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Dr MACKAY in the Chair.

Triangles in Multiple Perspective, viewed in connection with Determinants of the third order.

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I. *Introductory.*

The theorems given in the present section are fundamental in the theory of triangles in multiple perspective. They are all perfectly well known, but are given here because without them the succeeding sections would be unintelligible.

Two triangles $A_1A_2A_3$, $B_1B_2B_3$ can be in perspective in six different ways, indicated by the following symbols, in which the A's are to be understood as connecting collinearly with the B's standing directly underneath.

$$(1)\ \frac{A_1A_2A_3}{B_1B_2B_3};\quad (2)\ \frac{A_1A_2A_3}{B_2B_3B_1};\quad (3)\ \frac{A_1A_2A_3}{B_3B_1B_2};$$

$$(4)\ \frac{A_1A_2A_3}{B_1B_3B_2};\quad (5)\ \frac{A_1A_2A_3}{B_2B_1B_3};\quad (6)\ \frac{A_1A_2A_3}{B_3B_2B_1}.$$

If the coordinates of B_1, B_2, B_3, with reference to the triangle A, be given by the first, second, and third rows of the determinant

$$\begin{vmatrix} b_{11}, & b_{12}, & b_{13}, \\ b_{21}, & b_{22}, & b_{23}, \\ b_{31}, & b_{32}, & b_{33}, \end{vmatrix} \equiv \Delta b,$$

the necessary and sufficient conditions that the various modes of perspective enumerated above may obtain, are :

$$\text{for (1)}\ b_{21}b_{32}b_{13} = b_{31}b_{12}b_{23};\quad \text{for (4)}\ b_{21}b_{12}b_{33} = b_{31}b_{22}b_{13};$$
$$\text{for (2)}\ b_{31}b_{12}b_{23} = b_{11}b_{22}b_{33};\quad \text{for (5)}\ b_{31}b_{22}b_{13} = b_{11}b_{32}b_{23};$$
$$\text{for (3)}\ b_{11}b_{22}b_{33} = b_{21}b_{32}b_{13};\quad \text{for (6)}\ b_{11}b_{32}b_{23} = b_{21}b_{12}b_{33}.$$

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If we say that a positive (or negative) term in the expansion of a determinant of the third order is the first, second, or third positive (or negative) term, according as it contains the first, second, or third element of the first column, we may express the foregoing conditions as follows. We have perspective of the first kind between the triangles A and B when the second and third positive terms of Δb are equal, of the second kind when the third and first positive terms are equal, of the third kind when the first and second positive terms are equal, of the fourth kind when the second and third negative terms are equal, of the fifth kind when the third and first negative terms are equal, and of the sixth kind when the first and second negative terms are equal.

If the two triangles are in perspective in any two out of the first three ways, they are also in perspective in the remaining way, *i.e.*, they are in *triple perspective.* This kind of triple perspective may be called *direct.* If the two triangles are in perspective in any two out of the last three ways, they are also in perspective in the remaining way, *i.e.*, they are as before in triple perspective. This kind of triple perspective may be called *perverse.* It is to be noted that there is no essential difference between direct and perverse triple perspective, it being merely a matter of how the vertices of the triangles are named whether the triple perspective be of the one kind or the other.

When any one of the first three modes of perspective occurs simultaneously with any one of the last three modes, we have *double* perspective. Two triangles may be in double perspective in nine not essentially different ways. When two triangles are in double perspective it may be noted that a certain vertex of the one is twice connected with a certain vertex of the other, and that the two centres of perspective lie on the connector of these vertices.

When the first three conditions hold good simultaneously with any one of the last three, or the last three with any one of the first three, we have *quadruple* perspective. Two triangles may obviously be in quadruple perspective in six ways.

When any five of the conditions hold good simultaneously, they all hold good, and thus the two triangles are in *sextuple* perspective. Two real triangles can be in single, double, triple, or quadruple perspective, but in the case of sextuple perspective, one of the two triangles must reduce to a point or else be imaginary, *i.e.*, have imaginary points for two of its vertices. If b_1, b_2, b_3 be the coordinates, with reference to a triangle A, of one of the vertices of a triangle B supposed to be in sextuple perspective with A, the other vertices of B either coincide with the point (b_1, b_2, b_3) or are the points $(b_1, \omega b_2, \omega^2 b_3)$ and $(b_1, \omega^2 b_2, \omega b_3)$, where ω and ω^2 are the imaginary cube roots of unity. It is clear that the side of B containing the imaginary vertices is real.

The coordinates used in the conditions given for the various kinds of perspective have been trilinear or areal, but it is important to observe that precisely similar results are obtained when tangential coordinates are used.

II.

I now proceed to the problem of finding the conditions for perspective between two triangles A and B the coordinates of whose vertices are given with reference to a third triangle C.

Let the coordinates of A_1, A_2, A_3 be given by the rows of the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \equiv \triangle a,$$

and those of B_1, B_2, B_3 by the rows of the determinant

$$\begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} \equiv \triangle b,$$

Let B_{ik} be the complementary minor of the element b_{ik} of $\triangle b$.

Then the equations of the sides of B are

$$B_2B_3 \equiv x_1 B_{11} + x_2 B_{12} + x_3 B_{13} = 0,$$
$$B_3B_1 \equiv x_1 B_{21} + x_2 B_{22} + x_3 B_{23} = 0,$$
$$B_1B_2 \equiv x_1 B_{31} + x_2 B_{32} + x_3 B_{33} = 0.$$

Hence the perpendiculars from A_1, A_2, A_3 to B_2B_3 are proportional to

$$a_{11}B_{11} + a_{12}B_{12} + a_{13}B_{13},\ \ a_{21}B_{11} + a_{22}B_{12} + a_{23}B_{13},\ \ a_{31}B_{11} + a_{32}B_{12} + a_{33}B_{13},$$

say to β_{11}, β_{12}, β_{13}.

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Similarly the perpendiculars from A_1, A_2, A_3 to B_3B_1 are proportional to

$$a_{11}B_{21}+a_{12}B_{22}+a_{13}B_{23},\ a_{21}B_{21}+a_{22}B_{22}+a_{23}B_{23},\ a_{31}B_{21}+a_{32}B_{22}+a_{33}B_{23},$$

$$\text{say to}\quad \beta_{21},\ \beta_{22},\ \beta_{23}.$$

Similarly the perpendiculars from A_1, A_2, A_3 to B_1B_2 are proportional to

$$a_{11}B_{31}+a_{12}B_{32}+a_{13}B_{33},\ a_{21}B_{31}+a_{22}B_{32}+a_{23}B_{33},\ a_{31}B_{31}+a_{32}B_{32}+a_{33}B_{33},$$

$$\text{say to}\quad \beta_{31},\ \beta_{32},\ \beta_{33}.$$

Hence the tangential coordinates of the sides of B, *with reference to A*, are given by the rows of the determinant

$$\begin{vmatrix} \beta_{11}, & \beta_{12}, & \beta_{13} \\ \beta_{21}, & \beta_{22}, & \beta_{23} \\ \beta_{31} & \beta_{32}, & \beta_{33} \end{vmatrix} \equiv \Delta\beta.$$

It follows from what was said in section I. that A and B are simply in perspective in the first, second, or third way, according as the second and third, third and first, or first and second positive terms of $\Delta\beta$ are equal, and that they are simply in perspective in the fourth, fifth, or sixth way according as the second and third, third and first, or first and second negative terms of $\Delta\beta$ are equal. The conditions for the various kinds of perspective are, in fact, exactly the same as those given in section I., except that for b we have throughout β. Thus, for example, the necessary and sufficient condition that A and B be directly triply in perspective is that the three positive terms of $\Delta\beta$ be equal, and the condition that they be perversely triply in perspective is that the three negative terms of $\Delta\beta$ be equal.

It will be noticed that

$$\Delta\beta \equiv \Delta a . \Delta' b,$$

where $\Delta' b$ is the adjugate of Δb. In order that the elements of the rows of $\Delta\beta$ may be the tangential coordinates of the sides of B with reference to A, the multiplication of Δa and $\Delta' b$ must be by rows, and, further, must be effected in such a manner that, in the extended array of $\Delta\beta$, the a's in each vertical column may have precisely the same suffixes. It must be observed also that no factors may be removed from the columns of Δa or $\Delta' b$ before multiplication, unless the same factor be removed from each, otherwise the results would be vitiated. If we removed, for example, a factor from the first column of Δa, the elements of its rows would no longer represent the coordinates of the vertices of A.

A determinant similar to $\Delta\beta$, viz. $\Delta\alpha$, the elements of whose rows are the tangential coordinates of the sides of A *with reference to B*, is obtained by multiplying in exactly the same way Δb and $\Delta' a$.

The conditions that A and B may be in perspective may, of course, be obtained, in another form, by writing down the conditions that the lines joining corresponding vertices may be concurrent. This is the method adopted by Rosanes, who seems to have been the first to deal with the subject of multiple perspective, in his paper, *Über Dreiecke in perspectivischer Lage*, *Math. Ann.*, Bd. II., p. 549.

The condition found in this way for the first mode of perspective

$$\begin{matrix} A_1A_2A_3 \\ B_1B_2B_3, \end{matrix}$$

is $$\Delta_1 \equiv \begin{vmatrix} a_{12}b_{13}-a_{13}b_{12}, & a_{13}b_{11}-a_{11}b_{13}, & a_{11}b_{12}-a_{12}b_{11} \\ a_{22}b_{23}-a_{23}b_{22}, & a_{23}b_{21}-a_{21}b_{23}, & a_{21}b_{22}-a_{22}b_{21} \\ a_{32}b_{33}-a_{33}b_{32}, & a_{33}b_{31}-a_{31}b_{33}, & a_{31}b_{32}-a_{32}b_{31} \end{vmatrix} \equiv 0.$$

For $\dfrac{A_1A_2A_3}{B_2B_3B_1}$ it is $\Delta_2 \equiv 0$, and for $\dfrac{A_1A_2A_3}{B_3B_1B_2}$, $\Delta_3 \equiv 0$, where Δ_2 is derived from Δ_1, and Δ_3 from Δ_2 by cyclical interchange of the first suffixes of the b's. Again for

$$\frac{A_1A_2A_3}{B_1B_3B_2},\quad \frac{A_1A_2A_3}{B_2B_1B_3},\quad \frac{A_1A_2A_3}{B_3B_2B_1},$$

the conditions are $\Delta_4 \equiv 0$, $\Delta_5 \equiv 0$, $\Delta_6 \equiv 0$ respectively, where Δ_4, Δ_5, Δ_6 are derived from Δ_1, Δ_2, Δ_3 respectively by causing the b's in the second rows of the latter to exchange suffixes with the b's directly underneath in the third rows.

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These conditions must of course be equivalent to those already found, and as a matter of fact

$$\Delta_1 \,.\, \Delta b \equiv \beta_{21}\beta_{32}\beta_{13} - \beta_{31}\beta_{12}\beta_{23},$$
$$\Delta_2 \,.\, \Delta b \equiv \beta_{31}\beta_{12}\beta_{23} - \beta_{11}\beta_{22}\beta_{33},$$
$$\Delta_3 \,.\, \Delta b \equiv \beta_{11}\beta_{22}\beta_{33} - \beta_{21}\beta_{32}\beta_{13},$$
$$\Delta_4 \,.\, \Delta b \equiv \beta_{21}\beta_{12}\beta_{33} - \beta_{31}\beta_{22}\beta_{13},$$
$$\Delta_5 \,.\, \Delta b \equiv \beta_{31}\beta_{22}\beta_{13} - \beta_{11}\beta_{32}\beta_{23},$$
$$\Delta_6 \,.\, \Delta b \equiv \beta_{11}\beta_{32}\beta_{23} - \beta_{21}\beta_{12}\beta_{33}.$$

From the above we deduce the following identities connecting the elements of any two determinants, Δa and Δb, of the third order,

$$\Delta_1 + \Delta_2 + \Delta_3 \equiv 0, \text{ and } \Delta_4 + \Delta_5 + \Delta_6 \equiv 0.$$

It may also be worth stating that if $\Delta_4 \equiv \Delta_5 \equiv \Delta_6$, the determinants Δa and $\Delta \beta$ either are circulants or can be reduced to circulants by the removal of common factors from their columns, and that if $\Delta_1 \equiv \Delta_2 \equiv \Delta_3$, the same determinants, when any two of their rows are interchanged, either are circulants or can be reduced to circulants by the removal of common factors from their columns.

III.

If $C_1C_2C_3$ be taken as the fundamental triangle, it is easy to show that all the triangles which are in direct triple perspective with C can have the coordinates of their vertices (or sides) thrown into the form

$$\begin{vmatrix} p_1q_1, & p_2q_2, & p_3q_3 \\ p_1q_2, & p_2q_3, & p_3q_1 \\ p_1q_3, & p_2q_1, & p_3q_2 \end{vmatrix}.$$

I have already made use of coordinates of this form in the paper on "Triangles triply in perspective" which I communicated to the Society two years ago, and more recently they have been employed by Mr Ferrari in a paper "Sur les triangles trihomologiques" in *Mathesis*, Jan. 1902, in the course of which he traverses some of the ground covered by my paper and adds some new results. When the triple perspective is perverse, the second and third rows in the above array must be interchanged.

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The foregoing coordinates, as Mr Ferrari has pointed out, may be regarded as the result of multiplying the coordinates of a point

$$P(p_1, \quad p_2, \quad p_3)$$

by those of a triangle Q, viz.,

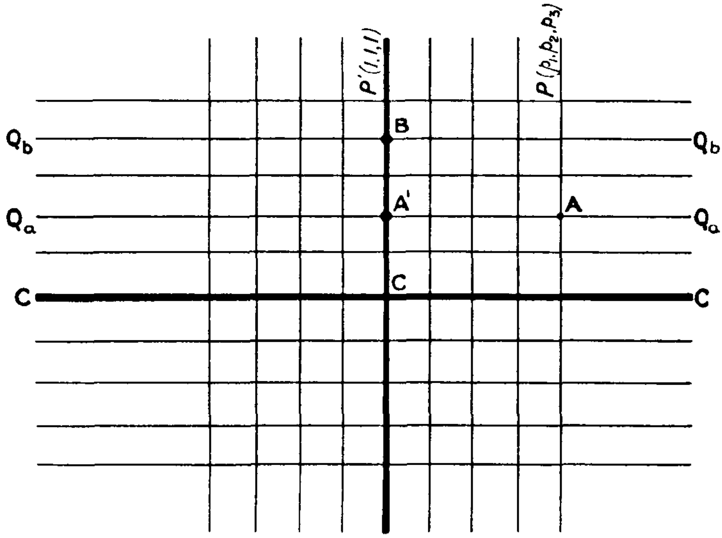
$$\begin{array}{c|ccc|} Q_1 & q_1, & q_2, & q_3 \\ Q_2 & q_2, & q_3, & q_1 \\ Q_3 & q_3, & q_1, & q_2 \end{array}\,.$$

When one of the vertices of Q is given the others are also given, and thus, with respect to the same fundamental triangle C, there are ∞^2 such triangles in the plane. As P can also assume ∞^2 positions, there are ∞^4 triangles in direct triple perspective with C. These triangles are also in perverse triple perspective with C if we take their vertices in a different order. Thus in all there are ∞^4 triangles triply in perspective with C.

If we regard P as fixed and Q as variable, we obtain a series of ∞^2 triangles which, adopting Mr Ferrari's notation, I shall call a Tp series. Every Tp series obviously includes the triangle $C_1C_3C_2$. The triangles Q form a Tp series, P being in this case the point (1, 1, 1). Following the nomenclature of my previous paper, I shall call this particular series the *isobaric* Tp series, although the name is not strictly appropriate unless the coordinates used be barycentric. The fundamental triangle C and any other triangle in triple perspective with it are sufficient to determine a Tp series, all the other triangles of the series, such as the triangles formed by their centres and axes of perspective, being derivable from the original two by purely linear constructions. Any Tp series can be projected into the isobaric Tp series by projecting P into the point (1, 1, 1), and in this way I proved, in the paper already referred to, a considerable number of theorems respecting the triangles of such a series, *inter alia* that any two of them are in perverse triple perspective with each other.

If we regard Q as fixed and P as variable, we obtain a series of ∞^2 triangles which Mr Ferrari has called a Tq series. The fundamental triangle C obviously belongs to only one Tq series, namely, that in which Q coincides with $C_1C_3C_2$. In this case all the triangles of the series coincide with $C_1C_3C_2$.

The ∞^4 triangles in triple perspective with C may be usefully represented by the subjoined diagram consisting of vertical and horizontal lines.



Each vertical line represents a Tp series, the thick vertical line representing the isobaric Tp series, for which P is the point (1, 1, 1). Each horizontal line represents a Tq series, the thick horizontal line representing the Tq series every triangle of which coincides with $C_1C_3C_2$. The ∞^4 triangles are then represented by the points of intersection of the lines, each line being supposed to have ∞^2 points. Every point on the thick horizontal line represents the same triangle $C_1C_3C_2$. Every point on the thick vertical line represents the defining triangle of the Tq series represented by the horizontal line passing through the point.

As has been said, all the triangles of any Tp series are triply in perspective with each other in pairs. The question now naturally arises whether any of the triangles of a Tq series are triply in perspective with each other. This question will be treated in connection with the more general question whether triangles which

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belong to different Tp and at the same time different Tq series can be triply in perspective with each other. Another form of this question is: Can two triangles A and B which are triply in perspective with a third triangle C, be triply in perspective with each other without belonging to the same Tp series with respect to C? Two cases fall to be considered, (1) when A, B, and C are supposed to be triply in perspective with each other in the same way, say directly, and (2) when two pairs are triply in perspective in one way, say A, C and B, C directly, and the remaining pair in a different way, say A, B perversely.

C being the fundamental triangle, let A (see diagram) belong to the Tp system defined by the point P(p_1, p_2, p_3), and to the Tq system defined by the triangle Qa, the coordinates of whose vertices are given by the rows of the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix};$$

and that B belongs to the Tp series defined by the point P$'(1, 1, 1)$, and to the Tq series defined by the triangle Qb, the coordinates of whose vertices are given by the rows of the determinant

$$\begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix}$$

There is obviously no loss of generality in making B belong to the isobaric Tp series, since every other Tp series can be projected into it. The coordinates of the vertices of A and B are given by the determinants

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array}\begin{vmatrix} p_1a_1, & p_2a_2, & p_3a_3 \\ p_1a_2, & p_2a_3, & p_3a_1 \\ p_1a_3, & p_2a_1, & p_3a_2 \end{vmatrix} \equiv \Delta pa \quad \text{and} \quad \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array}\begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix} \equiv \Delta b.$$

A and B are in direct triple perspective with C. If now it be supposed that A and B are also in direct triple perspective with each other, it follows from section II that the positive terms of the following determinant are equal,

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$$\Delta pa\,.\,\Delta' b \equiv \begin{vmatrix} p_1a_1B_1+p_2a_2B_2+p_3a_3B_3, & p_1a_2B_1+p_2a_3B_2+p_3a_1B_3, & p_1a_3B_1+p_2a_1B_2+p_3a_2B_3 \\ p_1a_1B_2+p_2a_2B_3+p_3a_3B_1, & p_1a_2B_2+p_2a_3B_3+p_3a_1B_1, & p_1a_3B_2+p_2a_1B_3+p_3a_2B_1 \\ p_1a_1B_3+p_2a_2B_1+p_3a_3B_2, & p_1a_2B_3+p_2a_3B_1+p_3a_1B_2, & p_1a_3B_3+p_2a_1B_1+p_3a_2B_2 \end{vmatrix}$$

where the B's are the complementary minors of the b's in Δb, or, what is the same thing, the tangential coordinates of the sides of B. Hence if Qa and Qb be regarded as fixed, *i.e.*, if the a's and B's be regarded as constant, we have the following equations for the determination of $p_1 : p_2 : p_3$,

$$\begin{aligned} &(p_1a_1B_1+p_2a_2B_2+p_3a_3B_3)(p_1a_2B_2+p_2a_3B_3+p_3a_1B_1)(p_1a_3B_3+p_2a_1B_1+p_3a_2B_2) \\ =&(p_1a_1B_2+p_2a_2B_3+p_3a_3B_1)(p_1a_2B_3+p_2a_3B_1+p_3a_1B_2)(p_1a_3B_1+p_2a_1B_2+p_3a_2B_3) \\ =&(p_1a_1B_3+p_2a_2B_1+p_3a_3B_2)(p_1a_2B_1+p_2a_3B_2+p_3a_1B_3)(p_1a_3B_2+p_2a_1B_3+p_3a_2B_1)\,. \end{aligned}$$

That is, the point P(p_1, p_2, p_3) may be any one of the nine points of intersection of the following system of cubics,

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_3-F_5)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_4-F_6) \\ &\qquad +p_1p_2p_3\{a_1^3(B_2^3-B_3^3)+a_2^3(B_3^3-B_1^3)+a_3^3(B_1^3-B_2^3)\}=0, \end{aligned}$$

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_5-F_1)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_6-F_2) \\ &\qquad +p_1p_2p_3\{a_1^3(B_3^3-B_1^3)+a_2^3(B_1^3-B_2^3)+a_3\,(B_2^3-B_3^3)\}=0, \end{aligned}$$

$$\begin{aligned} &(p_2^2p_3+p_3^2p_1+p_1^2p_2)(F_1-F_3)+(p_2p_3^2+p_3p_1^2+p_1p_2^2)(F_2-F_4) \\ &\qquad +p_1p_2p_3\{a_1^3(B_1^3-B_2^3)+a_2^3(B_2^3-B_3^3)+a_3^3(B_3^3-B_1^3)\}=0, \end{aligned}$$

where
$$\begin{aligned}
F_1 &\equiv a_2^2a_3\, B_2^2B_3 + a_3^2a_1\, B_3^2B_1 + a_1^2a_2\, B_1^2B_2\,,\\
F_2 &\equiv a_2\, a_3^2B_2\, B_3^2 + a_3\, a_1^2B_3\, B_1^2 + a_1\, a_2^2B_1\, B_2^2,\\
F_3 &\equiv a_2^2a_3\, B_3^2B_1 + a_3^2a_1\, B_1^2B_2 + a_1^2a_2\, B_2^2B_3\,,\\
F_4 &\equiv a_2\, a_3^2B_3\, B_1^2 + a_3\, a_1^2B_1\, B_2^2 + a_1\, a_2^2B_2\, B_3^2,\\
F_5 &\equiv a_2^2a_3\, B_1^2B_2 + a_3^2a_1\, B_2^2B_3 + a_1^2a_2\, B_3^2B_1\,,\\
F_6 &\equiv a_2\, a_3^2B_1\, B_2^2 + a_3\, a_1^2B_2\, B_3^2 + a_1\, a_2^2B_3\, B_1^2.
\end{aligned}$$

It is at once evident that these three cubics are circumscribed to C. Hence P may coincide with any of the three vertices of C. Again, if $p_1 : p_2 : p_3 = \lambda_1 : \lambda_2 : \lambda_3$ be a solution of the cubics, it is evident that $p_1 : p_2 : p_3 = \lambda_2 : \lambda_3 : \lambda_1$ and $p_1 : p_2 : p_3 = \lambda_3 : \lambda_1 : \lambda_2$ are also solutions. We cannot have $\lambda_1 : \lambda_2 : \lambda_3 = \lambda_2 : \lambda_3 : \lambda_1 = \lambda_3 : \lambda_1 : \lambda_2$, except in the cases where $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$. But none of these is a solution of the cubics. Hence we conclude that of the six undetermined positions of P, if one be real all are real, and if one be imaginary all are imaginary, otherwise we should have the cubics intersecting in an odd number of imaginary points. Again, any two of the cubics are equivalent to two equations of the form

$$p_2^2p_3 + p_3^2p_1 + p_1^2p_2 - k_1p_1p_2p_3 = 0,$$

and $$p_2\,p_3^2 + p_3\,p_1^2 + p_1\,p_2^2 - k_2p_1p_2p_3 = 0,$$

where k_1, k_2 are constants which can readily be determined.

Eliminating p_1 from these equations, and removing the factors which indicate that the cubics are circumscribed to C, we obtain the sextic

$$(p_2^3 - k_2p_2^2p_3 + k_1p_2p_3^2 - p_3^3)^2 = 0.$$

The fact that the expression on the left hand is a complete square indicates that the three cubics have contact of the first kind with each other at three points, *i.e.*, that the six undetermined positions of P reduce to three distinct points. Again, since the equation

$$p_2^3 - k_2p_2^2p_3 + k_1p_2p_3^2 - p_3^3 = 0$$

is known to have at least one real solution, giving a real position of P, it follows from what has been already said that the other positions of P are also real. Hence we conclude that the three cubics intersect in general in six real and distinct points, three of which, however, coincide with the vertices of C. When P coincides with a vertex of C, the triangle A obviously reduces to a point-triangle coinciding with the same vertex. Hence, excluding point-triangles, we may say that in general and in the absence of any particular relations connecting the a's of A with the b's of B, there are three and only three positions of P which give us a triangle A directly triply in perspective with B, and satisfying the condition that it shall belong to a certain Tq series different from that of B, and at the same time to some Tp series (distinct for each of the three cases) different from that of B. Thus on the diagram there are on the horizontal line marked Qa, three positions of A, none of them on the same vertical line with B, which represent triangles directly in triple perspective with the triangle represented by B.

I now turn to the special case where A and B, while still belonging to different Tp series, belong to the same Tq series.

If we retain the same coordinates as before for the vertices of A, those of B, which may now be denoted by A′ (see diagram) become

$$\begin{array}{c} A_1' \\ A_2' \\ A_3' \end{array} \begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix} \equiv \Delta a.$$

We therefore obtain for the determination of the values $p_1 : p_2 : p_3$ the same system of intersecting cubics as we obtained on page 125, except that in the values there given for the F's, and in the coefficients of $p_1 p_2 p_3$, A is everywhere substituted for B, A_κ being the complementary minor of a_κ in Δa.

The resulting relations* which can be established between the

* For example, distinguishing the new F's by dashes, and remembering that

$$\left.\begin{array}{r} A_\kappa \equiv a_\lambda a_\mu - a^2_\kappa, \\ \text{and} \quad A_\lambda A_\mu - A^2_\kappa \equiv a_\kappa \,.\, \Delta a, \end{array}\right. \quad \begin{pmatrix} \kappa = 1, 2, 3 \\ \lambda = 2, 3, 1 \\ \mu = 3, 1, 2 \end{pmatrix}$$

we have

$$F_5' - F_1' \equiv -(F_2' - F_4') \equiv -a_1 a_2 a_3 \,.\, \Delta a^2,$$

$$F_1' - F_3' \equiv -(F_6' - F_2') \equiv -(a_1 a_2 a_3 \,.\, \Sigma a_1^3 - \Sigma a_2^3 a_3^3)\Delta a,$$

$$F_3' - F_5' \equiv -(F_4' - F_6') \equiv (F_6' - F_2') - (F_5' - F_1') \equiv (F_2' - F_4') - (F_1' - F_3')$$

$$\equiv a_1^3 a_2^3 a_3^3 \begin{vmatrix} 1/a_1, & 1/a_2, & 1/a_3 \\ 1/a_2, & 1/a_3, & 1/a_1 \\ 1/a_3, & 1/a_1, & 1/a_2 \end{vmatrix} \,.\, \Delta a,$$

$$\Sigma a_1^3 (A_3^3 - A_1^3) \equiv -\Sigma a_1^3 (A_1^3 - A_2^3) \equiv (\Sigma A_1^6 - \Sigma A_2^3 A_3^3)/\Delta a,$$

$$\text{and} \quad \Sigma a_1^3 (A_2^3 - A_3^3) \equiv 0.$$

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coefficients of the cubics, though not devoid of interest in themselves, do not seem to affect the result already obtained in the more general case, viz., that the three cubics intersect in three real and distinct points, in addition to the vertices of C. Hence we conclude that every triangle of a Tq series is directly triply in perspective with three other triangles, point-triangles being excluded, of the same series.

I shall now discuss the second case of the general problem, where A and B, two triangles belonging to different Tp and Tq series, are supposed to be in triple perspective with each other not directly but perversely. In this case, the coordinates of the vertices of A and B being the same as before (see page 124), we have the negative terms of the determinant $\Delta pa \,.\, \Delta' b$ equal. Hence we have the following equations for the determination of $p_1 : p_2 : p_3$, the a's and the B's being regarded as constant,

$$
\begin{aligned}
&(p_1a_1\mathrm{B}_1 + p_2a_2\mathrm{B}_2 + p_3a_3\mathrm{B}_3)(p_1a_2\mathrm{B}_3 + p_2a_3\mathrm{B}_1 + p_3a_1\mathrm{B}_2)(p_1a_3\mathrm{B}_2 + p_2a_1\mathrm{B}_3 + p_3a_2\mathrm{B}_1)\\
= &(p_1a_1\mathrm{B}_2 + p_2a_2\mathrm{B}_3 + p_3a_3\mathrm{B}_1)(p_1a_2\mathrm{B}_1 + p_2a_3\mathrm{B}_2 + p_3a_1\mathrm{B}_3)(p_1a_3\mathrm{B}_3 + p_2a_1\mathrm{B}_1 + p_3a_2\mathrm{B}_2)\\
= &(p_1a_1\mathrm{B}_3 + p_2a_2\mathrm{B}_1 + p_3a_3\mathrm{B}_2)(p_1a_2\mathrm{B}_2 + p_2a_3\mathrm{B}_3 + p_3a_1\mathrm{B}_1)(p_1a_3\mathrm{B}_1 + p_2a_1\mathrm{B}_2 + p_3a_2\mathrm{B}_3).
\end{aligned}
$$

That is, the point P(p_1, p_2, p_3) may be any one of the nine points of intersection of the following system of cubics,

$$
\begin{aligned}
&p_2^2p_3\,(f_3 - f_5) + p_3^2p_1\,(f_5 - f_1) + p_1^2p_2\,(f_1 - f_3)\\
&+ p_2\,p_3^2(f_4 - f_6) + p_3\,p_1^2(f_6 - f_2) + p_1\,p_2^2(f_2 - f_4) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&p_2^2p_3\,(f_5 - f_1) + p_3^2p_1\,(f_1 - f_3) + p_1^2p_2\,(f_3 - f_5)\\
&+ p_2\,p_3^2(f_6 - f_2) + p_3\,p_1^2(f_2 - f_4) + p_1\,p_2^2(f_4 - f_6) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&p_2^2p_3\,(f_1 - f_3) + p_3^2p_1\,(f_3 - f_5) + p_1^2p_2\,(f_5 - f_1)\\
&+ p_2\,p_3^2(f_2 - f_4) + p_3\,p_1^2(f_4 - f_6) + p_1\,p_2^2(f_6 - f_2) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &\equiv a_2^2a_3\,\mathrm{B}_2^2\mathrm{B}_3 + a_3^2a_1\,\mathrm{B}_1^2\mathrm{B}_2 + a_1^2a_2\,\mathrm{B}_3^2\mathrm{B}_1\,,\\
f_2 &\equiv a_2\,a_3^2\mathrm{B}_1\,\mathrm{B}_2^2 + a_3\,a_1^2\mathrm{B}_3\,\mathrm{B}_1^2 + a_1\,a_2^2\mathrm{B}_2\,\mathrm{B}_3^2\,,\\
f_3 &\equiv a_2^2a_3\,\mathrm{B}_1^2\mathrm{B}_2 + a_3^2a_1\,\mathrm{B}_3^2\mathrm{B}_1 + a_1^2a_2\,\mathrm{B}_2^2\mathrm{B}_3\,,\\
f_4 &\equiv a_2\,a_3^2\mathrm{B}_3\,\mathrm{B}_1^2 + a_3\,a_1^2\mathrm{B}_2\,\mathrm{B}_3^2 + a_1\,a_2^2\mathrm{B}_1\,\mathrm{B}_2^2\,,\\
f_5 &\equiv a_2^2a_3\,\mathrm{B}_3^2\mathrm{B}_1 + a_3^2a_1\,\mathrm{B}_2^2\mathrm{B}_3 + a_1^2a_2\,\mathrm{B}_1^2\mathrm{B}_2\,,
\end{aligned}
$$

and

$$
f_6 \equiv a_2\,a_3^2\mathrm{B}_2\,\mathrm{B}_3^2 + a_3\,a_1^2\mathrm{B}_1\,\mathrm{B}_2^2 + a_1\,a_2^2\mathrm{B}_3\,\mathrm{B}_1^2\,.
$$

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Six of the points of intersection of the above cubics are evident from inspection, viz., the three vertices of the fundamental triangle C, and the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, and $(1, \omega^2, \omega)$. When P coincides with any one of the vertices of C, the triangle A, as we have already seen, reduces to that vertex. When it coincides with the point $(1, 1, 1)$, A belongs to the same Tp series as B, which is contrary to the supposition. When it coincides with either of the points $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, A again belongs, as can readily be shown, to the same Tp series as B, and, further, has imaginary vertices. Thus the six positions of P which have been found give us no real triangle A, with distinct vertices and belonging to a different Tp series from B.

With regard to the remaining three positions of P, it is evident from the equations that if $p_1 : p_2 : p_3 = \lambda_1 : \lambda_2 : \lambda_3$ be one solution, then $p_1 : p_2 : p_3 = \lambda_2 : \lambda_3 : \lambda_1$ and $p_1 : p_2 : p_3 = \lambda_3 : \lambda_1 : \lambda_2$ are also solutions. Hence these three positions of P must all be real, for if one were imaginary the remaining two would be so also, and the cubics would intersect in four real and five imaginary points. Again, the three undetermined positions of P do not in general coincide with any one of the six positions that have been determined, for it can be shown that the cubics do not touch each other at any of these six points, and that, in the absence of particular relations connecting the a's and the B's, no one of these six points is a double point. Further, the three undetermined positions do not in general coincide with each other, for in order that $\lambda_1 : \lambda_2 : \lambda_3 = \lambda_2 : \lambda_3 : \lambda_1 = \lambda_3 : \lambda_1 : \lambda_2$, we must have $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$. That is, the three undetermined positions cannot coincide with each other unless they coincide with one of the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, which, as we have just seen, is not the case.

Thus we are led to the conclusion that in general there are three and only three positions of P which give us a real triangle A in perverse triple perspective with B, possessing distinct vertices and satisfying the condition that it shall belong to a certain Tq series different from that of B and at the same time to some Tp series (distinct for each of the three cases) different from that of B. Combining this with the result obtained in the case when A and B were supposed to be in direct triple perspective, we conclude that if any two Tq series be selected, every triangle of the one is in triple perspective with six triangles of the other, the restriction that triangles belonging to the same Tp series be excluded from consideration being observed. When this restriction is removed we can say that every triangle of a Tq series is in triple perspective with seven triangles of any other Tq series, directly with three and perversely with four.

In the special case when A and B, while still belonging to different Tp series, belong to the same Tq series, we may as before suppose that B becomes A′, the coordinates of whose vertices are given by the rows of Δa. In this case the three cubics which by their intersection determine the nine possible positions of P are the same as those already given on page 128, except that A must be substituted throughout for B in the values there given for the f's. If the new f's be distinguished by dashes, we readily verify that

$$f_1' - f_3' \equiv -a_1(a_2^3 - a_3^3)A_1 \,.\, \Delta a \equiv f_4' - f_6',$$

$$f_3' - f_5' \equiv -a_2(a_3^3 - a_1^3)A_2 \,.\, \Delta a \equiv f_6' - f_2',$$

and $$f_5' - f_1' \equiv -a_3(a_1^3 - a_2^3)A_3 \,.\, \Delta a \equiv f_2' - f_4'.$$

These relations between the elements and the complementary minors of a persymmetric determinant of the type Δa seem in themselves noteworthy. They lead at once to the further identity

$$a_1(a_2^3 - a_3^3)A_1 + a_2(a_3^3 - a_1^3)A_2 + a_3(a_1^3 - a_2^3)A_3 \equiv 0.$$

The identities obviously still hold when, by having two of its rows interchanged, Δa is converted into a circulant of the third order.

As a result of these identities the equations of the three intersecting cubics may be written

$$U_1 \equiv p_1(p_2^2 + p_3^2)\alpha_3 + p_2(p_3^2 + p_1^2)\alpha_1 + p_3(p_1^2 + p_2^2)\alpha_2 = 0,$$

$$U_2 \equiv p_1(p_2^2 + p_3^2)\alpha_1 + p_2(p_3^2 + p_1^2)\alpha_2 + p_3(p_1^2 + p_2^2)\alpha_3 = 0,$$

and $$U_3 \equiv p_1(p_2^2 + p_3^2)\alpha_2 + p_2(p_3^2 + p_1^2)\alpha_3 + p_3(p_1^2 + p_2^2)\alpha_1 = 0,$$

where $\alpha_1 + \alpha_2 + \alpha_3 \equiv 0$.

U_1, U_2, U_3 evidently intersect in the same six points as the cubics obtained in connection with the more general case, viz., in the three vertices of C, and in the points $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$.

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Further,

$$\frac{dU_\kappa}{dp_1} \equiv (p_2^2+p_3^2)a_\mu + 2p_1p_2a_\kappa + 2p_3p_1a_\lambda,$$

$$\frac{dU_\kappa}{dp_2} \equiv 2p_1p_2a_\mu + (p_3^2+p_1^2)a_\kappa + 2p_2p_3a_\lambda,$$

and $$\frac{dU_\kappa}{dp_3} \equiv 2p_3p_1a_\mu + 2p_2p_3a_\kappa + (p_1^2+p_2^2)a_\lambda,$$

$(\kappa=1,\ 2,\ 3;\ \lambda=2,\ 3,\ 1;\ \mu=3,\ 1,\ 2)$.

Therefore when $p_1=p_2=p_3=1$,

$$\frac{dU_\kappa}{dp_1} = \frac{dU_\kappa}{dp_2} = \frac{dU_\kappa}{dp_3} = 2(a_\kappa + a_\lambda + a_\mu) = 0.$$

U_1, U_2, U_3, consequently, have a double point at (1, 1, 1). Their nine points of intersection have therefore been determined, since the double point counts as four intersections. It thus appears that if we exclude imaginary and point triangles, we have only one triangle A (viz., that derived from A′ by the multiplication of the coordinates of its vertices by (1, 1, 1)) which belongs to the same Tq series as A′ and is yet in perverse triple perspective with it. But since in this case A is just the same as A′, it also must be excluded. Thus we reach the conclusion that in any Tq series (that series alone excepted whose constituent triangles all coincide with the fundamental triangle C) no two triangles are in perverse triple perspective with each other. Combining this with the result obtained for the case when A and A′ were supposed to be in direct triple perspective, we conclude that in any Tq series every triangle is in triple perspective with three and no more than three other triangles of the same series, and that the perspective is direct.

It has been proved that in general every triangle of a Tq series is in triple perspective with seven and no more than seven triangles of any other Tq series. When certain particular relations exist between the coordinates of the two triangles which define the two Tq series, this number may be exceeded. A case of this arises in the following way.

Let A′ be the isobaric triangle and A_1 any other triangle of the Tq series defined by the triangle whose vertices have for coordinates

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the rows of the determinant Δa, *i.e.*, by A′. This series I shall call the Tq (a) series. The coordinates of the vertices of A′ and A_1 are

$$\begin{array}{c} A_1' \\ A_2' \\ A_3' \end{array}\begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix}, \quad \begin{array}{c} A_{11} \\ A_{12} \\ A_{13} \end{array}\begin{vmatrix} p_1a_1, & p_2a_2, & p_3a_3 \\ p_1a_2, & p_2a_3, & p_3a_1 \\ p_1a_3, & p_2a_1, & p_3a_2 \end{vmatrix}.$$

Let B_1 be the triangle obtained by joining the corresponding vertices of A′ and A. The equations of the sides of B_1 will be

$$A_1'A_{11} \equiv x_1(p_2 - p_3)/a_1 + x_2(p_3 - p_1)/a_2 + x_3(p_1 - p_2)/a_3 = 0,$$
$$A_2'A_{12} \equiv x_1(p_2 - p_3)/a_2 + x_2(p_3 - p_1)/a_3 + x_3(p_1 - p_2)/a_1 = 0,$$
$$A_3'A_{13} \equiv x_1(p_2 - p_3)/a_3 + x_2(p_3 - p_1)/a_1 + x_3(p_1 - p_2)/a_2 = 0.$$

The form of the coefficients shows, as Mr Ferrari has pointed out, that B_1 is in direct triple perspective with the fundamental triangle C. By forming the coordinates of its vertices we find that it belongs to the Tq series defined by the triangle

$$\begin{vmatrix} \dfrac{A_1}{a_1}, & \dfrac{A_2}{a_2}, & \dfrac{A_3}{a_3} \\ \dfrac{A_2}{a_2}, & \dfrac{A_3}{a_3}, & \dfrac{A_1}{a_1} \\ \dfrac{A_3}{a_3}, & \dfrac{A_1}{a_1}, & \dfrac{A_2}{a_2} \end{vmatrix},$$

a series which I shall denote by the symbol T$q(b)$, and to the Tp series defined by the point

$$P\left(\frac{1}{p_2 - p_3}, \quad \frac{1}{p_3 - p_1}, \quad \frac{1}{p_1 - p_2}\right).$$

An infinity of values of $\lambda_1 : \lambda_2 : \lambda_3$ can be found to satisfy the equation

$$\lambda_1(p_2 - p_3) + \lambda_2(p_3 - p_1) + \lambda_3(p_1 - p_2) = 0.$$

For all these values,

$\lambda_1 a_1$, $\lambda_2 a_2$, $\lambda_3 a_3$ satisfy the equation of the first side of B_1,

$\lambda_1 a_2$, $\lambda_2 a_3$, $\lambda_3 a_1$ satisfy the equation of the second side of B_1,

and $\lambda_1 a_3$, $\lambda_2 a_1$, $\lambda_3 a_2$ satisfy the equation of the third side of B_1.

That is to say, B_1 is circumscribed not only to the triangles A′ and A_1 of the T$q(a)$ series, but to a whole infinity of triangles of that series.

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Again, by joining the vertices of A′ to the corresponding vertices of A_2 any other triangle of the Tq (a) series which is not inscribed in B_1, we obtain another triangle B_2 which also belongs to the T$q(b)$ series, and is circumscribed to an infinity of triangles of the T$q(a)$ series. Proceeding in this way we can form an infinity of triangles, B_1, B_2, $B_3 \ldots B_\infty$, all belonging to the series T$q(b)$ and each of them circumscribed to an infinity of triangles of the series T$q(a)$, thus exhausting all the triangles of that series. The triangles B_1, B_2, $B_3 \ldots B_\infty$ are all circumscribed to the triangle A′. By starting with some other triangle of the T$q(a)$ series than A′, and by joining its vertices to the corresponding vertices of other triangles of the series, we find in the same way that it also is inscribed in an infinity of triangles belonging to the T$q(b)$ series. Hence we reach the conclusion that every triangle of the T$q(b)$ series is circumscribed to an infinity of triangles of the T$q(a)$ series, and that every triangle of the latter series is inscribed in an infinity of triangles of the former. Since two triangles such that one is inscribed in the other are necessarily in triple perspective with each other, it follows that for every two Tq series, the coordinates of the vertices of whose defining triangles are related as in T$q(a)$ and T$q(b)$, every triangle of the one is in triple perspective (perversely) with an infinity of triangles of the other.

The coordinates given above for P in the case of the triangle B_1 show that for the triangles of the subseries B_1, B_2, $B_3 \ldots B_\infty$, the locus of P is the circumconic

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=0.$$

Three of the triangles of the series T$q(b)$ reduce to point-triangles coinciding with the vertices of the fundamental triangle C, viz., the three triangles obtained by making P coincide with these vertices in succession. It is easy to see that these are the only point-triangles in the series T$q(b)$. Since A′ is in direct triple perspective, and consequently in perspective of the first kind, with three other triangles of the series T$q(a)$, three of the triangles of the subseries B_1, B_2, $B_3 \ldots B_\infty$ must reduce to points, viz., the centres of perspective corresponding to these three perspectives of the first kind. Hence the three point-triangles of the series T$q(b)$ are included in the subseries B_1, B_2, $B_3 \ldots B_\infty$, and consequently in every other such subseries of $\mathrm{T}q(b)$. Therefore we infer that every triangle of the $\mathrm{T}q(a)$ series, and consequently of every other $\mathrm{T}q$ series, is in simple perspective of the first kind with three infinities of triangles of the same series, with respect to the three vertices of C respectively as common centres of perspective.

A fuller exploration of the ∞^4 triangles in triple perspective with a given triangle would, no doubt, lead to many other interesting relations subsisting between them, but for the present those given may suffice.

A considerable number of elementary theorems on determinants arise in connection with the study of these triangles. The following contingent identities connecting the elements a_κ and the complementary minors A_κ of a persymmetric determinant of three elements, a_1, a_2, a_3, and consequently of the circulant obtained by transposing two of its rows, may serve as a specimen. Let A and A′ be two triangles belonging to the same $\mathrm{T}q$ series, the latter being the isobaric or defining triangle of the series, and let the coordinates of their vertices be given by the rows of the following determinants,

$$\mathrm{A}\ \begin{vmatrix} p_1a_1, & p_2a_2, & p_3a_3 \\ p_1a_2, & p_2a_3, & p_3a_1 \\ p_1a_3, & p_2a_1, & p_3a_2 \end{vmatrix} \equiv \Delta pa, \qquad \mathrm{A}'\ \begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_1 \\ a_3, & a_1, & a_2 \end{vmatrix} \equiv \Delta a.$$

Then the determinant $\Delta pa\,.\,\Delta' a$, obtained by multiplying by rows Δpa and the adjugate of Δa, has for its first, second, and third positive terms, and its first, second, and third negative terms respectively

$$\left.\begin{aligned} \alpha_1 &\equiv \Pi(p_1a_\kappa A_\kappa + p_2a_\lambda A_\lambda + p_3a_\mu A_\mu), \\ \alpha_2 &\equiv \Pi(p_1a_\kappa A_\lambda + p_2a_\lambda A_\mu + p_3a_\mu A_\kappa), \\ \alpha_3 &\equiv \Pi(p_1a_\kappa A_\mu + p_2a_\lambda A_\kappa + p_3a_\mu A_\lambda), \end{aligned}\right\} \left(\begin{aligned} \kappa &= 1,\ 2,\ 3 \\ \lambda &= 2,\ 3,\ 1 \\ \mu &= 3,\ 1,\ 2 \end{aligned}\right)$$

$$\alpha_4 \equiv (p_1a_1A_1 + p_2a_2A_2 + p_3a_3A_3)(p_1a_2A_3 + p_2a_3A_1 + p_3a_1A_2)(p_1a_3A_2 + p_2a_1A_3 + p_3a_2A_1),$$

$$\alpha_5 \equiv (p_1a_1A_2 + p_2a_2A_3 + p_3a_3A_1)(p_1a_2A_1 + p_2a_3A_2 + p_3a_1A_3)(p_1a_3A_3 + p_2a_1A_1 + p_3a_2A_2),$$

$$\alpha_6 \equiv (p_1a_1A_3 + p_2a_2A_1 + p_3a_3A_2)(p_1a_2A_2 + p_2a_3A_3 + p_3a_1A_1)(p_1a_3A_1 + p_2a_1A_2 + p_3a_2A_3)$$

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The tangential coordinates of the sides of A are given by the rows of the determinant

$$\begin{vmatrix} \dfrac{A_1}{p_1}, & \dfrac{A_2}{p_2}, & \dfrac{A_3}{p_3} \\ \dfrac{A_2}{p_1}, & \dfrac{A_3}{p_2}, & \dfrac{A_1}{p_3} \\ \dfrac{A_3}{p_1}, & \dfrac{A_1}{p_2}, & \dfrac{A_2}{p_3} \end{vmatrix}$$

which though not strictly speaking the adjugate of Δpa, being in fact the adjugate after each row has been divided by $p_1p_2p_3$, may be denoted by $\Delta' pa$. The first, second, and third positive terms and the first, second, and third negative terms of the determinant $\Delta a \,.\, \Delta' pa$, where the multiplication is effected in precisely the same manner as in the case of $\Delta pa \,.\, \Delta' a$, are easily seen to be exactly the same as α_1, α_2, α_3, α_4, α_5, α_6, respectively, except that p_κ is everywhere replaced by $1/p_\kappa$. They may be denoted by α_1', α_2', α_3', α_4', α_5', α_6'. Now if A and A$'$ be in perspective in any way, two of the α's must be equal, and also two of the α''s. For example, suppose that we have the perspective

$$\frac{A_1\,A_2\,A_3}{A_2'A_3'A_1'}. \quad \text{This is the same as the perspective} \quad \frac{A_1'A_2'A_3'}{A_3\,A_1\,A_2}.$$

Hence in this case $\alpha_3 = \alpha_1$, and also $\alpha_1' = \alpha_2'$. Hence, generally, if $\alpha_1 = \alpha_2$, then $\alpha_3' = \alpha_1'$, if $\alpha_2 = \alpha_3$, $\alpha_3' = \alpha_2'$, if $\alpha_3 = \alpha_1$, then $\alpha_1' = \alpha_2'$, if $\alpha_4 = \alpha_5$, then $\alpha_6' = \alpha_4'$, if $\alpha_5 = \alpha_6$, then $\alpha_6' = \alpha_5'$, if $\alpha_6 = \alpha_4$, then $\alpha_4' = \alpha_5'$, and conversely. We have already seen that the relation $\alpha_4 = \alpha_5 = \alpha_6$, and consequently also the relation $\alpha_4' = \alpha_5' = \alpha_6'$, can be true, zero values of the p's being excluded, only when

$$p_1 : p_2 : p_3 = 1 : 1 : 1 \text{ or } 1 : \omega : \omega^2 \text{ or } 1 : \omega^2 : \omega.$$

It may be worth stating as affording an exercise in determinants that if we eliminate p_1 by the dialytic method from the cubic equations U_1, U_2 given on page 130, and remove the factor p_2p_3, we obtain the determinant

$$\begin{vmatrix} 0 & , & p_2\alpha_1 + p_3\alpha_2 \,, & (p_2^2 + p_3^2)\alpha_3 & , & p_2\alpha_2 + p_3\alpha_1 \\ p_2\alpha_1 + p_3\alpha_2, & & (p_2^2 + p_3^2)\alpha_3, & p_2p_3(p_2\alpha_2 + p_3\alpha_1), & & 0 \\ p_2\alpha_2 + p_3\alpha_3, & & (p_2^2 + p_3^2)\alpha_1, & p_2p_3(p_2\alpha_3 + p_3\alpha_2), & & 0 \\ 0 & , & p_2\alpha_2 + p_3\alpha_3 \,, & (p_2^2 + p_3^2)\alpha_1 & , & p_2\alpha_3 + p_3\alpha_2 \end{vmatrix},$$

which, since the cubics have a double point at (1, 1, 1), and intersect besides at $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$, must be equal to $(p_2-p_3)^3(p_2^3-p_3^3)$, when the condition

$$a_1+a_2+a_3=0,$$

is fulfilled.

IV.

I conclude with a note on a paper by Mr Eugen Jahnke in *Crelle's Journal* (Bd. 123, Heft 1, p. 42) entitled "Eine dreifach perspectiven Dreiecken zugehörige Punktgruppe," in which the author studies a group of 9 points, V_{ik}, called by him the *Veronese* points, arising in connection with two directly perspective triangles $A_1A_2A_3$ and $B_1B_2B_3$, and defined as follows, the brackets containing the pairs of lines whose intersections give the points :—

$$V_{11}(A_2B_3,\ A_3B_2),\quad V_{12}(A_3B_2,\ A_1B_1),\quad V_{13}(A_1B_1,\ A_2B_3),$$
$$V_{21}(A_2B_2,\ A_3B_1),\quad V_{22}(A_3B_1,\ A_1B_3),\quad V_{23}(A_1B_3,\ A_2B_2),$$
$$V_{31}(A_2B_1,\ A_3B_3),\quad V_{32}(A_3B_3,\ A_1B_2),\quad V_{33}(A_1B_2,\ A_2B_1).$$

If C_1, C_2, C_3 be the centres of perspective of $A_1A_2A_3$ and $B_1B_2B_3$ (C_1 lying on A_1B_1, C_2 on A_2B_2, and C_3 on A_3B_3), then, as is well known, the three triangles A, B, C are triply in perspective in pairs, the centres of perspective of each two being the vertices of the third. Triangles related in this way are said by Mr Jahnke to be in *desmic* position.

A further group of 18 points, P_i, Q_i, R_i, X_i, Y_i, Z_i $(i=1, 2, 3)$ arises in connection with the following theorem given by Mr Jahnke :

$A_1A_2A_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{22}V_{33},\ P_1C_3C_2;\quad V_{12}V_{23}V_{31},\ C_1C_3P_2;\quad V_{13}V_{21}V_{32},\ C_1P_3C_2;$$
$$V_{11}V_{32}V_{23},\ X_1B_2B_3;\quad V_{13}V_{31}V_{22},\ B_1X_2B_3;\quad V_{12}V_{33}V_{21},\ B_1B_2X_3;$$

$B_1B_2B_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{22}V_{33},\ Q_1C_2C_3;\quad V_{12}V_{23}V_{31},\ C_1Q_2C_3;\quad V_{13}V_{21}V_{32},\ C_1C_2Q_3;$$
$$V_{11}V_{31}V_{21},\ Y_1A_2A_3;\quad V_{12}V_{32}V_{22},\ A_1Y_2A_3;\quad V_{13}V_{33}V_{23},\ A_1A_2Y_3;$$

and $C_1C_2C_3$ is in desmic position with the six pairs of triangles

$$V_{11}V_{31}V_{21},\ R_1A_3A_2;\quad V_{12}V_{32}V_{22},\ A_1A_3R_2;\quad V_{13}V_{23}V_{33},\ A_1R_3A_2;$$
$$V_{11}V_{23}V_{32},\ Z_1B_2B_3;\quad V_{13}V_{22}V_{31},\ B_1Z_2B_3;\quad V_{12}V_{21}V_{33},\ B_1B_2Z_3.$$

Mr Jahnke gives numerous theorems regarding the V-points, and one regarding the points of the second group, viz., that the following triads of points are collinear :

$$C_i,\ P_i,\ Q_i;\quad A_i,\ Y_i,\ R_i;\quad B_i,\ Z_i,\ X_i \qquad (i=1,\ 2,\ 3).$$

I add the following :

(1) The triangles A, B, C, P, Q, R, X, Y, Z are triply in perspective with each other in pairs, and in fact belong to the same Tp series with respect to any one of them as triangle of reference ;

(2) The triads A, P, X ; B, Q, Y : C, R, Z are in desmic position ;

(3) The three triangles $V_{11}V_{22}V_{33}$, $V_{23}V_{31}V_{12}$, $V_{32}V_{13}V_{21}$ belong to the same Tq series with respect to A. Since, as Mr Jahnke has shown, these triangles are also in direct triple perspective with B, we have here an illustration of the fact already proved, that a triangle of one Tq series is in direct triple perspective with three triangles of any other Tq series.

The proof of these propositions, which I omit, may be obtained by projecting the figure so that the coordinates of the vertices of B with respect to A are given by the rows of a determinant of the form

$$\begin{vmatrix} b_1, & b_2, & b_3 \\ b_2, & b_3, & b_1 \\ b_3, & b_1, & b_2 \end{vmatrix},$$

and by calculating the coordinates of the vertices of the other triangles involved. It will be found that the coordinates of the vertices of P, Q, R, X, Y, Z are the rows of determinants which either are circulants or can be converted into circulants by interchanging the last two rows.