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A CLASS OF ADDITION THEOREMS†

BY

H. M. SRIVASTAVA, J.-L. LAVOIE AND RICHARD TREMBLAY

ABSTRACT. Recently, H. M. Srivastava extended certain interesting generating functions of L. Carlitz to the forms:

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_r+\lambda_r n)}(x_1+ny_1,\ldots,x_s+ny_s)\frac{t^n}{n!},$$

where $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ and $\{g_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_s)\}_{n=0}^{\infty}$ are general one- and many-parameter sequences of functions. In the present paper some general addition formulas for analogous sequences of functions are derived, and a number of interesting applications of the main results are given.

1. Introduction and the main results. Motivated by Carlitz's generating functions for certain one- and two-parameter coefficients [1, p. 521, Theorem 1 and Equation (2.10)], Srivastava [4] has recently derived a class of generating functions for some general one- and many-parameter sequences of functions [*op. cit.*, p. 472, Equations (1.7) and (1.11)]. The object of the present paper is to give several general addition formulas for certain sequences of functions analogous to those considered by Srivastava [4]. Our main results are contained in the following

THEOREM. *Let $B(z)$ and $z^{-1}C(z)$ be arbitrary functions which are analytic in the neighborhood of the origin, and assume (for the sake of simplicity) that*

$$B(0)=C'(0)=1. \tag{1.1}$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ by means of

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)\frac{z^n}{n!}=[B(z)]^{\alpha}\exp(xC(z)), \tag{1.2}$$

where α and x are arbitrary complex numbers independent of z.

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Then, for arbitrary parameters λ *and* y,

$$f_n^{(\alpha+\lambda\gamma)}(x+\gamma y)=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}f_k^{(\alpha-\lambda k)}(x-ky)f_{n-k}^{(\lambda k+\lambda\gamma)}(ky+\gamma y), \tag{1.3}$$

provided that $\mathrm{Re}(\gamma)>0$.

More generally, let the functions $A(z)$, $B(z)$, $z^{-1}C(z)$ *and* $z^{-1}D(z)$ *be analytic about the origin such that*

$$A(0)=B(0)=C'(0)=D'(0)=1, \tag{1.4}$$

and define the sequence of functions $\{g_n^{(\alpha,\beta,\gamma)}(x,y,u,v)\}_{n=0}^{\infty}$ *by means of*

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta,\gamma)}(x,y,u,v)\frac{z^n}{n!}=[A(z)]^{\alpha}[B(z)]^{\beta}[\Omega(z)]^{\gamma}\exp(xC(z)+yD(z)), \tag{1.5}$$

where α, β, γ, x, y, u *and* v *are arbitrary complex numbers independent of* z, *and* (*for convenience*)

$$\Omega(z)=1+z\left(u\frac{B'(z)}{B(z)}+vC'(z)\right). \tag{1.6}$$

Then

$$\begin{aligned} &g_n^{(\alpha_1+\alpha_2,\beta_1+\beta_2-nu,\gamma_1+\gamma_2+1)}(x_1+x_2-nv,y_1+y_2,u,v)\\ &\quad=\sum_{k=0}^{n}\binom{n}{k}g_k^{(\alpha_1,\beta_1-ku,\gamma_1+1)}(x_1-kv,y_1,u,v)\\ &\qquad\cdot g_{n-k}^{(\alpha_2,\beta_2-(n-k)u,\gamma_2+1)}(x_2-(n-k)v,y_2,u,v).\end{aligned} \tag{1.7}$$

REMARK 1. The definitions (1.2) and (1.5) are essentially analogous to Srivastava's generating functions [4, p. 472, Equation (1.6) *with* $A(z)\equiv 1$] and [4, p. 472, Equation (1.10) *with* $r=s=2$], respectively.

REMARK 2. The choice of 1 in the conditions (1.1) and (1.4), as also in Srivastava's conditions [4, p. 471, Equation (1.5); p. 472, Equation (1.9)], is merely a convenient one; in fact, any nonzero constant values may be assumed for $A(0)$, $B(0)$, $C'(0)$, $D'(0)$, *et cetera.*

REMARK 3. The addition formula (1.3) does not seem to follow readily from the general result (1.7). Consequently, we present *independent* proofs of these addition formulas.

2. Proof of the addition formula (1.3). Making use of Srivastava's theorem [4, p. 472, Equation (1.7)] we find from the definition (1.2) that

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^n}{n!}=\frac{[B(\zeta)]^{\alpha}\exp(xC(\zeta))}{1-\zeta\{\lambda[B'(\zeta)/B(\zeta)]+yC'(\zeta)\}}, \tag{2.1}$$

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where

$$\zeta = t[B(\zeta)]^{\lambda} \exp(yC(\zeta)). \tag{2.2}$$

Rewriting (2.2) in the form

$$t = \zeta[B(\zeta)]^{-\lambda} \exp(-yC(\zeta)), \tag{2.3}$$

and differentiating both sides with respect to ζ, we get

$$\frac{dt}{d\zeta} = [B(\zeta)]^{-\lambda} \exp(-yC(\zeta))\left[1-\zeta\left\{\lambda\frac{B'(\zeta)}{B(\zeta)}+yC'(\zeta)\right\}\right]. \tag{2.4}$$

Now multiply both sides of (2.1) by $t^{\gamma}\,dt$ and apply (2.4), (2.3) and the definition (1.2) to the resulting right-hand side of (2.1), successively. We thus obtain

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^{\gamma+n}}{n!}\,dt = \sum_{n=0}^{\infty} f_n^{(\alpha-\lambda(\gamma+1))}(x-(\gamma+1)y)\frac{\zeta^{\gamma+n}}{n!}\,d\zeta. \tag{2.5}$$

Integrating both sides of (2.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)&\left.\frac{t^{\gamma+n+1}}{n!\,(\gamma+n+1)}\right|_0^t \\ &= \sum_{n=0}^{\infty} f_n^{(\alpha-\lambda(\gamma+1))}(x-(\gamma+1)y)\left.\frac{\zeta^{\gamma+n+1}}{n!\,(\gamma+n+1)}\right|_0^{\zeta},\end{aligned} \tag{2.6}$$

so that, if $\mathrm{Re}(\gamma) > -1$,

$$\begin{aligned}\sum_{n=0}^{\infty} f_n^{(\alpha-\lambda(\gamma+1))}&(x-(\gamma+1)y)\frac{\zeta^{\gamma+n+1}}{n!\,(\gamma+n+1)} = \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^{\gamma+n+1}}{n!\,(\gamma+n+1)} \\ &= \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{\zeta^{\gamma+n+1}}{n!\,(\gamma+n+1)}[B(\zeta)]^{-\lambda(\gamma+n+1)} \\ &\quad \times\exp(-y(\gamma+n+1)C(\zeta)), \quad \text{by (2.3)}, \\ &= \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\sum_{N=0}^{\infty} f_N^{(-\lambda(\gamma+n+1))}(-(\gamma+n+1)y)\frac{\zeta^{\gamma+N+n+1}}{N!\,n!\,(\gamma+n+1)}, \quad \text{by (1.2)}, \\ &= \sum_{N=0}^{\infty}\frac{\zeta^{\gamma+N+1}}{N!}\sum_{n=0}^{N}(\gamma+n+1)^{-1}\binom{N}{n} f_n^{(\alpha+\lambda n)}(x+ny) f_{N-n}^{(-\lambda(\gamma+n+1))}(-(\gamma+n+1)y),\end{aligned}$$

and the addition formula (1.3) follows upon equating the coefficients of $\zeta^{\gamma+N+1}$ (and upon introducing some notational changes for the sake of convenience).

3. Proof of the addition formula (1.7). By Taylor's theorem, (1.5) readily yields

$$g_n^{(\alpha,\beta,\gamma)}(x, y, u, v) = D_z^n\{[A(z)]^{\alpha}[B(z)]^{\beta}[\Omega(z)]^{\gamma} \exp(xC(z)+yD(z))\}|_{z=0}, \tag{3.1}$$

where, as usual, $D_z = d/dz$.

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Now we turn to the known result [3, p. 288, Equation (1.2)]

$$D^{\alpha}_{g(z)}\{f(z)\}=D^{\alpha}_{h(z)}\left\{\frac{f(z)g'(z)}{h'(z)}\left(\frac{h(z)-h(w)}{g(z)-g(w)}\right)^{\alpha+1}\right\}\bigg|_{w=z},\tag{3.2}$$

where $D^{\alpha}_{g(z)}\{f(z)\}$ denotes the *fractional derivative* (of *order* α) of $f(z)$ with respect to $g(z)$.

For $\alpha=n$ and $g(z)=z$, we obtain the following special case of interest to us:

$$D^{n}_{z}\{f(z)\}|_{z=0}=D^{n}_{h(z)}\left\{\frac{f(z)}{h'(z)}\left(\frac{h(z)}{z}\right)^{n+1}\right\}\bigg|_{z=0},\tag{3.3}$$

provided that $h(z)$ has a *simple* zero at the origin.

Letting

$$h(z)=z[B(z)]^{u}\exp(vC(z)),\tag{3.4}$$

so that

$$h'(z)=[B(z)]^{u}\Omega(z)\exp(vC(z)),\tag{3.5}$$

and from (3.3) we have

$$D^{n}_{z}\{f(z)\}|_{z=0}=D^{n}_{h(z)}\left\{\frac{f(z)}{\Omega(z)}[B(z)]^{nu}\exp(nvC(z))\right\}\bigg|_{z=0},\tag{3.6}$$

where $\Omega(z)$ and $h(z)$ are given by (1.6) and (3.4), respectively.

It follows from (3.1) and (3.6) that

$$\begin{aligned}g_n^{(\alpha,\beta,\gamma)}(x,y,u,v)=D^{n}_{h(z)}\{&[A(z)]^{\alpha}[B(z)]^{\beta+nu}[\Omega(z)]^{\gamma-1}\\ &\times\exp((x+nv)C(z)+yD(z))\}|_{z=0},\end{aligned}\tag{3.7}$$

where $h(z)$ is given, as before, by (3.4).

By setting

$$\alpha=\alpha_1+\alpha_2,\qquad \beta=\beta_1+\beta_2-nu,\qquad \gamma=\gamma_1+\gamma_2+1\tag{3.8}$$

and

$$x=x_1+x_2-nv,\qquad y=y_1+y_2,\tag{3.9}$$

and then using (3.3) in conjunction with Leibniz's formula for the nth derivative of the product of two functions, the addition formula (1.7) is readily obtained.

4. Applications to Bernoulli and Euler polynomials. For the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ defined by

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{z^n}{n!}=\left(\frac{z}{e^z-1}\right)^{\alpha}\exp(xz),\tag{4.1}$$

we have, upon comparing (4.1) with (1.2) and (1.5),

$$B(z)=z/(e^z-1),\qquad C(z)=z,\qquad f_n^{(\alpha)}(x)\to B_n^{(\alpha)}(x) \tag{4.2}$$

and

$$\begin{cases} A(z)=1,\qquad B(z)=z/(e^z-1),\qquad C(z)=z,\qquad y=0,\qquad u=v=-1,\\ \qquad g_n^{(\alpha,\beta,\gamma)}(x,0,-1,-1)\to B_n^{(\beta+\gamma)}(x), \end{cases} \tag{4.3}$$

respectively.

From (1.3) and (1.7) we thus obtain the addition formulas:

$$B_n^{(\alpha+\lambda\gamma)}(x+\gamma y)=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}B_k^{(\alpha-\lambda k)}(x-ky)B_{n-k}^{(\lambda k+\lambda\gamma)}(ky+\gamma y),\ \mathrm{Re}(\gamma)>0; \tag{4.4}$$

$$B_n^{(\alpha+\beta+n+1)}(x+y+n)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha+k+1)}(x+k)B_{n-k}^{(\beta+n-k+1)}(y+n-k). \tag{4.5}$$

Both (4.4) and (4.5) are believed to be new. As a matter of fact, they do not seem to follow from the familiar addition theorem:

$$B_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x)B_{n-k}^{(\beta)}(y), \tag{4.6}$$

which is an immediate consequence of the definition (4.1). Such addition theorems as (4.6) do indeed hold true for the sequences generated by (1.2) and (1.5), and we have

$$f_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}f_k^{(\alpha)}(x)f_{n-k}^{(\beta)}(y) \tag{4.7}$$

and

$$\begin{aligned} &g_n^{(\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2)}(x_1+x_2,\,y_1+y_2,\,u,\,v)\\ &\qquad=\sum_{k=0}^{n}\binom{n}{k}g_k^{(\alpha_1,\beta_1,\gamma_1)}(x_1,\,y_1,\,u,\,v)\,g_{n-k}^{(\alpha_2,\beta_2,\gamma_2)}(x_2,\,y_2,\,u,\,v), \end{aligned} \tag{4.8}$$

which obviously contain (4.6) and many other results scattered in the literature.

Since

$$B_n^{(n+1)}(x)=\prod_{k=1}^{n}(x-k)=(-1)^n(1-x)_n, \tag{4.9}$$

our addition theorem (4.5) reduces, when $\alpha=\beta=0$, to the well-known result:

$$(x+y)_n=\sum_{k=0}^{n}\binom{n}{k}(x)_k(y)_{n-k}. \tag{4.10}$$

Next we consider the generalized Euler polynomials defined by

$$\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{z^n}{n!}=\left(\frac{2}{e^z+1}\right)^{\alpha}\exp(xz), \tag{4.11}$$

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which, when compared with (1.2), yields at once the addition theorem:

$$(4.12)\ E_n^{(\alpha+\lambda\gamma)}(x+\gamma y)=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}E_k^{(\alpha-\lambda k)}(x-ky)E_{n-k}^{(\lambda k+\lambda\gamma)}(ky+\gamma y),\ \mathrm{Re}(\gamma)>0.$$

By comparing the generating functions (1.5) and (4.11) it is not difficult to see that the addition theorem (1.7) does not apply to the case of the generalized Euler polynomials.

5. Other applications. We begin by recalling the generating function [5, p. 78, Equation (3.2)]

$$(5.1)\qquad \sum_{n=0}^{\infty} G_n^{(\alpha)}(x^{1/r}, r, p, s)z^n=(1-sz)^{-\alpha/s}\exp(px[1-(1-sz)^{-r/s}]),$$

where $G_n^{(\alpha)}(x, r, p, s)$ are the polynomials considered by Srivastava and Singhal [5] in an attempt to present a unified study of the various known generalizations of the classical Hermite and Laguerre polynomials; here the parameters α, p, r and s are essentially arbitrary (with, of course, $r, s\neq 0$).

A comparison of (5.1) with (1.2) and (1.5) yields the following connections:

$$(5.2)\qquad \begin{cases} B(z)=(1-sz)^{-1/s}, \qquad C(z)=p[1-(1-sz)^{-r/s}],\\ f_n^{(\alpha)}(x)\to n!\, G_n^{(\alpha)}(x^{1/r}, r, p, s)\end{cases}$$

and

$$(5.3)\qquad \begin{cases} A(z)=1, \qquad B(z)=(1-sz)^{-1/s}, \qquad C(z)=p[1-(1-sz)^{-r/s}],\\ y=0, \qquad u=s, \qquad v=0,\\ g_n^{(\alpha,\beta,\gamma)}(x, 0, s, 0)\to n!\, G_n^{(\beta+s\gamma)}(x^{1/r}, r, p, s).\end{cases}$$

It follows from (1.3) that

$$(5.4)\qquad G_n^{(\alpha+\lambda\gamma)}([x+\gamma y]^{1/r}, r, p, s)$$
$$=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}G_k^{(\alpha-\lambda k)}([x-ky]^{1/r}, r, p, s)G_{n-k}^{(\lambda k+\lambda\gamma)}([ky+\gamma y]^{1/r}, r, p, s),\ \mathrm{Re}(\gamma)>0,$$

while (1.7) leads to the interesting addition theorem:

$$(5.5)\quad G_n^{(\alpha+\beta+s)}([x+y]^{1/r}, r, p, s)=\sum_{k=0}^{n}G_k^{(\alpha-ks+s)}(x^{1/r}, r, p, s)G_{n-k}^{(\beta+ks+s)}(y^{1/r}, r, p, s).$$

The polynomials $G_n^{(\alpha)}(x, r, p, s)$ can be specialized to a large number of familiar systems of polynomials by appealing to the relationships given by Srivastava and Singhal [5, p. 76]. For example, if we suitably make use of the known relationships [5, p. 76, Equations (1.8) and (1.9)]

$$(5.6)\qquad G_n^{(0)}(x, 2, 1, -1)=G_n^{(1-n)}(x, 2, 1, 1)=\frac{(-x)^n}{n!}H_n(x)$$

and

$$G_n^{(\alpha+n)}(x, 1, 1, -1) = G_n^{(\alpha+1)}(x, 1, 1, 1) = L_n^{(\alpha)}(x) \tag{5.7}$$

in our addition formulas (5.4) and (5.5), we shall obtain the corresponding (presumably new) results for the classical Hermite and Laguerre polynomials. The details may be omitted.

Yet another set of interesting special cases of our addition formulas (5.4) and (5.5) would result if we let $p = r = 1$ and apply the readily verifiable relationship

$$G_n^{(\alpha+1)}(x, 1, 1, s) = s^n Y_n^\alpha(x; s), \tag{5.8}$$

where $Y_n^\alpha(x; s)$ are one of the two classes of the *biorthogonal* polynomials introduced by Konhauser [2] for $\alpha > -1$ and $s = 1, 2, 3, \ldots$. We thus find from (5.4) that

$$Y_n^{\alpha+\lambda\gamma}(x+\gamma y; s) = \sum_{k=0}^{n} \frac{\gamma+n}{\gamma+k} Y_k^{\alpha-\lambda k}(x-ky; s) Y_{n-k}^{\lambda(k+\gamma)-1}(ky+\gamma y; s), \tag{5.9}$$
$$s = 1, 2, 3, \ldots; \operatorname{Re}(\gamma) > 0,$$

while (5.5) gives us

$$Y_n^{\alpha+\beta+s+1}(x+y; s) = \sum_{k=0}^{n} Y_k^{\alpha-ks+s}(x; s) Y_{n-k}^{\beta+ks+s}(y; s), \qquad s = 1, 2, 3, \ldots. \tag{5.10}$$

Notice that, since [*cf.* (5.7) and (5.8)]

$$Y_n^\alpha(x; 1) = L_n^{(\alpha)}(x), \tag{5.11}$$

the special cases of (5.9) and (5.10) when $s = 1$ will naturally yield the aforementioned results involving Laguerre polynomials. Putting $s = 1$ in, for instance, (5.10) and applying (5.11), we immediately arrive at the elegant result

$$L_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha-k)}(x) L_{n-k}^{(\beta+k)}(y), \tag{5.12}$$

which can, of course, be derived directly from a known generating function for the *modified* Laguerre polynomials $L_n^{(\alpha-n)}(x)$.

We remark in passing that a number of additional applications of our theorem can be given by using some of the examples considered earlier by Carlitz [1].

6. A direct proof of (3.3). In view of the fundamental importance of the derivative formula (3.3) in our derivation of the main addition theorem (1.7), we find it worthwhile to give a simple proof of (3.3) *without* using (3.2).

Let $g(z)$ be a univalent function in a domain $\mathscr{D}$ of the complex z-plane.

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Thus, for every point $z = z_0$ in $\mathcal{D}$, we have

$$g(z) = g(z_0) \Rightarrow z = z_0 \tag{6.1}$$

and

$$D^n_{g(z)}\{f(g(z))\}|_{g(z)=g(z_0)} = D^n_{g(z)}\{f(g(z))\}|_{z=z_0}. \tag{6.2}$$

From Taylor's series for $f(z)$ we readily have

$$f(g(z)) = f(g(z_0)) + \sum_{n=1}^{\infty} D^{n-1}_{g(z)}\{f'(g(z))\}\Big|_{z=z_0} \frac{[g(z)-g(z_0)]^n}{n!}, \tag{6.3}$$

while Lagrange's expansion theorem [6, p. 133] yields

(6.4)
$$G(z) = G(z_0) + \sum_{n=1}^{\infty} D^{n-1}_z \left\{G'(z)\left(\frac{z-z_0}{g(z)-g(z_0)}\right)^n\right\}\Big|_{z=z_0} \frac{[g(z)-g(z_0)]^n}{n!}, \ g'(z_0) \neq 0.$$

Setting $G(z) = f(g(z))$ in (6.4), we obtain

(6.5)
$$f(g(z)) = f(g(z_0)) + \sum_{n=1}^{\infty} D^{n-1}_z \left\{f'(g(z))g'(z)\left(\frac{z-z_0}{g(z)-g(z_0)}\right)^n\right\}\Big|_{z=z_0} \frac{[g(z)-g(z_0)]^n}{n!},$$

and, on comparing (6.3) and (6.5), we arrive at the desired result (3.3) in its essentially *equivalent* form:

$$D^{n-1}_{g(z)}\{H(z)\}|_{z=z_0} = D^{n-1}_z \left\{H(z)g'(z)\left(\frac{z-z_0}{g(z)-g(z_0)}\right)^n\right\}\Big|_{z=z_0}, \tag{6.6}$$

where, for convenience, $H(z) = f'(g(z))$, $g(z)$ being univalent in $\mathcal{D}$, $g'(z_0) \neq 0$, and n is a positive integer.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VICTORIA, VICTORIA V8W 2Y2, CANADA.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ LAVAL, QUÉBEC G1K 7P4, CANADA.

DÉPARTMENT DES SCIENCES ÉCONOMIQUES, UNIVERSITÉ DU QUÉBEC, CHICOUTIMI, QUÉBEC G7H 2P9, CANADA.