

A COMMENT ON FINITE NILPOTENT GROUPS OF DEFICIENCY ZERO

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A finite group is said to have deficiency zero if it can be presented with an equal number of generators and relations. Finite metacyclic groups of deficiency zero have been classified, see [1] or [6]. Finite non-metacyclic groups of deficiency zero, which we denote by  $FD_0$ -groups, are relatively scarce. In [3] I. D. Macdonald introduced a class of nilpotent  $FD_0$ -groups all having nilpotent class  $\leq 8$ . The largest nilpotent class known for a Macdonald group is 7 [4]. Only a finite number of nilpotent  $FD_0$ -groups, other than the Macdonald groups, seem to be known [5], [7]. In this note we exhibit a class of  $FD_0$ -groups which contains nilpotent groups of arbitrarily large nilpotent class.

For any natural number  $n$  define the group  $G_n$  by

$$G_n = \langle a, b \mid a^{4n-2}ba = b^2ab, a^4b^4 = 1 \rangle.$$

We shall use the notation  $C_n$  for the cyclic group of order  $n$  and the notation  $[x, y]$  for  $x^{-1}y^{-1}xy$ . Commutators of higher weight we define by

$$[x_1, x_2, \dots, x_n] = [[x_1, x_2, \dots, x_{n-1}], x_n].$$

LEMMA 1. In  $G_n$  the following relations hold:

- (i)  $b^{4n-2}ab = a^2ba$
- (ii)  $a^{32n} = b^{32n} = 1$ .

**Proof.** (i) This follows immediately.

(ii) We have  $bab^{-1-4n}a^{-1}b^{-1} = a^2b$  and, raising this to the fourth power,

$$(1) \quad b^{4+16n}(a^2b)^4 = 1$$

and so

$$(2) \quad [b, (a^2b)^4] = 1.$$

Now

$$(3) \quad a^2b^2a^2 = a^{-1}ba^{-1}b^{-1} \cdot b^{4n-4}.$$

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For, using Lemma 1(i)

$$\begin{aligned} a^2b^2(a^2ba)b^{-1}a &= a^2b^2 \cdot b^{4n-2}abb^{-1}a \\ &= a^2b^{4n}a^2 \\ &= b^{4n-4}. \end{aligned}$$

Also

$$(4) \quad b^2a^2b^2 = aba^{-1}b \cdot b^{4n}.$$

Now, using (2), (3), (4) and the relation  $a^4b^4 = 1$ , we have

$$\begin{aligned} ((a^2b)^4b^4)^2 &= (a^2b)^3a^2b(a^2b)^4b^8 \\ &= (a^2b)^3a^2(a^2b)^4b^9 \\ &= a^2ba^2b(a^2b^2a^2)ba^2ba^2b^6 \\ &= a^2ba^2ba^{-1}baba^2b^{4n+2} \\ &= a^2ba(b^2a^2b^2)aba^2b^2 \\ &= a^2bab^2a^2b^2ab(a^{-1}ba^{-1}b^{-1} \cdot b^{4n-4})a^{-2} \\ &= a^2ba(b^2a^2b^2 \cdot aba^{-1}b \cdot b^{4n-4})a^{-1}b^{-1}a^{-2} \\ &= 1. \end{aligned}$$

But  $((a^2b)^4b^4)^2 = 1$  together with (1) yields  $b^{32n} = 1$  as required.

**THEOREM 1.** *The group  $G_n$  is nilpotent and has order  $2^{10n}$ .*

**Proof.** Let  $r = aba^{-1}b^{-1}$ ,  $s = a^2ba^{-1}b^{-1}a^{-1}$ ,  $t = a^4$  and put  $H = \langle r, s, t \rangle$ . We enumerate the cosets of  $H$  in  $G_n$ . The enumeration is made easier by introducing the redundant generator  $a^3ba^{-1}b^{-1}a^{-2}$  into  $H$ . To see that this generator is redundant notice that, using (3) and  $a^4b^4 = 1$ , we have

$$(5) \quad aba^{-1}b^{-1} = b^{-2}a^{-2} \cdot a^{4n}.$$

Now use (5) to obtain

$$\begin{aligned} aba^{-1}b^{-1} \cdot a^3ba^{-1}b^{-1}a^{-2} &= b^{-2}aba^{-1}b^{-1}a^{-2} \cdot a^{4n} \\ &= b^{-2}(b^{-2}a^{-2})a^{-2} \cdot a^{8n} \\ &= a^{8n}. \end{aligned}$$

Hence  $a^3ba^{-1}b^{-1}a^{-2} = r^{-1}t^{2n} \in H$ .

Using integers to denote coset representatives we have the following table:

- |                                  |                              |
|----------------------------------|------------------------------|
| 1. $a = 2$                       | 1. $b = r^{-1} \cdot 4$      |
| 2. $a = 5$                       | 2. $b = 3$                   |
| 3. $a = s^{-1} \cdot 6$          | 3. $b = st^{-n-1} \cdot 7$   |
| 4. $a = 3$                       | 4. $b = r^2t^{-n-1} \cdot 5$ |
| 5. $a = 7$                       | 5. $b = 6$                   |
| 6. $a = rt^{-2n} \cdot 8$        | 6. $b = r^{-1}t^n \cdot 1$   |
| 7. $a = t \cdot 1$               | 7. $b = 8$                   |
| 8. $a = r^{-1}st^{2n+1} \cdot 4$ | 8. $b = s^{-1}t^n \cdot 2$   |

Note that we have used the fact that  $t$  is central, see Lemma 1 (ii).

We next use the Reidemeister–Schreier algorithm to obtain a presentation for  $H$  on the generators  $r, s, t$ . A considerable simplification is possible using the relations  $[r, t] = [s, t] = t^{8n} = 1$  which we know hold in  $H$ . The following presentation is obtained.

$$H = \langle r, s, t \mid s^2rsr^{-1} = 1, s^{-1}r^2sr^2 = 1, sr^3s^{-1}r = t^{4n}, sr^{-1}s^{-1}r = t^{4n}, r^{-1}s^2r^{-1}sr^2s = 1 [r, t] = [s, t] = t^{8n} = 1 \rangle$$

This presentation can be simplified using straightforward manipulations to give

$$H = \langle r, s, t \mid r^4 = s^4 = t^{8n} = 1, [r, s] = t^{4n}, [r, t] = [s, t] = 1 \rangle.$$

Now  $H/H' \cong C_4 \times C_4 \times C_{4n}$  and  $H' \cong C_2$ . This proves that the order of  $G_n$  is  $2^{10}n$  and that the period of  $a$  is  $32n$ .

The group  $G_n$  is nilpotent of class 6 and soluble of class 3 for any  $n$ . The first two factors of the derived series have rank 2 while the third is cyclic of order 2 generated by  $(a^2b)^4b^4$ .

By Lemma 1 (i)  $G_n$  admits an automorphism  $\theta$  of order 2 induced by the map which interchanges  $a$  and  $b$ . Form  $E_n$ , the split extension of  $G_n$  by  $\theta$ . Let  $\theta$  be induced by conjugation by an element  $x$  so that  $E_n$  is generated by  $x$  and  $y$  where  $a = y, b = yxy^{-1}$  and  $x^2 = 1$ . It is now easy to show that  $E_n$  has the presentation

$$\langle x, y \mid x^2 = 1, (yx)^2 = (xy^2)^2(xy)^2y^{4n}, (xy^4)^2 = 1 \rangle.$$

LEMMA 2. *The group  $E_{2n}$  is finite and has a 2-generator 2-relation presentation.*

**Proof.** Since  $G_{2n}$  may be presented with the symmetrical presentation

$$\langle a, b \mid b^{4n-2}ab^5 = a^{4n-2}ba, a^{4n-2}ba^5 = b^{4n-2}ab \rangle,$$

the result follows.

The presentation for  $G_{2^n}$  given in the above proof shows it to be a member of the class studied by Campbell in [2].

The groups  $E_{2^k}$  are nilpotent  $FD_0$ -groups whose class can be made arbitrarily large as the next theorem shows.

**THEOREM 2.** *For  $n = 2^k$ ,  $k \geq 1$ ,  $E_n$  is a  $FD_0$ -group which is nilpotent of class  $\geq 4 + k$ .*

**Proof.** Clearly  $y^4$  belongs to the derived group of  $E_n$ . Also

$$[y^4, \underbrace{x, x, \dots, x}_t] = y^\alpha$$

where  $\alpha = 4 \cdot (-2)^t$ .

So, choosing  $t = 2 + k$ , the result follows.

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