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ALGEBRAICALLY CLOSED REGULAR RINGS

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Introduction. In this paper all rings are commutative and have a unity. All ring homomorphisms preserve the unity. We let $\mathbf{L}$ denote the standard language for rings with two distinct constants, 0 and 1, playing the role of the zero and the unity respectively. A ring is *regular* if it satisfies the axiom $(\forall r)(\exists r')(rr'r = r)$ and it is *algebraically closed* if, for each integer $n \geqq 1$, it satisfies the sentence

$$(\forall r_0) \ldots (\forall r_{n-1})(\exists x)(r_0 + r_1x + \ldots + r_{n-1}x^{n-1} + x^n = 0).$$

Throughout this paper $\mathscr{Z}$ denotes the ring of integers. The definitions of a *theory*, an *elementary extension*, a *universal extension*, a *model-complete theory*, an *elementary equivalence*, and a *complete theory*, are given in [**2**]. If a ring S is an elementary extension of the ring R we write $R \prec S$. If S is a universal extension of R we write $R \subseteq_\forall S$. If R and S are elementarily equivalent we write $R \equiv S$. For a ring R let $\mathbf{B}(R) = \{e \in R : e^2 = e\}$. By introducing suitable new additions on $\mathbf{B}(R)$ (see [**4**, Section 1]), it may be viewed as either a Boolean algebra or as a Boolean ring. Further, where R and S are rings and $\sim$ is any one of the relations $\cong$, $\equiv$, $\prec$, or $\subseteq_\forall$, $\mathbf{B}(R) \sim \mathbf{B}(S)$ as Boolean algebras if and only if $\mathbf{B}(R) \sim \mathbf{B}(S)$ as Boolean rings. Also, if $R \sim S$ then $\mathbf{B}(R) \sim \mathbf{B}(S)$. We thus may (and do) view $\mathbf{L}$ as also containing these new additions on idempotents, as ternary relations on the universe, without altering any of the relations $\cong$, $\equiv$, $\prec$, or $\subseteq_\forall$, amongst rings. For a ring R we shall sometimes augment $\mathbf{L}$ to a language $\mathbf{L}(\mathbf{B}(R))$ by adding the elements of $\mathbf{B}(R)$ as constants, and the diagrams of $\mathbf{B}(R)$, both as a Boolean algebra and as a Boolean ring, as axioms. In this case a ring S is a model of $\mathbf{L}(\mathbf{B}(R))$ if and only if $\mathbf{B}(R) \subseteq \mathbf{B}(S)$ both as Boolean rings and Boolean algebras. For a ring R, let $X(R)$ denote the Stone space of the Boolean algebra $\mathbf{B}(R)$. (If R is regular, this is also the spectrum of R.) For any ring T and topological space X, let $\mathbf{C}(X, T)$ denote the ring of all continuous functions from X to T, where T has the discrete topology.

In [**4**, Theorem 3.1] and [**5**, Theorem 3.6] continuous function representations were given for certain regular algebraically closed algebras R over some field L. More specifically, paraphrasing [**4**, Theorem 3.1], if R is also algebraic over L, then $R \cong \mathbf{C}(X(R), F)$, where F is the algebraic closure of L. In [**5**, Theorem

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3.6] we showed that if $\mathbf{B}(R)$ is atomless, we can drop the hypothesis that R be algebraic over L, and still conclude that $R \equiv \mathbf{C}(X(R),F)$.

In this paper we generalize these results by dropping the requirements that R be an algebra over some field and (required only in [**5**]) that $\mathbf{B}(R)$ be atomless. However any ring admits a unique, natural, $\mathscr{Z}$-algebra structure, so that suitable analogues of the above results may be stated with $\mathscr{Z}$ in place of L. In this paper all rings are also viewed, without further comment, as $\mathscr{Z}$-algebras. We may not, however, represent R in terms of some $\mathbf{C}(X(R),F)$, since the elements of R may fail to have some common zero or prime additive order. To state and prove our results we require the representation of rings due to R. S. Pierce [**11**] which associates with each ring R a sheaf of rings $k(R)$ over $X(R)$ and a natural equivalence $R \cong \Gamma(X(R), k(R))$, where, for any subset $U \subseteq X(R)$, $\Gamma(U,k(R))$ denotes the ring of all continuous sections of $k(R)$ over U. The ring R is regular if and only if each stalk $(k(R))_x$ of $k(R)$ is a field, and is algebraically closed if and only if each $(k(R))_x$ is too. A brief description of this representation and the relevant notation and terminology, occurs in [**4**, Section 1].

In Section 1 we show that if R and S are algebraically closed regular rings such that $X(R) = X(S)$, $\mathrm{char}(k(R)_x) = \mathrm{char}(k(S)_x)$ for all $x \in X(R)$ (where $\mathrm{char}(L')$ denotes the characteristic of a field L'), R and S are algebraic over $\mathscr{Z}$, and $\mathbf{B}(R)$ and $\mathbf{B}(S)$ satisfy a condition related to but weaker than self-injectivity, then $R \cong S$. In Section 2 we show that if the hypothesis on $\mathbf{B}(R)$ and $\mathbf{B}(S)$, and the assumption that R and S be algebraic over $\mathscr{Z}$, are dropped, then we still have $R \equiv S$. Using these results and known embedding theorems, any regular ring can be embedded in a ring whose structure is given by one of these results.

Results similar to those described from Section 2 have recently been obtained by S. Comer [**7**, Theorem 1.1], using the proofs of the Feferman-Vaught results [**9**]. Our methods and preliminary results are different, and demonstrate that tests similar to Robinson's model-completeness test and the prime model test for complete theories, may be fruitfully applied to some theories that are neither model-complete nor have a prime model.

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1. Structure results for reduced algebraic $\mathscr{Z}$-algebras. We need the following definitions to state our results.

Definition 1.1. A sheaf K of fields over a Boolean space X is an *algebraic closure* of a sheaf k of fields over X if:

(i) k is a subsheaf of K, and

(ii) for each $x \in X$, K_x is the algebraic closure of k_x.

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The algebraic closure of a sheaf need neither exist [**4**, example after 3.4], nor be unique (example after Corollary 1.7).

Definition 1.2. (i) For a field L, let $\operatorname{char}(L)$ denote the characteristic of L.

(ii) For a sheaf k of fields over a Boolean space X, let $\operatorname{char}(k)\colon X \to N$, the natural numbers, be given by $x \to \operatorname{char}(k_x)$.

(iii) An *admissable characteristic function*, over a Boolean space X, is a continuous function $g\colon X \to N$, where N is topologized as the one-point compactification of $N - \{0\}$, such that, for each $x \in X$, $f(x)$ is either zero or prime.

The function $\operatorname{char}(k)$ is clearly an admissable characteristic function, since the sections of k over X have *clopen* (i.e. closed and open) support.

Definition 1.3. Let λ be a cardinal number.

(i) A Boolean space X has the λ-disjointness property if, whenever U and V are disjoint subsets of X each of which can be expressed as some union of fewer than λ clopen subsets of X, then $\operatorname{cl}(U) \cap \operatorname{cl}(V) = \emptyset$.

(ii) A ring R is λ-self-injective if, for each ideal I in R that can be generated by fewer than λ-elements and each $f \in \operatorname{Hom}_R(I, R)$, there exists $f' \in \operatorname{Hom}_R(R, R)$ such that $f'|I = f$.

The concepts from Definition 1.3 are dealt with in [**6**]. In particular, if R is a Boolean ring, then R is λ-self-injective if and only if its Stone space $X(R)$ has the λ-disjointness property.

Our results in this section are all corollaries of the following theorem.

THEOREM 1.4. *Let g be an admissable characteristic function over X, where X is a Boolean space. Then:*

(i) *There exists a unique* (to within isomorphism over X) *sheaf $\mathscr{P}(g)$ over X such that $\mathscr{P}(g)_x$ is the prime field of characteristic $g(x)$, for all $x \in X$.*

(ii) *The sheaf $\mathscr{P}(g)$ has an algebraic closure, $\mathscr{C}(g)$.*

(iii) *Suppose that X has the $\aleph_1$-disjointness property. Then the sheaf $\mathscr{C}(g)$ is unique, to within an isomorphism over X.*

Before proving this theorem, we obtain some corollaries.

Notation. The above meanings for $\mathscr{P}(g)$ and $\mathscr{C}(g)$ are retained throughout this paper. For a sheaf k of fields over X, let $\mathscr{P}(k)$ denote $\mathscr{P}(\operatorname{char}(k))$ and let $\mathscr{C}(k)$ denote $\mathscr{C}(\operatorname{char}(k))$. For a regular ring R, find the sheaf of fields $k(R)$ over some Boolean space $X(R)$ such that $R \cong \Gamma(X(R), k(R))$, and let $\operatorname{char}(R) = \operatorname{char}(k(R))$, $\mathscr{P}(R) = \mathscr{P}(k(R))$, and $\mathscr{C}(R) = \mathscr{C}(k(R))$.

This is a slight abuse of notation since, unless X and $X(R)$ are "nice", the sheaves $\mathscr{C}(g)$, $\mathscr{C}(k)$, and $\mathscr{C}(R)$, need not be unique.

COROLLARY 1.5. *Suppose that the ring R is regular, and algebraic over the integers $\mathscr{Z}$, that $\mathbf{B}(R)$ is $\aleph_1$-self-injective, and that $k(R)$ has an algebraic closure. Then there is an embedding $R \to \Gamma(X(R), \mathscr{C}(R))$, and $\mathscr{C}(R)$ depends only on* $\operatorname{char}(R)$.

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Proof. Since R is regular, $k(R)$ is a sheaf of fields over $X(R)$. Any algebraic closure K, of $k(R)$, is also an algebraic closure of $\mathscr{P}(R)$, since the hypothesis that R is algebraic over $\mathscr{Z}$ implies that K_x is algebraic over $\mathscr{P}(R)_x$, for all $x \in X$. By Theorem 1.4 (iii) this closure is unique, so that $K \cong \mathscr{C}(R)$. Thus there is an embedding

$$R \cong \Gamma(X(R), k(R)) \subseteq \Gamma(X(R), K) \cong \Gamma(X(R), \mathscr{C}(R)).$$

Remark. If R and $\Gamma(X(R), k(R))$ are identified, the above embedding acts like the identity on idempotents.

The above result gives some sufficient conditions for R to be embedded in a ring which does not depend upon the topology of $k(R)$, but merely upon $\mathbf{B}(R)$ and $\mathrm{char}(R)$. In spite of this, we shall see in the example at the end of this paper, it is possible to have R and S both satisfying the hypothesis from Corollary 1.5, $X(R) = X(S)$ (i.e. $\mathbf{B}(R) = \mathbf{B}(S)$), $(k(R))_x = (k(S))_x$ for all $x \in X(R)$, yet $R \ncong S$. This reflects the fact that if $\Phi: K \rightarrow K'$ is an isomorphism, where K and K' are the algebraic closures of $k(R)$ and $k(S)$ respectively, $\Phi|k(R)$ need not have $k(S)$ as its range. However the most common condition ensuring that the closures K and K' exist would be that $\mathbf{B}(R)$ is self-injective (see [**4**, Lemma 2.5 and Theorem 2.7]). In this case, we note in Corollary 1.9, $R \cong S$ does hold.

COROLLARY 1.6. *Suppose that the rings R and S are regular, algebraically closed, and algebraic over $\mathscr{Z}$, that $X(R) = X(S)$ (i.e., $\mathbf{B}(R) = \mathbf{B}(S)$), that $\mathrm{char}(R) = \mathrm{char}(S)$, and that $\mathbf{B}(R)$ and $\mathbf{B}(S)$ are $\aleph_1$-self-injective. Then $R \cong S$ by an isomorphism which acts as the identity on idempotents.*

Proof. Since R and S are algebraically closed, the sheaves $k(R)$ and $k(S)$ are algebraic closures of $\mathscr{P}(R) \cong \mathscr{P}(S)$. Thus, by Theorem 1.4.(iii) $k(R) \cong k(S)$ over X. This induces the required isomorphism

$$R \cong \Gamma(X(R), k(R)) \cong \Gamma(X(S), k(S)) \cong S.$$

COROLLARY 1.7. *Suppose that R and S are algebraically closed regular rings such that $X(R) = X(S)$ (i.e., $\mathbf{B}(R) = \mathbf{B}(S)$), $\mathrm{char}(R) = \mathrm{char}(S)$, and $\mathbf{B}(R)$ is $\aleph_1$-self-injective. Then there exists an algebraically closed regular ring T such that $\mathbf{B}(R) = \mathbf{B}(T) = \mathbf{B}(S)$, and there are embeddings $T \rightarrow R$ and $T \rightarrow S$, which act like the identity on idempotents.*

Proof. Let

$$\mathscr{R}' = \{r(x): x \in X, r \in \Gamma(X, k(R)), \text{and} f(r) = 0 \text{ for some } f(Y) \in \mathscr{Z}[Y]\},$$

and similarly define $\mathscr{S}'$. These are subsheaves of $k(R)$ and $k(S)$ respectively and, in fact, are algebraic closures of $\mathscr{P}(R)$ and $\mathscr{P}(S)$ respectively. Thus, by Theorem 1.4.(iii), $\mathscr{R}'$ and $\mathscr{S}'$ are isomorphic over X. Thus $T = \Gamma(X, \mathscr{R}')$, with the obvious embeddings, satisfies the lemma.

The following example shows that the $\aleph_1$-disjointness property and $\aleph_1$-self-injective conditions appearing in 1.4–1.7 are not irrelevant.

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Example. We construct an admissible characteristic function g, over some Boolean space X, such that $\mathscr{P}(g)$ has at least two nonisomorphic algebraic closures, $\mathscr{C}$ and $\mathscr{C}'$. The rings $R = \Gamma(X, \mathscr{C})$ and $S = \Gamma(X, \mathscr{C}')$ would be counterexamples to Corollaries 1.5–1.7, if the hypothesis that $\mathbf{B}(R)$ and $\mathbf{B}(S)$ are $\aleph_1$-self-injective were dropped. Let $X = \omega + 1$, and let g be the function with domain X such that $g(\omega) = 0$ and, for $n \in \omega$, $g(n)$ is the nth prime that can be written as $4m + 1$, for some integer $m \geqq 2$. This defines an admissible characteristic function over X. Let $f(Y) = Y^2 + 1$ and

$$U = \{x \in X: f(Y) \text{ has a root in } \mathscr{P}(g)_x\}.$$

A classic result of number theory asserts that $U = X - \{\omega\} = \omega$. Let τ and $\tau' \in \Gamma(U, \mathscr{P}(g))$ be such that $\tau(n)$ and $\tau'(n)$ are roots of $f(Y)$ and $\tau(n) = (-1)^n\tau'(n)$, for all $n \in U$. From [**4**, Theorem 2.7 and the proof of Lemma 2.2] we obtain algebraic closures $\mathscr{C}$ and $\mathscr{C}'$ of $\mathscr{P}(g)$ such that τ and τ' may be extended to, and identified with, $\tau \in \Gamma(X, \mathscr{C})$ and $\tau' \in \Gamma(X, \mathscr{C}')$ satisfying $f(\tau(\omega)) = 0 = f(\tau'(\omega))$. We show that $\mathscr{C}$ and $\mathscr{C}'$ are not isomorphic. Suppose that Φ: $\mathscr{C} \to \mathscr{C}'$ is a sheaf isomorphism. Then $\Phi(\tau(\omega)) = \pm\tau'(\omega)$, since Φ restricted to any $\mathscr{C}_x$ simply permutes the roots of any polynomial over the prime subfield, and hence of any polynomial over $\mathscr{Z}$. If $\Phi(\tau(\omega)) = \tau'(\omega)$ then, by the continuity of Φ, $\Phi(\tau(n)) = \tau'(n) = (-1)^n\tau(n)$, for all sufficiently large n. This is a contradiction, since any isomorphism acts as the identity on a prime subfield. A similar contradiction results if $\Phi(\tau(\omega)) = -\tau'(\omega)$.

Proof of Theorem 1.4.

(i) Let $\{k_x: x \in X\}$ be a collection of prime fields such that $g(x) = \operatorname{char}(k_x)$, for all $x \in X$. Let $\mathscr{P}(g) = k$, where k is the disjoint union $k = \bigcup\{k_x: x \in X\}$. We establish the result by showing that k may be topologized in exactly one way so as to be a sheaf of fields over X. For $n \in \mathscr{Z}$ let $\mathbf{n}$: $X \to k$ be given by $\mathbf{n}(x) = n \cdot 1_x$, where 1_x is the identity in k_x. Let $\mathbf{n}(x)/\mathbf{m}(x)$ be an arbitrary element of k, where n, $m \in \mathscr{Z}$ and $x \in X$. If k were a sheaf, then sets of the form

$$\mathbf{S}(n, m, x, N) = \{\mathbf{n}(y)/\mathbf{m}(y): y \in N\},$$

where N is a neighborhood of x in X such that $\mathbf{m}(y) \neq 0$ for $y \in N$, would be open neighborhoods of $\mathbf{n}(x)/\mathbf{m}(x)$ in k_x. Since g is continuous, $\mathbf{S}(n, m, x, N)$ is defined for all sufficiently small neighborhoods of x. It is now easy to verify that k, with the (unique) topology generated by $\mathbf{S}(n, m, x, N)$, where n, $m \in \mathscr{Z}$, $m \neq 0$, $x \in N$, and N is sufficiently small, is a sheaf of fields over X.

(ii) We establish the existence of $\mathscr{C}(g)$ by constructing an algebraic closure for $\mathscr{P}(g) = k$. Let $Y = g(X)$. Then Y, as a subspace of N (see Definition 1.2 (iii)), is a Boolean space and the inclusion map $h : Y \to N$ is an admissible characteristic function, so that the sheaf $\mathscr{P}(h)$ over Y exists, by (i). Since Y is countable, [**4**, Theorem 2.7] yields an algebraic closure $\mathscr{C}(h)$, over Y, of $\mathscr{P}(h)$. Then

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$$\mathscr{C}(g) = X +_g \mathscr{C}(h) = \{(x, s) : x \in X, s \in \mathscr{C}(h), \text{ and } s \in \mathscr{C}_{g(x)}(h)\},$$

topologized as a subset of $X \times \mathscr{C}(h)$, is the required sheaf.

(iii) Let $\mathscr{C}$ and $\mathscr{C}'$ be two algebraic closures for $\mathscr{P}(g) = k$. For each n, where n is zero or prime, let $V_n = \{x \in X : \text{char}(g(x)) = n\}$, F_n be the algebraic closure of the prime field of characteristic n, and $U = X - V_0$. The V_p are clopen, for p prime, so that V_0 is closed. By [**4**, Lemma 3.1] and (i) we have:

(1)* $\quad \mathscr{C}|V_n \cong V_n \times F_n \cong \mathscr{C}'|V_n,$

where $V_n \times F_n$ denotes the simple F_n-sheaf over V_n. Let

$$\mathbf{F} = \{(\mathscr{S}, \Psi) : \mathscr{S} \text{ is a subsheaf of } \mathscr{C}, \text{ each } \mathscr{S}|V_n \text{ is a simple sheaf,}$$
$$\text{and } \Psi : \mathscr{S} \to \mathscr{C}' \text{ is a sheaf embedding.}\}$$

Define a partial ordering $\leqq$ on F by

$$(\mathscr{S}, \Psi) \leqq (\mathscr{S}', \Psi') \text{ if and only if } \mathscr{S} \text{ is a subsheaf of } \mathscr{S}' \text{ and } \Psi'|\mathscr{S} = \Psi.$$

By Zorn's lemma pick a maximal element $(\mathscr{M}, \Psi)$ of $\mathbf{F}$. We shall show that $\mathscr{M} = \mathscr{C}$ and conclude, since each $\mathscr{C}_x$ is algebraic over $\mathscr{P}_x(g)$, that $\Psi : \mathscr{C} \to \mathscr{C}'$ is an isomorphism.

For $i \neq 0$ and $g(x) = i$, the maximality of $(\mathscr{M}, \Psi)$, the clopenness of V_i, (1)*, and [**4**, Lemma 3.1], ensure that

$$\mathscr{M}|V_i \cong V_i \times \mathscr{C}_x \cong V_i \times F_i.$$

Thus if $\mathscr{M} \neq \mathscr{S}$, then $\mathscr{M}|V_0 \cong V_0 \times L$ for some proper subfield L of F_0. Let $\alpha \in F - L$ and let $\boldsymbol{\alpha} \in \Gamma(V_0, \mathscr{M})$ be such that $\boldsymbol{\alpha}(v) = \alpha$ for all $v \in V_0$. Let $m(Y)$ be the minimal polynomial of α over L and let $\mathbf{m}(Y) \in \Gamma(V_0, \mathscr{M})[Y]$ be such that $\mathbf{m}(Y)(v) = m(Y)$ for all $v \in V_0$. Since V_0 is closed we may extend $\boldsymbol{\alpha}$ and $\mathbf{m}(Y)$ to, and identify them with, elements of $\Gamma(X, \mathscr{M})$ and $\Gamma(X, \mathscr{M})[Y]$ respectively, satisfying $\mathbf{m}(\boldsymbol{\alpha})(x) = 0$, for all $x \in X$. Let

$$\mathscr{N} = \left\{ \sum_{i=0}^{n} \sigma_i(x) \boldsymbol{\alpha}^i(x) : x \in X \text{ and each } \sigma_i \in \Gamma(X, \mathscr{M}) \right\},$$

where $n = \deg(\mathbf{m}(Y))$. Then $\mathscr{N}$ is a subsheaf of $\mathscr{C}$ and $\mathscr{M}$ is a proper subsheaf of $\mathscr{N}$. We shall extend Ψ to an embedding $\Phi : \mathscr{N} \to \mathscr{C}'$. Since Ψ induces an embedding $\Gamma(X, \mathscr{M}) \to \Gamma(X, \mathscr{C}')$, we may factor $\mathbf{m}(Y)$ in $\Gamma(X, \mathscr{C}')$, obtaining

$$\mathbf{m}(Y) = (Y - \tau_1) \ldots (Y - \tau_n).$$

Whenever $1 \leqq i \leqq n$ let

$$W_i = \{x \in U : \Psi(\boldsymbol{\alpha}(x)) = \tau_i(x)\} - \cup\{W_j : 1 \leqq j < i\}.$$

Since $W_i \cap V_j$ is always clopen, each W_i is an $\aleph_1$-set. Hence we have $\bar{W}_i \cap \bar{W}_j = \emptyset$ whenever $i \neq j$, and

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$$\bar{U} = \bar{W}_1 \cup \ldots \cup \bar{W}_n.$$

As a result there exists $\sigma \in \Gamma(\bar{U}, \mathscr{C}')$ satisfying

(2)* $\quad \Psi(\alpha(x)) = \sigma(x),$

for all $x \in \bar{U}$. It is standard that σ may be extended to and identified with an element of $\Gamma(X, \mathscr{C}')$ satisfying $\mathbf{m}(\sigma)(x) = 0$, for all $x \in X$. The required embedding $\Phi : \mathscr{N} \to \mathscr{C}'$ is defined to be the sheaf morphism such that

(3)* $\quad \Phi|\mathscr{M} = \Psi,$

and

(4)* $\quad \Psi(\alpha(x)) = \sigma(x), \quad$ for all $x \in X$.

Since (2)* holds, (3)* and (4)* are mutually consistent. This contradicts the maximality of $\mathscr{M}, \Psi$). Hence $\Psi : \mathscr{C} \to \mathscr{C}'$ is a sheaf isomorphism.

A Boolean space X is called extremally disconnected if $\bar{U} \cap \bar{V} = \emptyset$ whenever U and V are open sets such that $U \cap V = \emptyset$ or, equivalently if $\bar{U}$ is clopen, for each open set U. By [**4**, Theorem 2.7], any sheaf of fields over an extremally disconnected space has an algebraic closure. The following corollaries result from this fact and arguments, which we omit, similar to those above.

Corollary 1.8. *Suppose that $\mathscr{R}$ and $\mathscr{S}$ are sheaves of fields over an extremally disconnected Boolean space X, that $\mathscr{R}_x = \mathscr{S}_x$, and $\mathscr{R}_x$ is algebraic over $\mathscr{P}(\mathscr{R})_x$, for all $x \in X$. Then $\mathscr{R}$ and $\mathscr{S}$ are isomorphic over X.*

It is well-known (see [**11**, Proposition 24.1] and [**10**, § 2.4]) that the Stone space of a Boolean algebra B is extremally disconnected if and only if B is complete (or equivalently, if B is self-injective as a ring).

Thus we may restate Corollary 1.8 as:

Corollary 1.9. *Suppose that the rings R and S are regular, algebraic over $\mathscr{Z}$, that $X(R) = X(S)$* (i.e., $\mathbf{B}(R) = \mathbf{B}(S)$), *that $\mathbf{B}(R)$ is complete, and that $k(R)_x = k(S)_x$, for all $x \in X(R)$. Then $R \cong S$ by some isomorphism which acts like the identity on idempotents.*

Analogues of Corollaries 1.8 and 1.9 hold if $\mathscr{P}(\mathscr{R})$ is replaced with a simple sheaf of fields, and $\mathscr{Z}$ is replaced with a field.

2. Elementary properties of algebraically closed regular rings. Let Σ_1 denote the theory of commutative algebraically closed regular rings. Let Σ_0 and Σ_n, where $n > 1$ is an integer, denote the theories

$$\Sigma_1 \cup \{(\forall x)(n'x = 0 \to x = 0) : n' > 1\}$$

and

$$\Sigma_1 \cup \{(\forall x)(nx = 0)\},$$

respectively. For any Boolean algebra B and $n \geqq 0$, let $\Sigma_n(B)$ denote the theory, in $\mathbf{L}$, of those models R of Σ_n such that $\mathbf{B}(R) \equiv B$.

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The main result in this section is:

THEOREM 2.1. *Let R and S be models of Σ_1 such that $X(R) = X(S)$* (i.e., $\mathbf{B}(R) = \mathbf{B}(S)$) *and* $\mathrm{char}(R) = \mathrm{char}(S)$. *Then $R \equiv S$ in $\mathbf{L}$ and in $\mathbf{L}(\mathbf{B}(R))$.*

Our proof of Theorem 2.1 (which is delayed until necessary preliminary results are available) rests upon Robinson's test for the model-completeness of a theory and a variant of the prime model test for complete theories. However two difficulties block a straightforward application of these tests to the theory $\Sigma_1(B)$:

(1) The theory $\Sigma_1(B)$ fails to be model complete, for some Boolean algebras, B. This is due to the scarcity of model-complete theories of Boolean algebras.

(2) The theory $\Sigma_1(B)$ fails to have a prime model since $\mathrm{char}(T)$ is not unique, where T is a model of $\Sigma_1(B)$ such that $\mathbf{B}(T) = B$. Worse still, the obvious path around this difficulty fails; by the example following 1.7, the theory $\Sigma_f(B) = \Sigma_1(B) \cup \mathbf{D}(\Gamma(X(B), \mathscr{P}(f))$ may fail to have a prime model, where $f = \mathrm{char}(R)$ and $\mathbf{D}(\sim)$ denotes the diagram of $\sim$.

The first difficulty is overcome by adding new predicates to $\mathbf{L}$, and axioms to Σ_1, to obtain a model-complete theory. To overcome the second difficulty, we construct models R', S', and T of Σ_1 such that $R \prec R'$, $S \prec S'$, $T \prec R'$, and $T \prec S'$.

The key technical lemma in overcoming both of these difficulties follows. We recall that a minimal nonzero element in a Boolean algebra is called an *atom*.

LEMMA 2.2. *Let R and S be models of Σ_1. Suppose that $R \subseteq S$ and each atom in $\mathbf{B}(R)$ is also an atom in $\mathbf{B}(S)$. Then $R \subseteq\forall S$.*

Proof. Let $\theta \in \mathbf{L}(R)$ be an arbitrary existential statement. It clearly suffices to show that

(α) if $S \models \theta$ then $R \models \theta$.

Using standard methods, such as those from the proof of [**2**, Theorem 9.4.5.], we may assume, without loss of generality, that θ is *primitive over* R. That is to say

(β) $\theta = (\exists b_1) \ldots (\exists b_m)(\phi \wedge \neg \psi_1 \wedge \ldots \wedge \neg \psi_n)$,

where m and n are positive integers dependent on θ, ϕ is a conjunction of atomic formulae, each ψ_i is atomic, and all constants occurring in ϕ belong to R. (An atomic formula in any theory of rings is in one of these forms: $u + v = w$, $u \cdot v = w$, or $u = v$.)

Let $X = X(R)$, $Y = X(S)$, $\mathscr{R} = k(R)$, and $\mathscr{S} = k(S)$. Let the natural isomorphisms $R \cong \Gamma(X, \mathscr{R})$ and $S \cong \Gamma(Y, \mathscr{S})$ be denoted by $r \rightarrow \bar{r}$ and $s \rightarrow \hat{s}$, respectively. We shall obtain and use criteria for $S \models \theta$, in terms of Y and $\mathscr{S}$. To do this, we need the following notation. For each $y \in Y$ let $\hat{\mathscr{R}}_y = \{\hat{a}(y) \colon a \in R\}$. This is a subfield of $\mathscr{S}_y$. Suppose that $\{a_1, \ldots, a_r\}$ is the set of

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constants from R occurring in θ. For each $x \in X$ and $y \in Y$, let $\bar{\phi}_x$ denote the formula

$$\phi(\bar{a}_1(x), \ldots, \bar{a}_r(x), b_1, \ldots, b_m)$$

and $\tilde{\phi}_y$ denote the formula

$$\phi(\hat{a}_1(y), \ldots, \hat{a}_r(y), b_1, \ldots, b_m).$$

A similar convention applies to $(\bar{\psi}_i)_x$ and $(\tilde{\psi}_i)_y$, when $1 \leqq i \leqq n$. Clearly

$$(\exists b_1) \ldots (\exists b_m)[\bar{\phi}_x \wedge \neg (\bar{\psi}_1)_x \wedge \ldots \wedge \neg (\bar{\psi}_n)x]$$

is a primitive formula over $\mathscr{R}_x$ and

$$(\exists b_1) \ldots (\exists b_m)[\tilde{\phi}_y \wedge \neg (\tilde{\psi}_1)_y \wedge \ldots \wedge \neg (\tilde{\psi}_n)_y]$$

is a primitive formula over $\hat{\mathscr{R}}_y$.

Since θ is a primitive formula, an easy modification of the proof of [**11**, Proposition 3.4] establishes:

Criteria 1. $S \models \theta$ if and only if

$$(\gamma) \qquad (\exists b_1) \ldots (\exists b_m)\tilde{\phi}_y$$

holds in $\mathscr{S}_y$ for all $y \in Y$, and there exists $y(1), \ldots, y(n) \in Y$ such that

$$(\delta(i)) \qquad (\exists b_1) \ldots (\exists b_m)[\tilde{\phi}_y \wedge \neg (\tilde{\psi}_i)_y]$$

holds in $\mathscr{S}_y$, when $y = y(i)$ and $1 \leqq i \leqq n$, and

$$(\epsilon(i_1, \ldots, i_s)) \quad (\exists b_1) \ldots (\exists b_m)[\tilde{\phi}_y \wedge \neg (\tilde{\psi}_{i_1})_y \wedge \ldots \wedge \neg (\tilde{\psi}_{i_s})_y]$$

holds in $\mathscr{S}_y$ for $y = y(i_1)$, whenever $y(i_1) = \ldots = y(i_s)$.

Criteria 2. This is the analogue of Criteria 1, with X in place of Y and $\mathscr{R}$ in place of $\mathscr{S}$, and determines when $R \models \theta$.

Since $R \subseteq S$ there exists, as in [**11**, Lemma 6.3], a continuous onto map f: $Y \to X$ and a map g: $Y \times R \to S$ satisfying (a) and (b).

(a) For each $y \in Y$, the map $g(y, -)$ is a ring homomorphism from $\mathscr{R}_{f(y)}$ to $\mathscr{S}_y$. This map is actually one-one, since $\mathscr{R}_{f(y)}$ is a field.

(b) $g(y, \bar{a}(f(y))) = \hat{a}(y)$, for each $y \in Y$ and $a \in R$.

We also have (c) and (d).

(c) By (a) and (b) there is an isomorphism $h_y : \hat{\mathscr{R}}_y \to \mathscr{R}_{f(y)}$, for each $y \in Y$, such that $h_y(\hat{a}(y)) = \bar{a}(f(y))$, each $a \in R$. The fields $\hat{\mathscr{R}}_y \cong \mathscr{R}_{f(y)}$ and $\mathscr{S}_y$ are algebraically closed, since R and S are algebraically closed rings.

(d) If $x \in X$ is isolated, then $|f^{-1}(x)| = 1$ and $f^{-1}(x)$ consists of an isolated point in Y. To see this let $a \in R$ be such that $\bar{a}(x) = 1$ and $\bar{a}(x') = 0$ when $x' \neq x$. By (b), $\hat{a}(y) = 1$ when $y \in f^{-1}(x)$ and $\hat{a}(y) = 0$ otherwise. Using this one sees that if (d) fails, $\hat{a}$ is not an atom in $\mathbf{B}(\Gamma(Y, \mathscr{S})) \cong \mathbf{B}(S)$, yet $\bar{a}$ is an atom in $\mathbf{B}(\Gamma(X, \mathscr{R})) \cong \mathbf{B}(R)$. This would contradict our hypothesis.

Now suppose that $S \models \theta$ and let $y(1), \ldots, y(n) \in Y$ be as in criteria 1. Since the theory of algebraically closed fields is model-complete (by [**2**, p. 197])

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$(\delta(i))$ also holds in $\hat{\mathscr{R}}_y$, when $y = y(i)$ and $1 \leqq i \leqq n$, and $\epsilon(i_1, \ldots, i_s)$ holds in $\hat{\mathscr{R}}_y$, for $y = y(i_1)$, whenever $y(i_1) = \ldots = y(i_s)$. Application of the isomorphisms h_y now shows that, since f is onto,

$$(\gamma') \qquad (\exists b_1) \ldots (\exists b_m)\bar{\phi}_x$$

holds in $\mathscr{R}_x$ for all $x \in X$,

$$(\delta'(i)) \qquad (\exists b_1) \ldots (\exists b_m)[\bar{\phi}_x \wedge \neg\ (\bar{\psi}_i)_x]$$

holds in $\mathscr{R}_x$ for $x = f(y_i)$ and $1 \leqq i \leqq n$, and $(\epsilon'(i_1, \ldots i_s))$

$$(\exists b_1) \ldots (\exists b_m)[\bar{\phi}_x \wedge \neg\ (\bar{\psi}_{i_1})_x \wedge \ldots \wedge \neg\ (\bar{\psi}_{i_s})_x]$$

holds in $\mathscr{R}_x$ if $x = f(y(i_1))$ and $y(i_1) = \ldots = y(i_s)$. We can not yet conclude from criteria 2 that $R \models \theta$, since the function f need not be one-one. To overcome this difficulty we shall replace $\{f(y(1)), \ldots, f(y(n))\}$ with some suitable subset $\{x(1), \ldots, x(n)\}$ of X. If $f(y(i))$ is isolated in X, let $x(i) = f(y(i))$. Now suppose that $f(y(i))$ is not isolated in X and that

$$\{y(j): f(y(j)) = f(y(i))\} = \{y(i_1), \ldots, y(i_{s'})\}.$$

By a paraphrase of [**11**, Lemma 10.5], there is a neighborhood N of $f(y(i))$ such that $(\delta'(i_j))$ holds in $\mathscr{R}_x$, when $x \in N$ and $1 \leqq j \leqq s'$. We assume, without loss of generality, that $f(y(j)) \in N$ inplies that $f(y(j)) = f(y(i))$, for $1 \leqq j \leqq n$. The set N is infinite, since $f(y(i))$ is not isolated, so that there exist distinct $x(i_1), \ldots, x(i_{s'}) \in X$ such that $(\delta'(i_j))$ holds in $\mathscr{R}_x$, when $x = x(i_j)$ and $1 \leqq j \leqq s'$.

Thus, by the construction of $\{x(1), \ldots, x(n)\}$ we have:

(e) $\delta'(i)$ holds in $\mathscr{R}_x$, when $x = x(i)$ and $1 \leqq i \leqq n$.

(f) If $x(i_1) = \ldots = x(i_s)$ (where $i_1 \neq i_s$), then $x(i_1)$ is isolated in X, $f(y(i_1)) = x(i_1)$, and, by (*d*) $y(i_1) = \ldots = y(i_s)$. Thus, as already noted, $(\epsilon'(i_1, \ldots, i_s))$ holds in $\mathscr{R}_x$ if $x = f(y(i_1)) = x(i_1)$.

By (e), (f), the sentence involving (γ'), and Criteria 2, $R \models \theta$. Thus, $R \subseteq\forall S$.

The above proof merely required that all of the stalks of R and S belong to some common model-complete class. In particular, if A is a Boolean ring, then each $k(A)_x$ is the two element field, since it satisfies $(\forall r)(r^2 - r = 0)$. Hence:

LEMMA 2.3. *Suppose that A and B are Boolean algebras (or Boolean rings) such that $A \subseteq B$ and each atom in A is an atom in B. Then $A \subseteq\forall B$.*

We overcome difficulty (1) with:

PROPOSITION 2.4. *Suppose that R and S are models of Σ_1 such that $R \subseteq S$ and $\mathbf{B}(R) \prec \mathbf{B}(S)$. Then $R \prec S$.*

Proof. Define a unary predicate $I(\sim)$ by $I(x) \leftrightarrow (x^2 = x)$. The definition of the relativization, $\phi^{(W)}$, of a formula ϕ with respect to a unary predicate W, occurs in [**2**, p. 249]. Augment $\mathbf{L}$ and Σ_1 to $\mathbf{L}^\#$ and $(\Sigma_1)^\#$ respectively by adding,

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for each formula $\phi \in \mathbf{L}$ with free variables $x_1, \ldots, x_n$, the predicate $\boldsymbol{\phi}$ to $\mathbf{L}$ and the axiom

$$(\forall x_1) \ldots (\forall x_n)[\boldsymbol{\phi} \leftrightarrow (I(x_1) \wedge \ldots \wedge I(x_n) \wedge \phi^{(I)})]$$

to Σ_1. With any model T of Σ_1 there is associated, in a natural way, a model $T^\#$ of $(\Sigma_1)^\#$ that has the same underlying set and ring operations as T. Indeed, any model of $(\Sigma_1)^\#$ has the form $T^\#$ for some model T of Σ_1. Our construction has achieved this: If $Q^\#$ and $T^\#$ are any models of $(\Sigma_1)^\#$, then an embedding $Q^\# \to T^\#$ is simply an embedding $Q \to T$ under which $\mathbf{B}(Q) \prec \mathbf{B}(T)$. Thus if $Q^\# \subseteq T^\#$, then each atom in $\mathbf{B}(Q)$ is an atom in $\mathbf{B}(T)$ so that, by Lemma 2.2, $Q \subseteq\forall\ T$ and, by a short argument involving the additional axioms, $Q^\# \subseteq\forall\ T^\#$. Robinson's test, as paraphrased in [**2**, Corollary 9.4.4] now applies, establishing that $(\Sigma_1)^\#$ is model-complete. The proposition now follows, for $R^\# \subseteq S^\#$ is an embedding (since $\mathbf{B}(R) \prec \mathbf{B}(S)$) so that $R^\# \prec S^\#$ and hence $R \prec S$.

The addition of new predicates to a theory to obtain a model-complete one first occurs in [**13**].

To overcome difficulty (2) we need:

LEMMA 2.5. *Let A be a Boolean ring and λ a cardinal number. Then A has an elementary embedding into some λ-self-injective Boolean ring B.*

Proof. To begin we construct, by transfinite induction, classes $\{A_\alpha\}$ and $\{B_\alpha\}$ of Boolean rings such that, whenever $\alpha < \alpha' \leqq \beta$,

(a) $A_0 = A$,
(b) $A_\alpha \subseteq B_\alpha \subseteq A_\beta \subseteq B_\beta$,
(c) $A_{\alpha'} \prec A_\beta$, and
(d) $B_{\alpha'}$ is the complete ring of quotients of $A_{\alpha'}$.

Suppose that suitable A_α and B_α have been defined for all $\alpha <$ some β. If β is a limit ordinal let

$$A_\beta = \text{inj lim } \{A_\alpha : \alpha < \beta\}.$$

It is standard [**2**, Theorem 4.2.1] that (c) holds, since $A_\alpha \prec A_{\alpha'}$ whenever $\alpha < \alpha' < \beta$. Now suppose that $\beta = \mu + 1$ for some ordinal μ. Note that $A_\mu \subseteq\forall\ B_\mu$ by Lemma 2.3 since, if there were an atom $a \in A_\mu$ such that $0 < b < a$ for some $b \in B_\mu$, we would have $(A_\mu)b \cap (A_\mu) = 0$, contrary to the fact that B_μ is a quotient ring of A_μ (see [**10**, 2.3 Proposition 6]). Thus, by [**2**, Lemma 9.3.9], there is an embedding $h_\mu : B_\mu \to (A_\mu)^{I/F}$ into some ultrapower of A_μ such that $h_\mu | A_\mu$ is the natural map $A_\mu \to (A_\mu)^{I/F}$. Let $A_\beta = (A_\mu)^{I/F}$. Making suitable identifications and choosing B_β to satisfy (d), (a) $-$ (d) are satisfied both when β is a limit or a successor ordinal.

To establish the lemma we fix a regular cardinal $\beta \geqq \lambda$ and show that A_β is λ-self-injective. It is, by (a) and (c), an elementary extension of A. Suppose that $f \in \operatorname{Hom}_{A_\beta}(I, A_\beta)$ where I is an ideal in A_β generated by some set S of

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cardinality $< \lambda \leqq \beta$. Since β is regular, $S \cup f(S) \subseteq A_\alpha \subseteq B_\alpha$, for some $\alpha < \beta$. Since B_α is self-injective, $f|B_\alpha \cap I$ can be extended to some map $f' \in \mathrm{Hom}_{B_\alpha}(B_\alpha, B_\alpha)$. Extend f' to $f'' \in \mathrm{Hom}_{A_\beta}(A_\beta, A_\beta)$ by setting $f''(a) = a \cdot f'(1)$, for all $a \in A_\beta$. Finally, $f''|I = f$ since $f''|S = f$. Hence A_β is λ-self-injective.

In [**8**] Ersov establishes that there is a Boolean ring B that is not elementarily equivalent to any countably complete Boolean ring. In particular B is not elementarily equivalent to any self-injective Boolean ring.

Proof of Theorem 2.1. By Lemma 2.5 find an $\aleph_1$-self-injective Boolean ring B such that $\mathbf{B}(R) = \mathbf{B}(S) \prec B$. Extend, by [**5**, Lemma 4.4] and its proof, this embedding to embeddings $R \rightarrow R'$ and $S \rightarrow S'$, where R' and S' are models of Σ_1 such that $\mathbf{B}(R') = B = \mathbf{B}(S')$. Since B is $\aleph_1$-self-injective there exists, by Corollary 1.7, a model T of Σ_1 such that $\mathbf{B}(T) = B$, $T \subseteq R'$, and $T \subseteq S'$. In fact $T \prec R'$, $T \prec S'$, $R \prec R'$, and $S \prec S'$, by Proposition 2.4. From these elementary embeddings we obtain $R' \equiv S'$ and $R \equiv S$, in the languages $\mathbf{L}(\mathbf{B}(R))$ and $\mathbf{L}$.

COROLLARY 2.6. *The theory $\Sigma_p(B)$ is complete, where p is zero or prime and B is a Boolean algebra.*

Proof. Let R and S be models of $\Sigma_p(B)$. Since $\mathbf{B}(R) \equiv \mathbf{B}(S)$ a standard application of Keisler's ultrapower theorem (see [**2**, Theorem 7.2.6]) produces models R' and S' of $\Sigma_p(B)$ such that $\mathbf{B}(R') \cong \mathbf{B}(S')$, $R \prec R'$, and $S \prec S'$. That $R \equiv S$, and thus $\Sigma_p(B)$ is complete, now follows from Theorem 2.1 since $\mathrm{char}(R')(x) = p = \mathrm{char}(S')(x)$, for all $x \in X(R')$.

COROLLARY 2.7. *Let R be a model of Σ_1. Then:*

(i) $\Gamma(X(R), \mathscr{C}(R) \prec R$, *where $\mathscr{C}(R)$ is any algebraic closure of $\mathscr{P}(R)$ which also is a subsheaf of $k(R)$.*

(ii) *If $\mathbf{B}(R)$ is $\aleph_1$-self-injective, then* $\Gamma(X(R), \mathscr{C}(R)) \prec R$, *where $\mathscr{C}(R)$ is the unique algebraic closure of $\mathscr{P}(R)$.*

Proof. This follows from Theorems 1.4 and 2.1.

Corollary 1.4. (ii) gives a sufficient condition for the ring R to be an elementary extension of some ring which does not depend upon the topology of $k(R)$, but merely upon $X(R)$ and $\mathrm{char}(R)$. By [**4**, Theorem 2.10], any regular ring R can be embedded in a model S of Σ_1, to which Corollary 2.7 (ii) applies. More specifically, S is algebraic over R and $\mathbf{B}(S)$ is the complete quotient ring of $\mathbf{B}(R)$. These facts generalize Corollaries 1.6 and 1.5 to regular rings that might not be algebraic over the integers.

Remark. If $\mathrm{char}(R)$ is a constant p, where p is zero or prime, then the hypothesis in Corollaries 1.5 $-$1.7 and 2.7 (ii) that $\mathbf{B}(R)$ (and $\mathbf{B}(S)$) be $\aleph_1$ self-injective may be removed. To see this for Corollary 1.6 we note that, by [**4**, Lemma 3.1], $R \cong \mathbf{C}(X(R), F)$ where F is the algebraic closure of the prime field of characteristic p. Similarly the rings T from Corollary 1.7 and

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$\Gamma(X(R), \mathscr{C}(R))$ from Corollaries 1.5 and 2.7 are isomorphic to $\mathbf{C}(X(R), F)$, in this situation.

The next example substantiates the discussion following Corollary 1.5 and establishes that the hypothesis in Corollary 1.6 and Theorem 2.1 that R and S be algebraically closed, can not be removed, even if char(R) is identically zero.

Example. We construct regular rings R and S, that are algebraic over $\mathscr{Z}$, such that $X(R) = X(S)$, $\mathbf{B}(R)$ znd $\mathbf{B}(S)$ are $\aleph_1$-self-injective, $k(R)_x = k(S)_x$ for all $x \in X(R)$, and $k(R)$ and $k(S)$ have algebraic closures, yet $R \not\equiv S$ in either $\mathbf{L}$ or $\mathbf{L}(\mathbf{B}(R))$. Consequently $R \not\cong S$. Let T_1 denote the field of real algebraic numbers and T_2 the rational numbers. Let X be the Boolean space constructed in [**6**, §3]. By its construction there exist disjoint open subsets U and V of X and a point $p \in X - (U \cup V)$ such that $U \cup V \cup \{p\} = X$ and $p \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Define functions τ and τ^* from X to T_1 by $\tau(x) = \tau^*(x) = \sqrt{2}$ when $x \in U \cup \{p\}$ and $\tau(x) = -\tau^*(x) = \sqrt{2}$ when $x \in V$. Much as in the example following [**4**, Lemma 2.2] there exist sheaves $\mathscr{R}$ and $\mathscr{S}$ over the Boolean space X such that $\mathscr{R}_x = T_1 = \mathscr{S}_x$ for $x \in U \cup V$, $\mathscr{R}_p = T_2(\sqrt{2}) = \mathscr{S}_p$, $\tau \in \Gamma(X, \mathscr{R})$, and $\tau^* \in \Gamma(X, \mathscr{S})$. Let $R = \Gamma(X, \mathscr{R})$ and $S = \Gamma(X, \mathscr{S})$. The rings R and S are algebraic over the rationals (and hence over $\mathscr{Z}$) since the stalks of $\mathscr{R}$ and $\mathscr{S}$ are, and since X is compact. By [**6**, Theorem 3.2] X has the $\aleph_1$-disjointness property so that $\mathbf{B}(R)$ and $\mathbf{B}(S)$ are $\aleph_1$-self-injective [**6**, Theorem 2.4]. The sheaf $k(R) = \mathscr{R}$ has an algebraic closure since it is actually a subsheaf of the simple sheaf $X \times T_1$. For any polynomial $f(Y) \in \mathscr{Z}[Y]$, $W = \{x \in X : f(Y)$ has a root in $\mathscr{S}_x\}$ equals $U \cup V$ or X. In either case there exists $\sigma \in \Gamma(W, \mathscr{S})$ such that $\sigma(w)$ is a root of $f(Y)$ for all $w \in W$. Hence, by [**4**, Lemma 2.2] $\mathscr{S}$ can be embedding in some sheaf $\mathscr{S}^1$ such that $\sigma \in \Gamma(X, \mathscr{S}^1)$ and $(\mathscr{S}^1)_x = \mathscr{S}_x(\sigma(x))$, for all $x \in X$. Continuing by transfinite induction, an algebraic closure for $\mathscr{S}$ can be constructed.

To see that $R \not\equiv S$ in $\mathbf{L}$ (and hence in $\mathbf{L}(\mathbf{B}(R))$ too), we define predicates $\mathbf{N}$ and $\mathbf{P}$ as follows:

$$\mathbf{N}(e) \leftrightarrow [e^2 = e \wedge \neg\ (\exists\, r)(r^2 = e + e + e)],\ \mathbf{P}(r) \leftrightarrow (\exists\, e)$$

$$(\forall f)(\exists\, s)[\mathbf{N}(e) \wedge (f^2 = f \wedge \neg\ \mathbf{N}(f) \wedge 0 < f < e) \rightarrow rf = s^2].$$

Intuitively, in R and S, if e is an idempotent then $\mathbf{N}(e)$ holds if and only if $e(p) = 1$, fails if and only if $e(p) = 0$, and $\mathbf{P}(r)$ holds if and only if there is a neighborhood N of p such that if $x \in N - \{p\}$, then $r(x)$ is positive in T_1. The rings R and S are not elementarily equivalent since the sentence

$$(\exists\, r)[\mathbf{P}(r) \wedge r^2 = 1 + 1]$$

holds in R but not in S.

Remark. The referee suggests that Lemma 2.5 might be a special case of material from [**8**].

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