ_i is an isomorphism. Hence the Lie algebras underlying $\mathscr{G}_i$, $1 \leqq i \leqq s$, are isomorphic to the Lie algebras $\mathfrak{g}_i$ respectively.

The other claim is obvious by (1).

(3) The claim that (c) $\Rightarrow$ (d) is ovbious by Proposition 7.4, (3) and Shur's Lemma, the claim that (b) $\Rightarrow$ (a) by (1), and the claims that (d) $\Rightarrow$ (b), (a) $\Rightarrow$ (c) by Theorem 5.5, (2). q.e.d.

Remark 7.6. Let $\mathscr{G}$, $\mathscr{G}_{\mathrm{deg}}$, $\mathscr{G}_i$, $1 \leqq i \leqq s$, be as given in (2) and $\hat{M}^n_{\mathscr{G}} \subset \boldsymbol{H}^n_{\mathscr{G}}$, $\hat{M}^{n_0}_{\mathscr{G}_{\mathrm{deg}}} \subset \boldsymbol{H}^{n_0}_{\mathscr{G}_{\mathrm{deg}}}$, $\hat{M}^{n_i}_{\mathscr{G}_i} \subset \boldsymbol{H}^{n_i}_{\mathscr{G}_i}$, $1 \leqq i \leqq s$, the complete totally real parallel submanifolds associated with $\mathscr{G}$, $\mathscr{G}_{\mathrm{deg}}$, $\mathscr{G}_i$ respectively. Then $\hat{M}^n_{\mathscr{G}} \subset \boldsymbol{H}^n_{\mathscr{G}}$ is holomorphically and linearly congruent to the product:

$$\hat{M}^{n_0}_{\mathscr{G}_{\mathrm{deg}}} \times \hat{M}^{n_1}_{\mathscr{G}_1} \times \cdots \times \hat{M}^{n_s}_{\mathscr{G}_s} \subset \boldsymbol{H}^{n_0}_{\mathscr{G}_{\mathrm{deg}}} \times \boldsymbol{H}^{n_1}_{\mathscr{G}_1} \times \cdots \times \boldsymbol{H}^{n_s}_{\mathscr{G}_s}\,.$$

Remark 7.7. We give a geometric view for the symmetric endomorphism $L_{\mathscr{V}(\mathscr{A})}$, which plays an important role for the decompositions of $\mathscr{A}$,

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$\mathscr{G}$. Define an element $\eta_{\mathscr{A}} \in A$ by $\eta_{\mathscr{A}} = n \cdot L_{\mathscr{V}(\mathscr{A})}(E)$. Let $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ be an orthonormal basis of A. Then we have

$$n \cdot \eta_{\mathscr{A}} = -\sum_{i=1}^{k} e_i \cdot e_i + \sum_{j=k+1}^{n} e_j \cdot e_j \tag{7.8}$$

and thus

$$n \cdot T_{\eta_{\mathscr{A}}} = L_{\mathscr{V}(\mathscr{A})} \tag{7.9}$$

by (7.1). Put $\boldsymbol{\eta}_{\mathscr{A}} = (\eta_{\mathscr{A}}, 0, -\eta_{\mathscr{A}}) \in \mathfrak{p}_A$. Then the vector $-J_{\mathfrak{p}_A}(\boldsymbol{\eta}_{\mathscr{A}})$ is the mean curvature vector at $\boldsymbol{\nu}$ of the complete totally real parallel submanifold $\hat{M}^n_{\mathscr{G}} \subset \boldsymbol{H}^n_{\mathscr{G}}$ by (6.11).

Now we apply Theorem 7.5 for the classification of r-dimensional complete parallel submanifolds of $\overline{M}^r(c)$, $c \neq 0$.

LEMMA 7.8. *Let $\mathscr{A}$ be an OJA with unity and $\mathscr{G}$ the HSGLA coming from $\mathscr{A}$. Assume that the symmetric bilinear form underlying $\mathscr{A}$ is positive definite. Then the Lie algebra underlying $\mathscr{G}$ is semi-simple.*

Proof. Note that the pseudo-Hermitian space $\boldsymbol{H}^n_{\mathscr{G}}$ is isometric to the Euclidean space $\boldsymbol{R}^{2n}$ and thus that the parallel submanifold $\hat{M}^n_{\mathscr{G}}$ is contained in a sphere of $\boldsymbol{R}^{2n}$. Hence the Lie algebra underlying the HSGLA $\mathscr{G}_{\hat{M}_{\mathscr{G}}}$ associated with $\hat{M}^n_{\mathscr{G}}$ is semi-simple (Ferus [3], Takeuchi [12]). This implies that the Lie algebra underlying $\mathscr{G}$ is semi-simple, since $\mathscr{G}$ is equivalent to $\mathscr{G}_{\hat{M}_{\mathscr{G}}}$ by Remark 6.5. q.e.d.

In our categories, an object is called *decomposable* if it is decomposed into the sum of two proper objects, and is called *indecomposable* if not so.

LEMMA 7.9. *Let $\mathscr{A} = (A, \langle\ \ \rangle)$ be an OJA with unity and $\mathscr{G} = (\mathfrak{g}_A = \sum (\mathfrak{g}_A)_\mu, \rho_A, J_{\mathfrak{p}_A}, \langle\ \ \rangle_{\mathfrak{p}_A})$ the HGSLA coming from $\mathscr{A}$. Then,*

(1) *the following five objects are indecomposable if one of them is so: OJA $\mathscr{A}$, HSGLA $\mathscr{G}$, OJTS $\mathscr{V}(\mathscr{A})$, OSGLA $\mathscr{O}(\mathscr{G})$, OSLA underlying $\mathscr{G}$. Moreover,*

(2) *$\mathscr{G}$ is indecomposable if either of the following conditions is satisfied:*

(a) *The Lie algebra underlying $\mathscr{G}$ is simple.*

(b) *$L^{\ell}_{\mathscr{V}(\mathscr{A})} = 0$ for some ℓ and the signature of $\langle\ \ \rangle$ is $(1, n-1)$.*

Proof. (1) It is obvious by Propsition 7.1, (1) (resp. Proposition 7.1, (2)) that $\mathscr{V}(\mathscr{A})$ (resp. $\mathscr{A}$) is indecomposable if and only if $\mathscr{O}(\mathscr{G})$ (resp. $\mathscr{G}$) is so. Note that the complex structure $J_{\mathfrak{p}_A}$ is given by $J_{\mathfrak{p}_A} = \mathrm{ad}\,(\boldsymbol{J}) | \mathfrak{p}_A$ and

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that the Lie subalgebras $(\mathfrak{g}_A)_\mu$ of $\mathfrak{g}_A$ are characterized as eigen spaces of $\mathrm{ad}(\nu)$ for eigen values μ respectively. Then it is obvious that $\mathscr{G}$ is indecomposable if and only if $\mathcal{O}(\mathscr{G})$ is so if and only if the OSLA underlying $\mathscr{G}$ is so.

(2) It is obvious that $\mathscr{G}$ is indecomposable if (a). We show that $\mathscr{G}$ is indecomposable if (b). Assume that $\mathscr{G}$ is decomposed into the sum $\mathscr{G}_1 \oplus \mathscr{G}_2$ of proper HSGLA's $\mathscr{G}_i$. Then we may assume that $\mathscr{G}_i$, $i = 1, 2$, come from OJA's $\mathscr{A}_i = (A_i, \langle\ \ \rangle_i)$ with unity respectively, such that $\mathscr{A}$ is equivalent to the sum $\mathscr{A}_1 \oplus \mathscr{A}_2$ by Proposition 7.1, (2), (b) and (3). Since the signature of $\langle\ \ \rangle$ is $(1, n-1)$, either of $\langle\ \ \rangle_i$, $i = 1, 2$, is positive definite. Assume that $\langle\ \ \rangle_1$ is so. Then the Lie algebra underlying $\mathscr{G}_1$ is semi-simple by Lemma 7.8. Hence $L_{\mathscr{V}(\mathscr{A}_1)}$ is non-degenerate on A_1 by Theorem 5.4, (4). This contradicts that $L^{\ell}_{\mathscr{V}(\mathscr{A})} = 0$. Hence $\mathscr{G}$ is indecomposable.

q.e.d.

Denote by $\mathscr{A}_c$ an OJA with unity satisfying the conditions (E_c1), (E_c2) and $\mathscr{G}_c$ the HSGLA coming from $\mathscr{A}_c$. Then, by Lemma 7.9 and Theorem 7.5, (2), an HSGLA $\mathscr{G}_c$ is indecomposable if and only if it satisfies either of the following:

(a) The Lie algebra underlying $\mathscr{G}_c$ is simple.

(b) $L^{\ell}_{\mathscr{V}(\mathscr{A}_c)} = 0$ for some ℓ.

The HSGLA $\mathscr{G}_c$ is called *simple* if it satisfies (a) and is called *almost nilpotent* if it satisfies (b). Note that $c < 0$ for an almost nilpotent HSGLA $\mathscr{G}_c$.

Fix a real number $c \neq 0$ and an integer $r \geqq 0$. Let $\mathscr{J} = ((\mathscr{G}_0)_{c_0}, \cdots, (\mathscr{G}_s)_{c_s})$ be an object consisting of indecomposable HSGLA's $(\mathscr{G}_i)_{c_i}$, $0 \leqq i \leqq s$, such that (i) $\sum_{i=0}^{s} 1/c_i = 1/c$, and (ii) the signature of the symmetric bilinear form underlying $\oplus_{i=0}^{s} (\mathscr{A}_i)_{c_i}$ is $(1, r)$ or $(0, r+1)$ according as $c < 0$ or $c > 0$. Two objects $\mathscr{J} = ((\mathscr{G}_0)_{c_0}, \cdots, (\mathscr{G}_s)_{c_s})$, $\mathscr{J}' = ((\mathscr{G}'_0)_{c'_0}, \cdots, (\mathscr{G}'_t)_{c'_t})$ satisfying the conditions (i), (ii), are said to be *equivalent* to each other if $s = t$ and there exists a permutation Σ such that $(\mathscr{G}'_{\Sigma(i)})_{c'_{\Sigma(i)}}$ is equivalent to $(\mathscr{G}_i)_{c_i}$ for any i. Here we note that $c'_{\Sigma(i)} = c_i$ for any i.

Now let us define an object $\mathscr{J}_M$ associated with an r-dimensional complete totally real parallel submanifold M^r of $\overline{M}^r(c)$. Let $\mathscr{A}_{\hat{M}}$ (resp. $\mathscr{G}_{\hat{M}}$) be the OJA (resp. HSGLA) associated with the complete inverse $\hat{M}^{r+1}$ of M^r, and $(\mathscr{A}_{\hat{M}})_{\deg} \oplus (\oplus_{i=1}^{s} (\mathscr{A}_{\hat{M}})_i)$ (resp. $(\mathscr{G}_{\hat{M}})_{\deg} \oplus (\oplus_{i=1}^{s} (\mathscr{G}_{\hat{M}})_i)$) the decomposition of $\mathscr{A}_{\hat{M}}$ (resp. $\mathscr{G}_{\hat{M}}$) given in Theorem 7.5. Denote by $E_{\deg}$, E_i, $1 \leqq i \leqq s$, the unities of $(\mathscr{A}_{\hat{M}})_{\deg}$, $(\mathscr{A}_{\hat{M}})_i$ respectively and by $\langle\ \ \rangle_{\deg}$, $\langle\ \ \rangle_i$, $1 \leqq i \leqq s$, the non-degenerate symmetric bilinear forms underlying $(\mathscr{A}_{\hat{M}})_{\deg}$, $(\mathscr{A}_{\hat{M}})_i$ respectively. Define real numbers c_i, $0 \leqq i \leqq s$, by $\langle E_{\deg}, E_{\deg}\rangle_{\deg} = 4/c_0$, $\langle E_i, E_i\rangle_i = 4/c_i$. Then, since $\mathscr{A}_{\hat{M}}$ satisfies the conditions $(E_c 1)$, $(E_c 2)$ by Lemma 6.2, OJA's $(\mathscr{A}_{\hat{M}})_{\deg}$ and $(\mathscr{A}_{\hat{M}})_i$, $1 \leqq i \leqq s$, satisfy the conditions $(\mathrm{E}_{c_0} 1)$, $(\mathrm{E}_{c_0} 2)$ and $(\mathrm{E}_{c_i} 1)$, $(\mathrm{E}_{c_i} 2)$ respectively, and moreover the object $((\mathscr{G}_{\hat{M}})_{\deg}, (\mathscr{G}_{\hat{M}})_1, \cdots, (\mathscr{G}_{\hat{M}})_s)$ satisfies the conditions (i), (ii). We denote this object by $\mathscr{J}_M$.

Conversely, let $\mathscr{J} = ((\mathscr{G}_0)_{c_0}, \cdots, (\mathscr{G}_s)_{c_s})$ be an object satisfying the conditions (i), (ii) and put $\mathscr{G}_c = \bigoplus_{i=0}^{s} (\mathscr{G}_i)_{c_i}$. Then $\mathscr{G}_c$ is the HSGLA coming from an OJA $\mathscr{A}_c$ with unity satisfying the conditions $(\mathrm{E}_c 1)$, $(\mathrm{E}_c 2)$. For simplicity, denote by $M^r_{\mathscr{J}} \subset \overline{M}^r_{\mathscr{J}}(c)$ the complete totally real parallel submanifold $M^r_{\mathscr{G}_c} \subset \overline{M}^r_{\mathscr{G}_c}(c)$ associated with the HSGLA $\mathscr{G}_c$.

THEOREM 7.10. (1) *Let M^r be an r-dimensional complete totally real parallel submanifold of $\overline{M}^r(c)$, $c \neq 0$. Then $M^r_{\mathscr{J}_M} \subset \overline{M}^r_{\mathscr{J}_M}(c)$ is holomorphically congruent to $M^r \subset \overline{M}^r(c)$.*

(2) *Let $\mathscr{J}$ is an object satisfying the conditions* (i), (ii). *Then $\mathscr{J}_{M^r_{\mathscr{J}}}$ is equivalent to $\mathscr{J}$.*

(3) *Let $\mathscr{J}$, $\mathscr{J}'$ be objects satisfying the conditions* (i), (ii). *Then $\mathscr{J}$ is equivalent to $\mathscr{J}'$ if and only if $M^r_{\mathscr{J}} \subset \overline{M}^r_{\mathscr{J}}(c)$ is holomorphically congruent to $M^r_{\mathscr{J}'} \subset \overline{M}^r_{\mathscr{J}'}(c)$.*

Proof. The claims (1), (2) are obvious by Theorem 6.3, (1), (2) respectively and the claim (3) by Theorem 6.3, (3) and Theorem 7.5, (3).

q.e.d.

Remark 7.11. Let $\mathscr{J} = ((\mathscr{G}_0)_{c_0}, \cdots, (\mathscr{G}_s)_{c_s})$ be an object satisfying the conditions (i), (ii). Assume that $c > 0$. Then all $(\mathscr{G}_i)_{c_i}$ are simple and all c_i are positive. Hence Theorem 7.10 is a reproduction of Theorem 3.1.

Assume that $c < 0$. Then there exists an index i such that $(\mathscr{G}_i)_{c_i}$ is simple or almost nilpotent with $c_i < 0$ and that $(\mathscr{G}_j)_{c_j}$, $j \neq i$, are simple with $c_j > 0$.

§ 8. Almost nilpotent HSGLA's $\mathscr{G}_c$

Let $\mathscr{A}_c$ be an OJA with unity satisfying the conditions $(\mathrm{E}_c 1)$, $(\mathrm{E}_c 2)$ and $\mathscr{G}_c$ the HSGLA coming from $\mathscr{A}_c$. If $\mathscr{G}_c$ is almost nilpotent, the Lie algebra underlying $\mathscr{G}_c$ is neither semi-simple nor solvable by Proposition 7.3, (3), the condition (SGL 4) and the property that $\mathrm{id} \in (\mathfrak{g}_c)_0$.

Now Cahen-Parker [2] has studied indecomposable effective OSLA's

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and HSLA's such that the Lie algebras underlying them are neither semi-simple nor solvable. We pick up some results which we need in this section.

Let $(\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$ be an OSLA. If the OSLA is effective and satisfies that $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$, the form $\langle\ \rangle_{\mathfrak{p}}$ on $\mathfrak{p}$ is uniquely extended into a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which is left invariant by ρ and which $\mathrm{ad}(T)$, $T \in \mathfrak{g}$, are skew symmetric for. We denote by $\langle\ \rangle$ the bilinear form on $\mathfrak{g}$.

Let $(\mathfrak{g}, \rho)$ be an SLA. A subspace $\mathfrak{s}$ of $\mathfrak{g}$ is called *ρ-invariant* if $\rho(\mathfrak{s}) = \mathfrak{s}$. A ρ-invariant subspace $\mathfrak{s}$ is decomposed into the sum of $\mathfrak{s} \cap \mathfrak{k}$, $\mathfrak{s} \cap \mathfrak{p}$. These subspaces $\mathfrak{s} \cap \mathfrak{k}$, $\mathfrak{s} \cap \mathfrak{p}$ are denoted by $\mathfrak{s}_k$, $\mathfrak{s}_p$ respectively. A Levi decomposition of $\mathfrak{g}$ into the sum of radical $\mathscr{R}$ and semi-simple subalgebra $\mathscr{S}$ is called *ρ-invariant* if $\rho(\mathscr{S}) = \mathscr{S}$. There always exists a ρ-invariant Levi decomposition of $\mathfrak{g}$.

LEMMA 8.1 (Cahen-Parker [2]). *Let $\mathscr{G} = (\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$ be an indecomposable effective OSLA satisfying that*

(0) $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$,

(i) *$\mathfrak{g}$ is neither semi-simple nor solvable,*

(ii) *$\mathscr{G}$ underlies an HSLA,*

(iii) *the signature of $\langle\ \rangle_{\mathfrak{p}}$ is $(2, 2r)$, $r \geqq 1$.*

Let $\mathfrak{g} = \mathscr{R} \oplus \mathscr{S}$ be a ρ-invariant Levi decomposition. Then,

(1) *the radical $\mathscr{R}$ is nilpotent and* $\dim \mathscr{S} = 3$,

(2) $C_{\mathfrak{k}}(\mathscr{S}) = \{X \in \mathfrak{k};\ [X, \mathscr{S}] = \{0\}\} = \{0\}$.

Moreover,

(3) *the nilpotent radical $\mathscr{R}$ is decomposed into the sum $\mathscr{V} \oplus \mathscr{S}'$ of mutually orthogonal ρ-invariant $\mathscr{S}$-modules $\mathscr{V}$, $\mathscr{S}'$ such that*

(a) *$\mathscr{S}'$ is the center of $\mathscr{R}$ and* $\dim \mathscr{S}' = 3$,

(b) *$\mathscr{V}$ is orthogonal to $\mathscr{S}$,*

(c) $[\mathscr{V}, \mathscr{V}] \subset \mathscr{S}'$,

(d) $\langle [\mathscr{S}_{\mathfrak{p}}, \mathscr{S}'_{\mathfrak{p}}], \mathscr{R}_{\mathfrak{k}} \rangle = \{0\}$.

Hereafter in this section we denote by $\mathscr{G}_c = (\mathfrak{g} = \sum \mathfrak{g}_\mu, \rho, J_{\mathfrak{p}}, \langle\ \rangle_{\mathfrak{p}})$ the HSGLA coming from an OJA $\mathscr{A}_c = (A, \langle\ \rangle)$ with the unity E satisfying the conditions $(\mathrm{E}_c 1)$, $(\mathrm{E}_c 2)$, and assume that $\mathscr{G}_c$ is almost nilpotent. Then we note that $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$ and $\dim A = r + 1 \geqq 2$. Since $c < 0$, the signature of $\langle\ \rangle$ is $(1, r)$ by $(\mathrm{E}_c 1)$ and thus that of $\langle\ \rangle_{\mathfrak{p}}$ is $(2, 2r)$. Hence the OSLA $(\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$ underlying $\mathscr{G}_c$ satisfies the conditions (0), (i), (ii), (iii). We apply Lemma 8.1 to this OSLA. Denote by $\mathscr{R}$ the radical of $\mathfrak{g}$ and put $\mathscr{S} = \{\boldsymbol{\varepsilon}_{-1} = (E, 0, 0), \boldsymbol{\varepsilon}_1 = (0, 0, E), \boldsymbol{\nu} = (0, -\mathrm{id}, 0)\}_{\boldsymbol{R}}$.

LEMMA 8.2. (1) *The subspace* $\mathscr{S}$ *is a* ρ*-invariant Lie subalgebra of* $\mathfrak{g}$ *isomorphic to* $sl(2, R)$ *and* $\mathfrak{g}$ *is the direct sum of* $\mathscr{R}, \mathscr{S}$, i.e., *the direct sum* $\mathfrak{g} = \mathscr{R} \oplus \mathscr{S}$ *is a* ρ*-invariant Levi decomposition.*

(2) $[T_X, T_Y] = 0$ *for* $X, Y \in A$.

Proof. (1) Put $e_1 = (E, 0, E)$, $e_2 = (E, 0, -E)$, $e_3 = \boldsymbol{\nu}$. Then we have $[e_1, e_2] = -e_3$, $[e_1, e_3] = e_2$, $[e_2, e_3] = e_1$. This implies that $\mathscr{S}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $sl(2, R)$. The claim that $\rho(\mathscr{S}) = \mathscr{S}$ is obvious by the definition of ρ and the claim that $\mathfrak{g} = \mathscr{R} \oplus \mathscr{S}$ by the simplicity of $\mathscr{S}$ and Lemma 8.1, (1).

(2) Since $(0, [T_X, T_Y], 0) \in \mathfrak{k}$ and $[(0, [T_X, T_Y], 0), \mathscr{S}] = \{0\}$, we have $(0, [T_X, T_Y], 0) \in C_{\mathfrak{k}}(\mathscr{S})$, and thus $[T_X, T_Y] = 0$ by Lemma 8.1, (2). q.e.d.

Let $\mathscr{R} = \mathscr{V} \oplus \mathscr{S}'$ be the decomposition of $\mathscr{R}$ given in Lemma 8.1, (3). Put

$$A_{\mathscr{S}} = \{E\}_{\boldsymbol{R}}\,, \qquad A_{\mathscr{R}} = \{X \in A;\ T_X \text{ is nilpotent.}\}\,,$$
$$A_{\mathscr{S}'} = \{\eta = \eta_{\mathscr{A}_c}\}_{\boldsymbol{R}}\,, \qquad A_{\mathscr{V}} = \{X \in A_{\mathscr{R}};\ \langle X, E\rangle = 0\}\,.$$

LEMMA 8.3. (1) *The subspaces* $\mathscr{S}, \mathscr{R}, \mathscr{S}', \mathscr{V}$ *are characterized by* $A_{\mathscr{S}}$, $A_{\mathscr{R}}$, $A_{\mathscr{S}'}$, $A_{\mathscr{V}}$ *respectively as follows*:

$$\mathscr{S} = \{(X, T_Y, Z) \in \mathfrak{g};\ X, Y, Z \in A_{\mathscr{S}}\}\,, \qquad \mathscr{R} = \{(X, T_Y, Z) \in \mathfrak{g};\ X, Y, Z \in A_{\mathscr{R}}\}\,,$$
$$\mathscr{S}' = \{(X, T_Y, Z) \in \mathfrak{g};\ X, Y, Z \in A_{\mathscr{S}'}\}\,, \qquad \mathscr{V} = \{(X, T_Y, Z) \in \mathfrak{g};\ X, Y, Z \in A_{\mathscr{V}}\}\,.$$

(2) $A = A_{\mathscr{S}} \oplus A_{\mathscr{R}}$, $A_{\mathscr{R}} = A_{\mathscr{V}} \oplus A_{\mathscr{S}'}$ *and* $\dim A_{\mathscr{S}} = 1$, $\dim A_{\mathscr{R}} = r$, $\dim A_{\mathscr{V}} = r - 1$, $\dim A_{\mathscr{S}'} = 1$.

Proof. It is obvious by the definition of $\mathscr{S}$ that $\mathscr{S}$ is characterized by $A_{\mathscr{S}}$ as above.

Recall that $\mathfrak{g}_\mu$, $\mu = 0, \pm 1$, are characterized as eigen spaces of $\mathrm{ad}(\boldsymbol{\nu})$ for eigen values μ respectively, and note that $\boldsymbol{\nu} \in \mathscr{S}$. Then, since $\mathscr{S}, \mathscr{R}$, $\mathscr{S}'$, $\mathscr{V}$ are $\mathscr{S}$-modules, we have

$$(8.1)\quad \begin{cases} \mathscr{R} = \sum_\mu \mathscr{R} \cap \mathfrak{g}_\mu\,, \quad \mathscr{S}' = \sum_\mu \mathscr{S}' \cap \mathfrak{g}_\mu\,, \quad \mathscr{V} = \sum_\mu \mathscr{V} \cap \mathfrak{g}_\mu\,, \\ \mathfrak{g}_\mu = \mathscr{S} \cap \mathfrak{g}_\mu \oplus \mathscr{R} \cap \mathfrak{g}_\mu\,, \quad \mathscr{R} \cap \mathfrak{g}_\mu = \mathscr{S}' \cap \mathfrak{g}_\mu \oplus \mathscr{V} \cap \mathfrak{g}_\mu\,, \quad \mu = 0, \pm 1\,. \end{cases}$$

Note that $\mathfrak{g}_0 = \{(0, T_X, 0);\ X \in A\}$ by Lemma 8.2, (2), and that $\mathrm{ad}(\boldsymbol{\varepsilon}_1)|\mathfrak{g}_{-1}$ (resp. $\mathrm{ad}(\boldsymbol{\varepsilon}_1)|\mathfrak{g}_0$) is a linear isomorphism of $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_0$) onto $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) given in the following.

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$$(8.2)\qquad \begin{cases} \operatorname{ad}(\varepsilon_1)(X, 0, 0) = (1/2)(0, T_X, 0) \\ (\text{resp. } \operatorname{ad}(\varepsilon_1)(0, T_X, 0) = (0, 0, X)) . \end{cases}$$

Define subspaces $\tilde{A}_{\mathscr{R}}$, $\tilde{A}_{\mathscr{S}'}$, $\tilde{A}_{\mathscr{V}}$ by $\mathscr{R} \cap \mathfrak{g}_{-1} = (\tilde{A}_{\mathscr{R}}, 0, 0)$, $\mathscr{S}' \cap \mathfrak{g}_{-1} = (\tilde{A}_{\mathscr{S}'}, 0, 0)$, $\mathscr{V} \cap \mathfrak{g}_{-1} = (\tilde{A}_{\mathscr{V}}, 0, 0)$. Then, since $\varepsilon_1 \in \mathscr{S}$, we have

$$\mathscr{R} = \{(X, T_Y, Z);\ X, Y, Z \in \tilde{A}_{\mathscr{R}}\}, \qquad \mathscr{S}' = \{(X, T_Y, Z);\ X, Y, Z \in \tilde{A}_{\mathscr{S}'}\},$$
$$\mathscr{V} = \{(X, T_Y, Z);\ X, Y, Z \in \tilde{A}_{\mathscr{V}}\}$$

by (8.1), (8.2). Here note that $\dim \tilde{A}_{\mathscr{R}} = r$, $\dim \tilde{A}_{\mathscr{S}'} = 1$, $\dim \tilde{A}_{\mathscr{V}} = r - 1$ since $\dim \mathscr{S} = \dim \mathscr{S}' = 3$ by Lemma 8.1.

To complete our proof, we may show that $\tilde{A}_{\mathscr{R}} = A_{\mathscr{R}}$, $\tilde{A}_{\mathscr{S}'} = A_{\mathscr{S}'}$, $\tilde{A}_{\mathscr{V}} = A_{\mathscr{V}}$.

Take an element $X \in \tilde{A}_{\mathscr{R}}$. Then we have $(0, T_X, 0) \in \mathscr{R}$. Since $\mathscr{R}$ is a 2-step nilpotent ideal by Lemma 8.1, we have

$$(T_X^3(Y), 0, 0) = \operatorname{ad}((0, T_X, 0))^3(Y, 0, 0) \in [\mathscr{R}, [\mathscr{R}, \mathscr{R}]] = \{0\}$$

for $Y \in A$ and thus $T_X^3 = 0$. This implies that $X \in A_{\mathscr{R}}$. Hence we have $\tilde{A}_{\mathscr{R}} \subset A_{\mathscr{R}}$. Note that $E \notin A_{\mathscr{R}}$ since $T_E = \mathrm{id}$ is not nilpotent, and thus that $\dim A_{\mathscr{R}} \leqq r$. Then we have $\tilde{A}_{\mathscr{R}} = A_{\mathscr{R}}$ since $\dim \tilde{A}_{\mathscr{R}} = r$.

Next we show that $\tilde{A}_{\mathscr{V}} = A_{\mathscr{V}}$. Note that $\dim A_{\mathscr{V}} \leqq r - 1$. In fact, we have $\eta \in A_{\mathscr{R}}$ since $(r+1)T_\eta = L_{\mathscr{V}(\mathscr{A}_C)}$ by (7.9). Moreover we have

$$(8.3)\qquad \begin{aligned} \langle \eta, E \rangle &= (1/(r+1))\langle -e_0 \cdot e_0 + \textstyle\sum_i e_i \cdot e_i, E \rangle \\ &= (1/(r+1))\{-\langle e_0, e_0 \rangle + \textstyle\sum_i \langle e_i, e_i \rangle\} = 1 \end{aligned}$$

by (7.8), where $\{e_0, e_1, \cdots, e_r\}$ denotes an orthonormal basis of A. Hence we have $\eta \notin A_{\mathscr{V}}$. This implies that $\dim A_{\mathscr{V}} \leqq r - 1$. Take an element $X \in \tilde{A}_{\mathscr{V}} \subset \tilde{A}_{\mathscr{R}} = A_{\mathscr{R}}$. Then we have $(X, 0, -X) \in \mathscr{V}$. Since $\mathscr{V}$ is orthogonal to $\mathscr{S}$ by Lemma 8.1, (3), (b), we have

$$\langle X, E \rangle = \langle (X, 0, -X), (E, 0, -E)_{\mathfrak{p}} \rangle = 0$$

and thus $X \in A_{\mathscr{V}}$. This implies that $\tilde{A}_{\mathscr{V}} \subset A_{\mathscr{V}}$. Since $\dim \tilde{A}_{\mathscr{V}} = r - 1$, we have $\tilde{A}_{\mathscr{V}} = A_{\mathscr{V}}$.

Finally we show that $\tilde{A}_{\mathscr{S}'} = A_{\mathscr{S}'}$. Note that

$$(8.4)\qquad \langle \tilde{A}_{\mathscr{S}'}, A_{\mathscr{R}} \rangle = \{0\} .$$

In fact, take $X \in \tilde{A}_{\mathscr{S}'}$, $Y \in A_{\mathscr{R}}$. Then we have $(0, T_X, 0) \in \mathscr{S}'_{\mathfrak{p}}$, $(Y, 0, Y) \in \mathscr{R}_{\mathfrak{k}}$. Hence we have

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$$0 = \langle [(E, 0, -E), (0, T_X, 0)], (Y, 0, Y)\rangle = -\langle (0, T_X, 0), [(E, 0, -E), (Y, 0, Y)]\rangle$$
$$= \langle (0, T_X, 0), (0, T_Y, 0)\rangle_{\mathfrak{p}} = \langle X, Y\rangle$$

by Lemma 8.1, (3), (d). This implies (8.4). Now, since $\langle E, \tilde{A}_{\mathscr{S}'}\rangle \neq \{0\}$, there exists an element $U \in A$ such that $\langle E, U\rangle = 0$, $E + U \in \tilde{A}_{\mathscr{S}'}$. Then we have

$$\langle U, A_{\mathscr{V}}\rangle = \{0\}\,, \quad \langle U, U\rangle = -(4/c)\,, \tag{8.5}$$

$$U \cdot U = -E - 2U\,. \tag{8.6}$$

In fact, we have

$$\{0\} = \langle E + U, A_{\mathscr{V}}\rangle = \langle U, A_{\mathscr{V}}\rangle$$

and

$$0 = \langle E + U, E + U\rangle = \langle E, E\rangle + \langle U, U\rangle = 4/c + \langle U, U\rangle$$

by (8.4). These imply (8.5). Since $\mathscr{S}'$ is the center of $\mathscr{R}$ by Lemma 8.1, (3), (a), we have

$$((E + U)\cdot(E + U), 0, 0) = [(0, T_{E+U}, 0), (E + U, 0, 0)] \in [\mathscr{S}', \mathscr{S}'] = \{0\}$$

and thus $(E + U)\cdot(E + U) = 0$. This implies (8.6). Put $e_0 = (\sqrt{-c}/2)E$, $e_1 = (\sqrt{-c}/2)U$ and let $\{e_2, \cdots, e_r\}$ be an orthonormal basis of $A_{\mathscr{V}}$. Then $\{e_0, e_1, \cdots, e_r\}$ is an orthonormal basis of A by (8.5). Note that

$$(e_i \cdot e_i, 0, 0) = [(0, T_{e_i}, 0), (e_i, 0, 0)] \in [\mathscr{V}, \mathscr{V}] \subset \mathscr{S}'$$

for $i \geqq 2$ by Lemma 8.1, (3), (c), and thus $e_i \cdot e_i \in \tilde{A}_{\mathscr{S}'}$, $i \geqq 2$. Then we have

$$\begin{aligned}(r + 1)\eta &= -e_0 \cdot e_0 + e_1 \cdot e_1 + \textstyle\sum_{i=2}^r e_i \cdot e_i \\ &= (c/2)(E + U) + \textstyle\sum_{i=2}^r e_i \cdot e_i \in \tilde{A}_{\mathscr{S}'}\end{aligned}$$

by (8.6). This implies that $A_{\mathscr{S}'} \subset \tilde{A}_{\mathscr{S}'}$. Since $\eta \neq 0$ and $\dim \tilde{A}_{\mathscr{S}'} = 1$, we have $A_{\mathscr{S}'} = \tilde{A}_{\mathscr{S}'}$. q.e.d.

LEMMA 8.4. *The Jordan product $\cdot$ on A and the non-degenerate symmetric bilinear form $\langle\ \ \rangle$ on A are given in the following.*

$$\begin{cases} E \cdot X = X \cdot E = X & \text{for } X \in A\,, \\ \eta \cdot Y = Y \cdot \eta = 0 & \text{for } Y \in A_{\mathscr{R}}\,, \\ Z \cdot W = W \cdot Z = \langle Z, W\rangle\eta & \text{for } Z, W \in A_{\mathscr{V}}\,, \end{cases} \tag{8.7}$$

and

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$$(8.8)\qquad \begin{cases} \langle E, E\rangle = 4/c\,, & \langle E, \eta\rangle = 1\,, \quad \langle E, A_{\mathscr{V}}\rangle = \{0\}\,, \\ \langle \eta, \eta\rangle = 0\,, & \langle \eta, A_{\mathscr{V}}\rangle = \{0\}\,. \end{cases}$$

Proof. Since $\mathscr{S}'$ is the center of $\mathscr{R}$, we have

$$(\eta\cdot Y, 0, 0) = [(0, T_\eta, 0), (Y, 0, 0)] \in [\mathscr{S}', \mathscr{R}] = \{0\}$$

for $Y \in A_{\mathscr{R}}$ and thus $\eta\cdot Y = 0$.

Let $Z, W \in A_{\mathscr{V}}$. Since $[\mathscr{V}, \mathscr{V}] \subset \mathscr{S}'$, we have

$$(Z\cdot W, 0, 0) = [(0, T_Z, 0), (W, 0, 0)] \in \mathscr{S}'$$

and thus $Z\cdot W \in A_{\mathscr{S}'}$. Put $Z\cdot W = \lambda(Z, W)\eta$. Then we have

$$\langle Z\cdot W, E\rangle = \lambda(Z, W)\langle \eta, E\rangle = \lambda(Z, W)$$

by (8.3) and thus $\lambda(Z, W) = \langle Z, W\rangle$. This implies that $Z\cdot W = \langle Z, W\rangle\eta$.

The other equations of (8.7) are obvious. In the equations of (8.8) the first is clear by (E_c 2), the second by (8.3), the third by the definition of $A_{\mathscr{V}}$, the forth and the fifth by (8.4). q.e.d.

Fix an integer $r \geqq 1$ and a negative number c. Let $\mathscr{P} = (P, \langle\ \ \rangle)$ be an object consisting of $(r-1)$-dimensional real vector space P and positive definite inner product $\langle\ \ \rangle$ on P. Let E, η be elements. Define an $(r+1)$-dimensional real vector space A by $A = \boldsymbol{R}\cdot E \oplus \boldsymbol{R}\cdot\eta \oplus P$. And define a product $\cdot$ on A by (8.7) and extend the inner product $\langle\ \ \rangle$ on P into a symmetric bilinear form on A by (8.8). This symmetric bilinear form on A will be also denoted by $\langle\ \ \rangle$. We denote this object $(A, \langle\ \ \rangle)$ by $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$.

THEOREM 8.5. (1) *The object $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$ is an OJA with the unity E satisfying the conditions* (E_c 1), (E_c 2), *such that $\eta_{\mathscr{A}(r,c,\mathscr{P},E,\eta)} = \eta$. Moreover the HSGLA $\mathscr{G}(r, c, \mathscr{P}, E, \eta)$ coming from $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$ is almost nilpotent.*

(2) *Let $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$, $\mathscr{A}(r', c', \mathscr{P}', E', \eta')$ be OJA's constructed as above. Then $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$ is equivalent to $\mathscr{A}(r', c', \mathscr{P}', E', \eta')$ if and only if $r = r'$ and $c = c'$.*

(3) *Let $\mathscr{A}_c = (A, \langle\ \ \rangle)$ be an OJA with the unity E satisfying the conditions* (E_c 1), (E_c 2). *Assume that the HSGLA $\mathscr{G}_c$ coming from $\mathscr{A}_c$ is almost nilpotent. Then $\mathscr{A}_c$ is equivalent to $\mathscr{A}(r, c, \mathscr{P}, E, \eta)$, where $r = \dim A - 1$ and $\mathscr{P} = (A_{\mathscr{V}}, \langle\ \ \rangle | A_{\mathscr{V}})$.*

(4) *Let $\mathscr{G}_c$ be as in* (3). *Then the submanifold $\hat{M}^{r+1}_{\mathscr{G}_c}$ is flat, and thus*

so is the submanifold $M^r_{\mathscr{G}_c}$.

Proof. The claim (1) is proved straightforwardly. The claim (2) is clear from the construction of objects and the claim (3) by Lemma 8.4.

We show the claim (4). Let $\mathscr{A}_{\hat{M}^{r+1}_{\mathscr{G}_c}}$ be the OJA associated with $\hat{M}^{r+1}_{\mathscr{G}_c}$. Then $\mathscr{A}_{\hat{M}^{r+1}_{\mathscr{G}_c}}$ is equivalent to $\mathscr{A}_c$ by Theorem 6.3, (2). Hence the curvature endomorphisms $\hat{R}(X, Y)$, $X, Y \in T_p(\hat{M}^{r+1}_{\mathscr{G}_c})$, of $\hat{M}^{r+1}_{\mathscr{G}_c}$ are identified with $[T_X, T_Y]$, $X, Y \in A$. Hence $\hat{M}^{r+1}_{\mathscr{G}_c}$ is flat by Lemma 8.2, (2). q.e.d.

§ 9. Simple HSGLA's $\mathscr{G}_c$

Let $\mathscr{A}_c = (A, \langle\ \ \rangle)$ be an OJA with the unity E satisfying the conditions (E_c1), (E_c2) and $\mathscr{G}_c = (\mathfrak{g} = \sum \mathfrak{g}_\mu, \rho, J_\mathfrak{p}, \langle\ \ \rangle_\mathfrak{p})$ the HSGLA coming from $\mathscr{A}_c$. Assume that $\mathscr{G}_c$ is simple. Then $\mathfrak{g}$ is a simple Lie algebra of non-compact type.

Assume that $c > 0$. Then $\langle\ \ \rangle$ is positive definite by (E_c1), and thus so is $\langle\ \ \rangle_\mathfrak{p}$. This HSGLA $\mathscr{G}_c$ is constructed from an object (D, c) of irreducible symmetric bounded domain D of tube type and positive number c (See Naitoh-Takeuchi [10] for the construction).

In this section we study simple HSGLA's $\mathscr{G}_c$ such that $c < 0$. Such an HSGLA $\mathscr{G}_c$ has the following properties:

(1) $(\mathfrak{g}, \rho, J_\mathfrak{p}, \langle\ \ \rangle_\mathfrak{p})$ is an HSLA such that the signature of $\langle\ \ \rangle_\mathfrak{p}$ is $(2, 2r)$, $r \geqq 0$.

(2) $\mathfrak{g} = \sum \mathfrak{g}_\mu$ is a GLA such that $\mathfrak{g}_\mu$, $\mu = 0, \pm 1$, are eigen spaces of $\mathrm{ad}(\boldsymbol{\nu})$ for eigen values μ respectively.

(3) $2 \dim \mathfrak{g}_{-1} = \dim \mathfrak{p}$.

Now Berger [1] has classified SLA's $(\mathfrak{g}, \rho)$ with simple Lie algebra $\mathfrak{g}$ of non-compact type and moreover has pointed out SLA's underlying HSLA's among them. Let $(\mathfrak{g}, \rho)$ be an SLA with simple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ by the involution ρ. Then the restriction $B_\mathfrak{p}$ of the Killing form B of $\mathfrak{g}$ is a non-degenerate symmetric bilinear form on $\mathfrak{p}$ (See Berger [1]). We list up Lie algebras $\mathfrak{g}$, Lie subalgebras $\mathfrak{k}$, and signatures of $B_\mathfrak{p}$ for SLA's $(\mathfrak{g}, \rho)$ underlying HSLA's in the following, (Table I).

Now let $\mathfrak{g}$ be a simple Lie algebra of non-compact type and e an element of $\mathfrak{g}$ such that the eigen values of $\mathrm{ad}(e)$ are $0, \pm 1$. Kobayashi-Nagano [6] has classified all such pairs $(\mathfrak{g}, e)$. Denote by $\mathfrak{g}_{-1}$ the (-1)-eigen space of $\mathrm{ad}(e)$. We list up Lie algebras $\mathfrak{g}$ and $\dim \mathfrak{g}_{-1}$ for pairs $(\mathfrak{g}, e)$ in the following, (Table II).

Table I.

$\mathfrak{g}$	$\mathfrak{k}$	signature of $B_{\mathfrak{p}}$
$\mathfrak{sl}(k, \boldsymbol{C})$	$\mathfrak{sl}(j, \boldsymbol{C}) \oplus \mathfrak{sl}(k-j, \boldsymbol{C}) \oplus \boldsymbol{C}$	$(2j(k-j), 2j(k-j))$
$\mathfrak{sl}(2k, \boldsymbol{R})$	$\mathfrak{sl}(k, \boldsymbol{C}) \oplus \boldsymbol{R}$	$(k(k-1), k(k+1))$
$\mathfrak{su}^*(2k)$	$\mathfrak{sl}(k, \boldsymbol{C}) \oplus \boldsymbol{R}$	$(k(k+1), k(k-1))$
$\mathfrak{su}(p, q)$	$\mathfrak{su}(i, j) \oplus \mathfrak{su}(p-i, q-j) \oplus \boldsymbol{R}$	$(2i(p-i) + 2j(q-j), 2\{(p+q)(i+j)$ $-(i+j)^2 - i(p-i) - j(q-j)\})$
$\mathfrak{so}(2k, \boldsymbol{C})$	$\mathfrak{sl}(k, \boldsymbol{C}) \oplus \boldsymbol{C}$	$(k(k-1), k(k-1))$
$\mathfrak{so}(k, \boldsymbol{C})$	$\mathfrak{so}(k-2, \boldsymbol{C}) \oplus \boldsymbol{C}$	$(2k-4, 2k-4)$
$\mathfrak{so}^*(2k)$	$\mathfrak{su}(j, k-j) \oplus \boldsymbol{R}$	$(2j(k-j), k(k-1) - 2j(k-j))$
$\mathfrak{so}^*(2k)$	$\mathfrak{so}^*(2k-2) \oplus \boldsymbol{R}$	$(2k-2, 2k-2)$
$\mathfrak{so}(2p, 2q)$	$\mathfrak{su}(p, q) \oplus \boldsymbol{R}$	$(p^2 + q^2 - (p+q), 2pq)$
$\mathfrak{so}(p+2, q)$	$\mathfrak{so}(p, q) \oplus \boldsymbol{R}$	$(2p, 2q)$
$\mathfrak{sp}(k, \boldsymbol{C})$	$\mathfrak{sl}(k, \boldsymbol{C}) \oplus \boldsymbol{C}$	$(k(k+1), k(k+1))$
$\mathfrak{sp}(k, \boldsymbol{R})$	$\mathfrak{su}(j, k-j) \oplus \boldsymbol{R}$	$(2j(k-j), k(k+1) - 2j(k-j))$
$\mathfrak{sp}(p, q)$	$\mathfrak{su}(p, q) \oplus \boldsymbol{R}$	$(p^2 + q^2 + (p+q), 2pq)$
$E_6^{\boldsymbol{C}}$	$\mathfrak{so}(10, \boldsymbol{C}) \oplus \boldsymbol{C}$	$(32, 32)$
E_6^2	$\mathfrak{so}^*(10) \oplus \boldsymbol{R}$	$(12, 20)$
E_6^2	$\mathfrak{so}(4, 6) \oplus \boldsymbol{R}$	$(16, 16)$
E_6^3	$\mathfrak{so}(10) \oplus \boldsymbol{R}$	$(0, 32)$
E_6^3	$\mathfrak{so}(2, 8) \oplus \boldsymbol{R}$	$(16, 16)$
E_6^3	$\mathfrak{so}^*(10) \oplus \boldsymbol{R}$	$(20, 12)$
$E_7^{\boldsymbol{C}}$	$E_6^{\boldsymbol{C}} \oplus \boldsymbol{C}$	$(54, 54)$
E_7^1	$E_6^2 \oplus \boldsymbol{R}$	$(24, 30)$
E_7^2	$E_6^3 \oplus \boldsymbol{R}$	$(22, 32)$
E_7^2	$E_6^2 \oplus \boldsymbol{R}$	$(30, 24)$
E_7^3	$E_6 \oplus \boldsymbol{R}$	$(0, 54)$
E_7^3	$E_6^3 \oplus \boldsymbol{R}$	$(32, 22)$
E_7^3	$E_6^4 \oplus \boldsymbol{R}$	$(26, 28)$

Note: Lie algebras $\mathfrak{g}$ are not necessarily simple.

LEMMA 9.1. *Let $M^r_{\mathscr{G}_c} \subset \overline{M}^r_{\mathscr{G}_c}(c)$ be the complete totally real parallel submanifold associated with the simple HSGLA $\mathscr{G}_c$, $c < 0$. Then the following three cases are possible*:

(a) *The submanifold $M^r_{\mathscr{G}_c}$ is isometric to the real hyperbolic space* $\mathrm{SO}(1, r)/\mathrm{SO}(r)$ *of constant sectional curvature $c/4$ and is totaly geodesic in* $\overline{M}^r_{\mathscr{G}_c}(c)$.

(b) $r = 1$, i.e., $M^1_{\mathscr{G}_c}$ *is a curve in* $\overline{M}^1_{\mathscr{G}_c}(c)$.

(c) $r = 0$, i.e., $M^0_{\mathscr{G}_c} = \overline{M}^0_{\mathscr{G}_c}(c)$ *is a point.*

Table II.

$\mathfrak{g}$	$\dim \mathfrak{g}_{-1}$	$\mathfrak{g}$	$\dim \mathfrak{g}_{-1}$
$\mathfrak{sl}(p+q, \boldsymbol{R})$	pq	$\mathfrak{sp}(n, n)$	$2n^2+n$
$\mathfrak{sl}(p+q, \boldsymbol{C})$	$2pq$	$\mathfrak{sp}(n, \boldsymbol{R})$	$n(n+1)/2$
$\mathfrak{su}^*(2p+2q)$	$4pq$	$\mathfrak{sp}(n, \boldsymbol{C})$	n^2
$\mathfrak{so}(n, n)$	$n(n-1)/2$	$\mathfrak{sp}(n, n)$	$2n^2+n$
$\mathfrak{so}(2n, \boldsymbol{C})$	$n(n-1)$	E_6^1	16
$\mathfrak{so}^*(4n)$	$n(2n-1)$	E_6^C	32
$\mathfrak{so}(p+1, q+1)$	$p+q$	E_6^4	16
$\mathfrak{so}(p+q+2, \boldsymbol{C})$	$2(p+q)$	E_7^1	27
$\mathfrak{so}(n, n)$	$n(n-1)/2$	E_7^C	54
$\mathfrak{su}(n, n)$	n^2	E_7^3	27

Proof. Since $\mathscr{G}_c$ is effective and $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$, the form $\langle\ \ \rangle_{\mathfrak{p}}$ is extented into a non-degenerate symmetric bilinear form $\langle\ \ \rangle$ on $\mathfrak{g}$ which $\mathrm{ad}(T)$, $T \in \mathfrak{g}$, are skew-symmetric for. Assume that the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ is complex simple. Then we have $\langle\ ,\ \rangle = \alpha B$ for some $\alpha \neq 0$ by Shur's lemma. Assume that $\mathfrak{g}^C$ is not complex simple. Then the real Lie algebra $\mathfrak{g}$ is isomorphic to the complexification $\mathfrak{h}^C$ of a compact simple Lie algebra $\mathfrak{h}$. Denote by $B_{\mathfrak{h}}$ the Killing form of $\mathfrak{h}$. Since $\mathrm{ad}(T)$, $T \in \mathfrak{g}$, are skew-symmetric for $\langle\ ,\ \rangle$, there exist real numbers a, b such that

$$\begin{aligned}\langle X+\sqrt{-1}\,Y, X'+\sqrt{-1}\,Y'\rangle = a\{B_{\mathfrak{h}}(X, X') - B_{\mathfrak{h}}(Y, Y')\} \\ + b\{B_{\mathfrak{h}}(X, Y') + B_{\mathfrak{h}}(X', Y)\}\end{aligned}$$

for $X, X', Y, Y' \in \mathfrak{h}$. The involution ρ commutes with a Cartan involution of $\mathfrak{g}$ (Berger [1]) and moreover the subalgebra $\mathfrak{k}$ is a complex Lie algebra for $\mathfrak{g}$ of this type contained in Table I. Hence we have $\mathfrak{p} = \mathfrak{h} \cap \mathfrak{p} \oplus \sqrt{-1}\,\mathfrak{h} \cap \mathfrak{p}$. Since $B_{\mathfrak{h}}$ is negative definite, the signature of $\langle\ \ \rangle_{\mathfrak{p}}$ is $(\dim \mathfrak{h} \cap \mathfrak{p}, \dim \mathfrak{h} \cap \mathfrak{p})$ by the above expression of $\langle\ ,\ \rangle$.

Now we pick up all possible cases satisfying the conditions (1), (2), (3) by using Table I, II. Then we can see that the Lie subalgebra $\mathfrak{k}$ is isomorphic to one of the following:

$$\mathfrak{so}(1, r) \oplus \boldsymbol{R} (r \geqq 2),\ \boldsymbol{C},\ \boldsymbol{R}.$$

The tangent space at ν of $\hat{M}^{r+1}_{\mathscr{G}_c}$ is identified with the subspace $\mathfrak{m} = \{(X, 0, X); X \in A\}$ of $\mathfrak{k}$. Then we note that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$. Put $p = \pi_{\mathscr{G}_c}(\nu) \in M^r_{\mathscr{G}_c}$ and $\tilde{\mathfrak{m}} = T_p(M^r_{\mathscr{G}_c})$. Denote by $\tilde{\mathfrak{k}}_0$ the holonomy algebra of curvature endomorphisms $R_p(X, Y)$, $X, Y \in \tilde{\mathfrak{m}}$. The direct sum $\tilde{\mathfrak{k}} = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{m}}$ is a Lie algebra with the following product [ ]:

$$[T, S] = T \circ S - S \circ T, \quad [T, X] = -[X, T] = T(X), \quad [X, Y] = -R_p(X, Y)$$

for $T, S \in \tilde{\mathfrak{k}}_0$, $X, Y \in \tilde{\mathfrak{m}}$. Then we can easily see that $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is isomorphic to $\tilde{\mathfrak{k}} \oplus \boldsymbol{R}$ and thus $\mathfrak{k}$ is isomorphic to $\tilde{\mathfrak{k}} \oplus \boldsymbol{R}$.

Assume that $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(1, r) \oplus \boldsymbol{R}$. Then $\tilde{\mathfrak{k}}$ is isomorphic to $\mathfrak{so}(1, r)$. Hence $M^r_{\mathscr{G}_c}$ is locally isometric to the riemannian symmetric space $SO(1, r)/SO(r)$ by Proposition 4.2. Then the submanifold $M^r_{\mathscr{G}_c}$ is totally geodesic in $\overline{M}^r_{\mathscr{G}_c}(c)$ (See the proof of Proposition 4.2). This implies that $M^r_{\mathscr{G}_c}$ is globally isometric to the real hyperbolic space of constant sectional curvature $c/4$.

Assume that $\mathfrak{k}$ is isomorphic to either $\boldsymbol{C}$ or $\boldsymbol{R}$. Then we have dim $\hat{M}^{r+1}_{\mathscr{G}_c} = 1, 2$ and thus $r = 0, 1$. q.e.d.

Now we study OJA's $\mathscr{A}_c$ and HSGLA's $\mathscr{G}_c$ for cases (a), (b), (c) respectively.

Case (a). Assume that $M^r_{\mathscr{G}_c}$ is totally geodesic in $\overline{M}^r_{\mathscr{G}_c}(c)$. Let $A = \{E\}_{\boldsymbol{R}} \oplus H$ be the orthogonal decomposition of A for $\langle \ \rangle$. Identify the tangent space $T_\nu(\hat{M}^{r+1}_{\mathscr{G}_c})$ with the subspace $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1) \cap \mathfrak{p}$ of $\mathfrak{p}$. Then, since $J_\mathfrak{p}(\boldsymbol{\nu}) = (E, 0, -E)$, the horizontal space $H_\nu(\hat{M}^{r+1}_{\mathscr{G}_c})$, the vertical space $V_\nu(\hat{M}^{r+1}_{\mathscr{G}_c})$ at $\boldsymbol{\nu} \in \hat{M}^{r+1}_{\mathscr{G}_c}$ are given by

$$H_\nu(\hat{M}^{r+1}_{\mathscr{G}_c}) = \{(X, 0, -X); X \in H\}, \qquad V_\nu(\hat{M}^{r+1}_{\mathscr{G}_c}) = \{(E, 0, -E)\}_{\boldsymbol{R}}$$

respectively. Denote by σ the second fundamental form at $\boldsymbol{\nu}$ of the submanifold $\hat{M}^{r+1}_{\mathscr{G}_c}$ in the pseudo-Hermitian space $F^{r+1}_{\mathscr{G}_c} = (\mathfrak{p}, J_\mathfrak{p}, \langle \ \rangle_\mathfrak{p})$. Then, since $M^r_{\mathscr{G}_c}$ is totally geodesic in $\overline{M}^r_{\mathscr{G}_c}(c)$, we have

$$\sigma((X, 0, -X), (Y, 0, -Y)) = -(c/4)\langle X, Y\rangle \boldsymbol{\nu}$$

for $X, Y \in H$ by (4.1), (4.6). On the other hand, the second fundamental form σ is also given by

$$\sigma((X, 0, -X), (Y, 0, -Y)) = (0, T_{X \cdot Y}, 0) \tag{9.1}$$

for $X, Y \in A$ (See the proof of Theorem 6.3, (1)). Hence we have $T_{X \cdot Y} = (c/4)\langle X, Y\rangle T_E$ for $X, Y \in H$. This implies that $X \cdot Y = (c/4)\langle X, Y\rangle E$ for $X, Y \in H$.

Fix an integer $r \geqq 1$ and a negative number c. Let $\mathscr{P} = (P, \langle \ \rangle)$ be an object of r-dimensional real vector space P and positive definite inner product $\langle\ \ \rangle$ on P. And let E be an element. Then put $A = \{E\}_R \oplus P$ and define a product $\cdot$ on A by

$$X \cdot Y = (c/4)\langle X, Y\rangle E, \qquad Z \cdot E = E \cdot Z = Z \tag{9.2}$$

for $X, Y \in P$, $Z \in A$ and moreover extend the inner product $\langle\ \ \rangle$ on P into a non-degenerate symmetric bilinear form $\langle\ \ \rangle$ on A by

$$\langle E, P\rangle = \{0\}, \qquad \langle E, E\rangle = 4/c. \tag{9.3}$$

Then the object $\mathscr{A}(r, c, \mathscr{P}, E) = (A, \langle\ \ \rangle)$ is an OJA with the unity E satisfying the conditions ($\mathrm{E}_c 1$), ($\mathrm{E}_c 2$). Moreover such two OJA's $\mathscr{A}(r, c, \mathscr{P}, E)$, $\mathscr{A}(r', c', \mathscr{P}', E')$ are equivalent to each other if and only if $r = r'$ and $c = c'$. Our OJA $\mathscr{A}_c$ is equivalent to $\mathscr{A}(r, c, \mathscr{H}, E)$, where $r = \dim A - 1$, $\mathscr{H} = (H, \langle\ \ \rangle|_H)$.

Denote by $\mathscr{G}(r, c, \mathscr{H}, E)$ the HSGLA coming from the OJA $\mathscr{A}(r, c, \mathscr{H}, E)$. Then $\mathscr{G}_c$ is equivalent to $\mathscr{G}(r, c, \mathscr{H}, E)$. We show that the Lie algebra $\mathfrak{g}$ underlying $\mathscr{G}(r, c, \mathscr{H}, E)$ is isomorphic to $\mathfrak{so}(r + 2, 1)$. Let $\{e_1, \cdots, e_r\}$ be an orthonormal basis of H and put $e_0 = (\sqrt{-c}/2)E$. Then $\{e_0, e_1, \cdots, e_r\}$ is an orthonormal basis of A. Define a matrix $\alpha(F) = (\alpha(F)_{ij}) \in M_{r+1}(\boldsymbol{R})$ for $F \in L$ by $F(e_i) = \sum_j \alpha(F)_{ij} e_j$, $0 \leqq i \leqq r$, and put

$$\beta(F) = \alpha(F) + \langle F(e_0), e_0\rangle 1_{r+1}$$

where 1_{r+1} denotes the unit matrix of degree $r + 1$. Then we can see that $\beta(F) \in \mathfrak{so}(r + 1)$ by (9.2). Put

$$\mathscr{X}(X) = \begin{pmatrix} x_0 \\ \vdots \\ \vdots \\ x_r \end{pmatrix}, \qquad \overline{\mathscr{X}(X)} = \begin{pmatrix} x_0 \\ -x_1 \\ \vdots \\ -x_r \end{pmatrix}$$

for $X = \sum_j x_j e_j \in A$. Define a linear mapping Φ of $\mathfrak{g}$ onto $\mathfrak{so}(r + 2, 1)$ by

$$\Phi(X, F, Y) = \left(\begin{array}{c|c|c} 0 & \overbrace{(\sqrt{-c}/4)({}^t\mathscr{X}(X) + {}^t\overline{\mathscr{X}(Y)})}^{r+1} & \langle F(e_0), e_0\rangle \\ \hline (\sqrt{-c}/4)(\mathscr{X}(X) + \overline{\mathscr{X}(Y)}) & -\beta(F) & (\sqrt{-c}/4)(\mathscr{X}(X) - \overline{\mathscr{X}(Y)}) \\ \hline \langle F(e_0), e_0\rangle & (\sqrt{-c}/4)({}^t\mathscr{X}(X) - {}^t\overline{\mathscr{X}(Y)}) & 0 \end{array}\right) \Bigg\} r+1$$

$$\in \mathfrak{so}(r + 2, 1)$$

for $(X, F, Y) \in \mathfrak{g}$. Then we can see that Φ is a Lie algebra isomorphism by (9.2). Hence $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(r + 2, 1)$.

Various objects underlying the HSGLA $\mathscr{G}(r, c, \mathscr{H}, E)$ may be expressed explicitly through the isomorphism Φ.

Case (b). Assume that $r = 1$, i.e., $M^1_{\mathscr{G}_c}$ is a curve in $\overline{M}^1_{\mathscr{G}_c}(c)$. Since the curve $M^1_{\mathscr{G}_c}$ is parallel in $\overline{M}^1_{\mathscr{G}_c}(c)$, it is a Frenet curve of osculating rank 0 or 1. Denote by $\kappa = \kappa_1 \geqq 0$ the Frenet curvature of $M^1_{\mathscr{G}_c}$ and by σ_M the second fundamental form of $M^1_{\mathscr{G}_c} \subset \overline{M}^1_{\mathscr{G}_c}(c)$. Then, since $M^1_{\mathscr{G}_c}$ is totally real, we have

$$\sigma_M(X, X) = \kappa |X| J_{\overline{M}}(X) \tag{9.4}$$

for a tangent vector field X of $M^1_{\mathscr{G}_c}$, where $J_{\overline{M}}$ denotes the complex structure of $\overline{M}^1_{\mathscr{G}_c}(c)$ and $|*|$ denotes the length of $*$. Let $A = \{E\}_R \oplus H$ be the orthogonal decomposition of A for $\langle \ \ \rangle$. Then the horizontal space $H_\nu(\hat{M}^2_{\mathscr{G}_c})$, the vertical space $V_\nu(\hat{M}^2_{\mathscr{G}_c})$ at ν are given as in the case (a). Denote by σ the second fundamental form at ν of $\hat{M}^2_{\mathscr{G}_c} \subset F^2_{\mathscr{G}_c}$. Then we have $\sigma(X, X) = (0, \kappa |X| T_X + (c/4)\langle X, X\rangle \,\mathrm{id}, 0)$ for $X \in H$ by (4.1), (4.6), (9.4). Hence we have $T_{X\cdot X} = \kappa |X| T_X + (c/4)\langle X, X\rangle T_E$ for $X \in H$ by (9.1). This implies that $X \cdot X = \kappa |X| X + (c/4)\langle X, X\rangle E$ for $X \in H$.

Fix a negative number c and a non-negative number κ. Let $\mathscr{P} = (P, \langle \ \ \rangle)$ be an object of 1-dimensional real vector space P and positive definite inner product $\langle \ \ \rangle$ on P. Then put $A = \{E\}_R \oplus P$ and define a product $\cdot$ on A by

$$X \cdot X = \kappa |X| X + (c/4)\langle X, X\rangle E, \qquad Y \cdot E = E \cdot Y = Y \tag{9.5}$$

for $X \in P$, $Y \in A$, and moreover extend the inner product $\langle \ \ \rangle$ on P into a non-degenerate symmetric bilinear form $\langle \ \ \rangle$ on A by

$$\langle E, P\rangle = \{0\}, \qquad \langle E, E\rangle = 4/c\,. \tag{9.6}$$

Then the object $\mathscr{A}(c, \kappa, \mathscr{P}, E) = (A, \langle \ \ \rangle)$ is an OJA with the unity E satisfying the conditions (E_c 1), (E_c 2). Moreover such two OJA's $\mathscr{A}(c, \kappa, \mathscr{P}, E)$, $\mathscr{A}(c', \kappa', \mathscr{P}', E')$ are equivalent to each other if and only if $c = c'$ and $\kappa = \kappa'$. Our OJA $\mathscr{A}_c$ is equivalent to $\mathscr{A}(c, \kappa, \mathscr{H}, E)$, where κ denotes the Frenet curvature of $M^1_{\mathscr{G}_c}$ and $\mathscr{H} = (H, \langle \ \ \rangle|_H)$.

Denote by $\mathscr{G}(c, \kappa, \mathscr{H}, E)$ the HSGLA coming from $\mathscr{A}(c, \kappa, \mathscr{H}, E)$ and by $\mathfrak{g}$ the Lie algebra underlying $\mathscr{G}(c, \kappa, \mathscr{H}, E)$. Let e_1 be a unit vector of H and put $e_0 = (\sqrt{-c}/2)E$. Then $\{e_0, e_1\}$ is an orthonormal basis of A. The matrix representation $D(\eta)$ of T_η for this basis is given by

$$D(\eta) = \begin{pmatrix} c/4 & \sqrt{-c}\kappa/4 \\ -\sqrt{-c}\kappa/4, & c/4 + \kappa^2/2 \end{pmatrix}$$

by (9.5), (9.6). Then we consider separately the following three cases (i), (ii), (iii).

(i) $\kappa = \sqrt{-c}$: In this case the eigen values of $D(\eta)$ are zero and thus T_η is nilpotent. This implies that the HSGLA $\mathscr{G}(c, \sqrt{-c}, \mathscr{H}, E)$ is equivalent to the almost nilpotent HSGLA $\mathscr{G}(2, c, \{0\}, E, \eta)$.

(ii) $\kappa > \sqrt{-c}$: In this case the matrix $D(\eta)$ has two distinct nonzero real eigen values λ_i, $i = 1, 2$. Let A_i, $i = 1, 2$, be eigen spaces of T_η for eigen values λ_i respectively. Since A is spanned over $\boldsymbol{R}$ by the vectors E, η, the subspaces A_i are ideals of A by (9.5), and thus A is decomposed into the sum $A_1 \oplus A_2$ of ideals A_i. Since T_η is non-degenerate on A, the decomposition of A induces a decomposition $\mathscr{A}_1 \oplus \mathscr{A}_2$ of the OJA $\mathscr{A}(c, \kappa, \mathscr{H}, E)$ by Lemma 7.2. Hence the HSGLA $\mathscr{G}(c, \kappa, \mathscr{H}, E)$ is decomposed into the sum $\mathscr{G}_1 \oplus \mathscr{G}_2$ of two simple HSGLA's $\mathscr{G}_i$. These HSGLA's $\mathscr{G}_i$ will be studied in the case (c).

(iii) $0 \leqq \kappa < \sqrt{-c}$: Define a complex structure j_κ on A by

$$\begin{cases} j_\kappa(e_0) = (1/\sqrt{-c-\kappa^2})(-\kappa e_0 + \sqrt{-c}\, e_1)\,, \\ j_\kappa(e_1) = (1/\sqrt{-c-\kappa^2})(-\sqrt{-c}\, e_0 + \kappa e_1)\,, \end{cases}$$

and denote by $\boldsymbol{C}_\kappa$ the 1-dimensional complex linear space (A, j_κ). Moreover identify $\boldsymbol{C}_\kappa$ with the complex space $\boldsymbol{C}$ by

$$\boldsymbol{C}_\kappa \ni x e_0 + y j_\kappa(e_0) \longleftrightarrow (\sqrt{-c}/2)(x + yi) \in \boldsymbol{C}\,.$$

Then the Jordan product $\cdot$ on $\boldsymbol{C}_\kappa$ corresponds with the canonical product $\cdot$ on $\boldsymbol{C}$ under this identification by (9.5). Hence the $\boldsymbol{R}$-linear endomorphisms $F \in L$ are $\boldsymbol{C}$-linear endomorphisms on $\boldsymbol{C}$. Let $\alpha(F)$ be a complex number such that $F(Z) = \alpha(F)\cdot Z$ for $Z \in \boldsymbol{C}$. Define a linear mapping Φ_κ of $\mathfrak{g}$ onto $\mathfrak{sl}(2, \boldsymbol{C})$ by

$$\Phi_\kappa((Z, F, W)) = \begin{pmatrix} -\alpha(F)/2 & -W/2 \\ Z/2 & \alpha(F)/2 \end{pmatrix}$$

for $(Z, F, W) \in \mathfrak{g}$. Then we can easily see that Φ_κ is a Lie algebra isomorphism. Hence $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, C)$.

Various objects underlying $\mathscr{G}(c, \kappa, \mathscr{H}, E)$ may be expressed explicitly through the isomorphism Φ_κ.

Finally we note that $\mathscr{G}(c, 0, \mathscr{H}, E)$ is equivalent to the HSGLA $\mathscr{G}(1, c, \mathscr{P}, E)$ in the case (a).

Case (c). Assume that $r = 0$. Then we have $A = \{E\}_R$ and $\langle E, E\rangle = 4/c$. Let $\mathscr{A}_{-c} = (A, -\langle\ \ \rangle)$ and $\mathscr{G}_{-c} = (\mathfrak{g} = \sum \mathfrak{g}_\mu, \rho, J_p, -\langle\ \ \rangle_p)$. Then $\mathscr{A}_{-c}$ (resp. $\mathscr{G}_{-c}$) is equivalent to the OJA (resp. HSGLA) corresponding to the object $(D^1, -c)$, where D^1 denotes the 1-dimensional symmetric bounded domain of tube type. Particularly, $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$. This OJA $\mathscr{A}_c$ (resp. HSGLA $\mathscr{G}_c$) will be denoted by $\mathscr{A}(c, E)$ (resp. $\mathscr{G}(c, E)$).

Summing up the above cases, we have the following

THEOREM 9.2. *Let $\mathscr{G}_c$ be the HSGLA coming from an OJA $\mathscr{A}_c$ with the unity E satisfying the conditions* (E_c1), (E_c2). *Assume that $c < 0$ and $\mathscr{G}_c$ is simple. Then $\mathscr{G}_c$ is equivalent to one of the following HSGLA's*:

$\mathscr{G}(r, c, \mathscr{P}, E)$ $(r \geqq 1)$ *in Case* (a), $\mathscr{G}(c, \kappa, \mathscr{P}, E)$ $(0 < \kappa < \sqrt{-c})$ *in Case* (b), (iii), *and* $\mathscr{G}(c, E)$ *in Case* (c).

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