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ERROR BOUNDS FOR COMPOUND POISSON APPROXIMATIONS OF THE INDIVIDUAL RISK MODEL

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ABSTRACT

The approximation of the individual risk model by a compound Poisson model plays an important role in computational risk theory. It is thus desirable to have sharp lower and upper bounds for the error resulting from this approximation if the aggregate claims distribution, related probabilities or stop-loss premiums are calculated.

The aim of this paper is to unify the ideas and to extend to a more general setting the work done in this connection by BÜHLMANN et al. (1977), GERBER (1984) and others. The quality of the presented bounds is discussed and a comparison with the results of HIPP (1985) and HIPP & MICHEL (1990) is made.

KEYWORDS

Individual risk model; compound Poisson approximation; error bound; stop-loss premium.

1. INTRODUCTION

In the individual risk model an exact calculation of the aggregate claims distribution and of associated functions, such as stop-loss premiums, is very time consuming. Therefore actuaries often prefer to use an approximative computation method or to replace the individual model by a collective model in which the aggregate claims distribution can easily be calculated for example by Panjer's recursion formula.

Of course, only approximations that are close enough to the original model will be of real interest. Theoretical error bounds are helpful in this regard since they give a quantitative measure to asses the quality of an approximation.

This paper examines the error caused by approximating the individual model by a compound Poisson model. Explicit lower and upper bounds are derived for the error in calculating the distribution function of aggregate claims, associated probabilities and net stop-loss premiums. The analysis applies to approximations with different values of the Poisson parameter and generalizes the results obtained by BÜHLMANN et al. (1977) and GERBER (1984). It is shown that the error bounds for the aggregate claims distribution and for the stop-loss premiums are minimized for different values of the Poisson parameter. As a special case, it also pointed out that the well-known upper bound of

GERBER (1984) for the stop-loss error in the classical compound Poisson approximation, can be improved by a factor 1/2.

To conclude, the quality of the error bounds is discussed and a comparison is made with the results at which HIPP (1985) and HIPP & MICHEL (1990) arrive by applying concentration functions. It turns out that the presented bounds are much easier to calculate, but are only useful for small and moderate portfolios. This is inherent to the method used to derive these bounds.

2. COMPOUND POISSON APPROXIMATIONS OF THE INDIVIDUAL MODEL

Consider a portfolio containing n independent policies labelled from 1 to n. Let p_i, with $0 < p_i < 1$, denote the probability that policy i produces no claim in a given period and $q_i = 1 - p_i$ the probability that the policy leads to at least one claim. Further, define G_i as the conditional distribution of the total claim amount of policy i in the period, given that at least one claim occurs. As usual only positive claims are considered, that is $G_i(0) = 0$.

With this notation the distribution F_i of the claim amount generated by an individual policy i can be written as

$$F_i = p_i I + q_i G_i, \qquad i = 1, 2, \ldots, n, \tag{1}$$

where I is the atomic distribution concentrated at zero.

In the individual risk model, the distribution F^{ind} of the aggregate claims of the portfolio is obtained by convoluting the n distributions (1)

$$F^{ind} = \mathop{*}_{i=1}^{n} F_i. \tag{2}$$

Now, suppose that one wants to approximate the individual model by a compound Poisson model. This can be done by replacing each distribution F_i by a compound Poisson distribution P_i, say with Poisson parameter $\lambda_i > 0$ and amount distribution Q_i

$$P_i = \sum_{k=0}^{\infty} e^{-\lambda_i} \frac{\lambda_i^k}{k!} Q_i^{*k}, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where by convention $Q_i^{*0} = I$.

The quality of the resulting approximation will depend on the choice of λ_i and Q_i. Several arguments can be used and in the literature different proposals for λ_i can be found, but Q_i is always taken identical to G_i. This assumption will also be made in the remainder of the paper.

By taking the convolution of the compound Poisson distributions (3), one obtains an approximation F^{cP} for the distribution F^{ind} of the aggregate claims of the portfolio

$$F^{cP} = \mathop{*}_{i=1}^{n} P_i = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} G^{*k}, \tag{4}$$

which is again a compound Poisson distribution with Poisson parameter

$$\lambda = \sum_{i=1}^{n} \lambda_i \tag{5}$$

and amount distribution

$$G = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i G_i . \tag{6}$$

In the following sections the individual model will be compared with compound Poisson approximations having different specifications of the parameters λ_i. This will be done by deriving upper and lower bounds for the error which emerges in calculating the distribution function of aggregate claims, probabilities for arbitrary events and stop-loss premiums.

3. BOUNDS FOR THE AGGREGATE CLAIMS DISTRIBUTION

First some lemmas are given that will be useful in the proof of Theorem 1.

Lemma 1: Let F, G and H be distribution functions and assume that there exist constants a and b such that for all s

$$a \leq F(s) - G(s) \leq b . \tag{7}$$

Then, one has for all s

$$a \leq F * H(s) - G * H(s) \leq b . \tag{8}$$

Proof: The proof of (8) follows immediately from (7) and

$$F * H(s) - G * H(s) = \int_{-\infty}^{\infty} [F(s-x) - G(s-x)] \, dH(x) .$$

Q.E.D.

Lemma 2: Let $F_1, F_2, \ldots, F_n$ and $G_1, G_2, \ldots, G_n$ be distribution functions satisfying for all s

$$a_i \leq F_i(s) - G_i(s) \leq b_i, \qquad i = 1, 2, \ldots, n . \tag{9}$$

Then, one has for all s

$$\sum_{i=1}^{n} a_i \leq \mathop{*}_{i=1}^{n} F_i(s) - \mathop{*}_{i=1}^{n} G_i(s) \leq \sum_{i=1}^{n} b_i . \tag{10}$$

Proof: The lemma is proved by induction. By assumption (10) holds for $n = 1$. Assume that it also holds for $n = k - 1$. Then, one has by using Lemma 1 twice

$$\sum_{i=1}^{k-1} a_i \leq \left(\mathop{*}_{i=1}^{k-1} F_i\right) * F_k(s) - \left(\mathop{*}_{i=1}^{k-1} G_i\right) * F_k(s) \leq \sum_{i=1}^{k-1} b_i$$

and

$$a_k \leq \left(\mathop{*}_{i=1}^{k-1} G_i\right) * F_k(s) - \left(\mathop{*}_{i=1}^{k-1} G_i\right) * G_k(s) \leq b_k .$$

Taking the sum shows that the result holds for $n = k$, which proves the lemma.

Q.E.D.

The following Theorem allows an assessment of the error in calculating the distribution function of aggregate claims, which results if the individual model is replaced by a compound Poisson model, as presented in Section 2.

According to common usage the positive and negative parts of a number c are denoted respectively by $(c)^+$ and $-(c)^-$, that is $(c)^+ = \max(c, o)$ and $(c)^- = \min(c, o)$. It is clear that $-(c)^- = (-c)^+$, $c = (c)^+ + (c)^-$ and $|c| = (c)^+ - (c)^-$.

Theorem 1: For all s one has

$$\sum_{i=1}^{n} (p_i - e^{-\lambda_i})^- \leq F^{ind}(s) - F^{cP}(s) \leq \sum_{i=1}^{n} [p_i - e^{-\lambda_i} + (q_i - \lambda_i e^{-\lambda_i})^+] . \tag{11}$$

Proof: According to Lemma 2 it is sufficient to prove (11) for the special case $n = 1$, so that the index i can be dropped in the remainder of the proof. Then, one has

$$F^{ind} = pI + qG$$

and

$$F^{cP} = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} G^{*k}$$

with $G(0) = 0$.

If $s < 0$, then $F^{ind}(s) = F^{cP}(s) = 0$ and (11) is satisfied since

$$p - e^{-\lambda} + (q - \lambda e^{-\lambda})^+ \geq 1 - e^{-\lambda} - \lambda e^{-\lambda} \geq 0 .$$

The case of interest is $s \geq 0$. Then, one has

$$F^{ind}(s) - F^{cP}(s) = p - e^{-\lambda} + (q - \lambda e^{-\lambda})\, G(s) - \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} G^{*k}(s)$$

$$\leq p - e^{-\lambda} + (q - \lambda e^{-\lambda})^+$$

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Since $G^{*k}(s) \le G(s)$, one has on the other hand

$$F^{ind}(s) - F^{cP}(s) \ge p - e^{-\lambda} + (q - 1 + e^{-\lambda})\, G(s)$$
$$\ge p - e^{-\lambda} + (e^{-\lambda} - p)^- = (p - e^{-\lambda})^-$$

which proves the theorem.

Q.E.D.

Remark that $F^{cP}(s) \le F^{ind}(s)$ for all s, if $\lambda_i \ge -\ln p_i$ for $i = 1, 2, \ldots, n$.
Further note that if $q_i \ge \lambda_i e^{-\lambda_i}$ for $i = 1, 2, \ldots, n$, the upper bound in (11) can be simplified by using the inequality

$$p_i - e^{-\lambda_i} + (q_i - \lambda_i e^{-\lambda_i})^+ < \lambda_i^2/2, \tag{12}$$

which follows from

$$1 - (1 + \lambda_i)\, e^{-\lambda_i} = 1 - e^{\ln(1+\lambda_i) - \lambda_i} < 1 - e^{-\lambda_i^2/2} < \lambda_i^2/2.$$

Now the error bounds (11) will be specified for some choices of the λ_i used in the literature; see e.g. GERBER (1979, Ch. 4) for a description of the first two cases.

Case 1: The most common assumption is

$$\lambda_i = q_i \qquad i = 1, 2, \ldots, n. \tag{13}$$

This means that the Poisson parameter is chosen such that the expected number of claims is the same in the two models. Since $e^{-\lambda_i} > 1 - \lambda_i = p_i$ and $q_i > \lambda_i e^{-\lambda_i}$ one gets from (11) that for all s

$$-\frac{1}{2}\sum_{i=1}^{n} q_i^2 < \sum_{i=1}^{n} (p_i - e^{-q_i}) \le F^{ind}(s) - F^{cP}(s) \tag{14}$$
$$\le \sum_{i=1}^{n} [1 - (1 + q_i)\, e^{-q_i}] < \frac{1}{2}\sum_{i=1}^{n} q_i^2.$$

To show the magnitude of the error simpler bounds have been added to the left and right side. The right bound follows from (12).

Case 2: An alternative is to put

$$\lambda_i = -\ln p_i \qquad i = 1, 2, \ldots, n. \tag{15}$$

Under this assumption the probability of no claims is the same in the two models.
Since $q_i = 1 - e^{\lambda_i} > \lambda_i e^{-\lambda_i}$ one has from (11) and (12) that for all s

$$0 \le F^{ind}(s) - F^{cP}(s) \le \sum_{i=1}^{n} (q_i + p_i \ln p_i) < \frac{1}{2}\sum_{i=1}^{n} (\ln p_i)^2. \tag{16}$$

This error bound is given in HIPP (1985, formula (5)).

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Case 3: As noticed by HIPP (1986) the first order approximation for the aggregate claims distribution suggested by KORNYA (1983) can also be seen as a compound Poisson distribution with

$$\lambda_i = q_i/p_i \qquad i = 1, 2, \ldots, n. \tag{17}$$

Since $e^{-\lambda_i} > 1+\lambda_i = 1/p_i$ one has $e^{-\lambda_i} < p_i$ and $q_i = \lambda_i p_i > \lambda_i e^{-\lambda_i}$. Hence, it follows from (11) and (12) that for all s

$$0 \leq F^{ind}(s) - F^{cP}(s) \leq \sum_{i=1}^{n} (p_i - e^{-q_i/p_i})/p_i < \frac{1}{2} \sum_{i=1}^{n} (q_i/p_i)^2. \tag{18}$$

To conclude this section it will be examined which choice of the λ_i is preferable in the sense that the difference between the upper and lower bound in (11) is minimized. To that end consider the magnitude of

$$f_L(\lambda) = (p-e^{-\lambda})^-$$
$$f_U(\lambda) = p-e^{-\lambda}+(q-\lambda e^{-\lambda})^+$$

and

$$f(\lambda) = f_U(\lambda) - f_L(\lambda) = (p-e^{-\lambda})^+ + (q-\lambda e^{-\lambda})^+ \tag{19}$$

for different values of $\lambda > 0$.

Since $f_U(\lambda)$ is an ever increasing function of λ and $f_L(\lambda) = 0$ for $\lambda \geq -\ln p$, the function $f(\lambda)$ attains its minimum at a value $\lambda^* \leq -\ln p$. In case $\lambda \leq -\ln p$, $f(\lambda)$ takes the form $q - \lambda e^{-\lambda}$ which is a decreasing function if $\lambda < 1$ and an increasing function if $\lambda > 1$. Hence, $f(\lambda)$ is minimized for

$$\lambda^* = \begin{cases} -\ln p & \text{if} \quad -\ln p < 1 \\ 1 & \text{if} \quad -\ln p \geq 1 \end{cases} \tag{20}$$

The condition $-\ln p < 1$ corresponds with $q < 1 - e^{-1} = 0{,}632121$. Remark that the commonly used compound Poisson approximation with $\lambda_i = q_i$ does not give rise to the smallest difference between the upper and lower bound in (11).

4. BOUNDS FOR PROBABILITIES OF ARBITRARY EVENTS

Let $\mu(F, A)$ denote the probability that a random variable with distribution function F assumes a value in a set A. By convention the word set serves as abbreviation for a Borel set on the real line.

Theorem 2 gives explicit bounds for the maximal difference of probabilities calculated respectively in the original model and in an associated compound Poisson model. The proof of the Lemmas 3 and 4 is similar to the proofs given in Section 3 and is therefore omitted. Note that bounds which hold for every set A must be symmetrical, since $\mu(F, A^C) = 1 - \mu(F, A)$ where A^C denotes the complement of A.

Lemma 3: Let F, G and H be distribution functions and assume that there exists a constant b such that for every set A

(21) $$|\mu(F, A) - \mu(G, A)| \leq b.$$

Then, one has for any set A

(22) $$|\mu(F * H, A) - \mu(G * H, A)| \leq b.$$

Lemma 4: Let $F_1, F_2, \ldots, F_n$ and $G_1, G_2, \ldots, G_n$ be distribution functions satisfying for all sets A

(23) $$|\mu(F_i, A) - \mu(G_i, A)| \leq b_i, \qquad i = 1, 2, \ldots, n.$$

Then, one has for all A

(24) $$\left|\mu\left(\mathop{*}_{i=1}^{n} F_i, A\right) - \mu\left(\mathop{*}_{i=1}^{n} G_i, A\right)\right| \leq \sum_{i=1}^{n} b_i.$$

Theorem 2: For all sets A one has

(25) $$|\mu(F^{ind}, A) - \mu(F^{cP}, A)| \leq \sum_{i=1}^{n} [(p_i - e^{-\lambda_i})^+ + (q_i - \lambda_i e^{-\lambda_i})^+].$$

Proof: According to Lemma 4 it is sufficient to prove (25) for $n = 1$.

Using the same notation as in the proof of Theorem 1 one has

$$\mu(F^{ind}, A) - \mu(F^{cP}, A) = p\mu(I, A) + q\mu(G, A) - \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \mu(G^{*k}, A).$$

From this it follows that

$$\begin{aligned}(p - e^{-\lambda})^- + (q - \lambda e^{-\lambda})^- - 1 + e^{-\lambda} + \lambda e^{-\lambda} &\leq \mu(F^{ind}, A) - \mu(F^{cP}, A) \\ &\leq (p - e^{-\lambda})^+ + (q - \lambda e^{-\lambda})^+,\end{aligned}$$

which proves the theorem.

Q.E.D.

Remark that for sets of the form $A =]-\infty, s]$ better bounds are given in Theorem 1.

Now, (25) will be examined in detail for some special choices of the λ_i.

Case 1: If $\lambda_i = q_i$, $i = 1, 2, \ldots, n$, one gets for all sets A

(26) $$|\mu(F^{ind}, A) - \mu(F^{cP}, A)| \leq \sum_{i=1}^{n} q_i(1 - e^{-q_i}) < \sum_{i=1}^{n} q_i^2.$$

This error bound was derived by GERBER (1984). In the special case of a quasi homogeneous individual model, i.e. a portfolio consisting of n policies with different claim probabilities q_i but identical claim amount distributions G_i, the above bound was improved by MICHEL (1987). He showed that in this case

$$(27) \qquad |\mu(F^{ind}, A) - \mu(F^{cP}, A)| \leq \sum_{i=1}^{n} q_i^2 \Big/ \sum_{i=1}^{n} q_i,$$

which furnishes a smaller bound than (26) if $\lambda = \sum_{i=1}^{n} q_i > 1$.

Case 2: For $\lambda_i = -\ln p_i$, $i = 1, 2, \ldots, n$, one has for all A

$$(28) \qquad |\mu(F^{ind}, A) - \mu(F^{cP}, A)| \leq \sum_{i=1}^{n} (q_i + p_i \ln p_i) < \frac{1}{2} \sum_{i=1}^{n} (\ln p_i)^2 .$$

Case 3: If $\lambda_i = q_i/p_i$, $i = 1, 2, \ldots, n$, one obtains as bound

$$(29) \qquad |\mu(F^{ind}, A) - \mu(F^{cP}, A)| \leq \sum_{i=1}^{n} (p_i - e^{-q_i/p_i})/p_i < \frac{1}{2} \sum_{i=1}^{n} (q_i/p_i)^2 .$$

This bound has been derived by HIPP (1986, formula (3)).

Finally, remark that the magnitude of the error bound (25) is determined by terms of the form

$$(30) \qquad f(\lambda) = (p - e^{-\lambda})^+ + (q - \lambda e^{-\lambda})^+$$

Hence, it follows from Section 3 that $f(\lambda)$ is minimized for the value λ^* given by (20).

5. BOUNDS FOR STOP-LOSS PREMIUMS

Let X be a random variable with distribution function F and finite mean μ. The stop-loss transform of F is defined by

$$(31) \qquad \Pi(F, t) = E[(X-t)^+] = \int_t^{\infty} (x-t)\, dF(x) .$$

Remark that in particular $\Pi(F, 0) = \mu$.

If F is the aggregate claims distribution of a portfolio during a certain period, then $\Pi(F, t)$ is the net stop-loss premium with retention limit t for that period.

In the following lemmas some basic stop-loss inequalities are derived. They are inspired by the pioneer work by BÜHLMANN et al. (1977). See also the textbooks of GERBER (1979, Section 7.3) and SUNDT (1991, Section 10.3).

Lemma 5: Let F, G and H be distribution functions and assume that there exist constants a and b such that for all t

(32) $$a \leq \Pi(F, t) - \Pi(G, t) \leq b.$$

Then, one has for all t

(33) $$a \leq \Pi(F * H, t) - \Pi(G * H, t) \leq b.$$

Proof: Let X, Y and Z be random variables with distribution functions F, G and H respectively and assume that Z is independent of X and Y. Taking the conditional expectation on Z yields

$$\begin{aligned}\Pi(F * H, t) - \Pi(G * H, t) &= E[E[(X+Z-t)^+ | Z]] - E[E[(Y+Z-t)^+ | Z]] \\ &= E[\Pi(F, t-Z)] - E[\Pi(G, t-Z)].\end{aligned}$$

Now, (33) follows immediately from the assumption (32).

Q.E.D.

Lemma 6: Let $F_1, F_2, \ldots, F_n$ and $G_1, G_2, \ldots, G_n$ be distribution functions satisfying for all t

(34) $$a_i \leq \Pi(F_i, t) - \Pi(G_i, t) \leq b_i.$$

Then, one has for all t

(35) $$\sum_{i=1}^{n} a_i \leq \Pi\left(\mathop{*}_{i=1}^{n} F_i, t\right) - \Pi\left(\mathop{*}_{i=1}^{n} G_i, t\right) \leq \sum_{i=1}^{n} b_i.$$

Proof: The inequalities (35) hold by assumption for $n = 1$ and are proved in general by induction. The proof is based on Lemma 5 and is similar to the proof of Lemma 2.

Q.E.D.

Lemma 7: Let F and G be distribution functions on the non-negative reals. Then one has for all t

(36) $$\Pi(F, t) + \Pi(G, t) + (t)^- \leq \Pi(F * G, t) \leq \Pi(F, t) + \Pi(G, 0).$$

Proof: It is easily to verify that for arbitrary t and non-negative x and y the following inequalities hold

$$(x-t)^+ + (y-t)^+ + (t)^- \leq (x+y-t)^+ \leq (x-t)^+ + y$$

This implies the assertion.

Q.E.D.

Lemma 8: Let F be a distribution function on the non-negative reals. Then, one has for $n = 1, 2, \ldots,$ and all t

(37) $$n\Pi(F, t) + (n-1)(t)^- \leq \Pi(F^{*n}, t) \leq (n-1)\,\Pi(F, 0) + \Pi(F, t).$$

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Proof: The lemma is proved by induction. Clearly (37) holds for $n = 1$. Now, assume that it holds for $n = k-1$. Then, the proof for $n = k$ follows by applying Lemma 7 to $F^{*(k-1)}$ and F.

Q.E.D.

Remark that under the assumption of the lemma only stop-loss premiums with a non-negative retention t are of real interest. It follows that for $t \geq 0$, (37) reduces to

$$n\Pi(F, t) \leq \Pi(F^{*n}, t) \leq (n-1)\,\Pi(F, 0) + \Pi(F, t). \tag{38}$$

The following theorem yields bounds for the error which results if a compound Poisson model is used for approximate computation of stop-loss premiums in the individual model.

Theorem 3: For all retentions t one has

$$\sum_{i=1}^{n} \mu_i[1-\lambda_i-e^{-\lambda_i}+(e^{-\lambda_i}-p_i)^-] \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq \sum_{i=1}^{n} \mu_i(q_i-\lambda_i)^+, \tag{39}$$

where μ_i denotes the mean of the conditional claim amount distribution G_i.

Proof: In view of Lemma 6 it is sufficient to give the proof for the special case $n = 1$, where

$$F^{ind} = pI + qG$$

is approximated by

$$F^{cP} = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} G^{*k}.$$

Since $G(0) = 0$, one has for $t \leq 0$

$$\Pi(F^{ind}, t) - \Pi(F^{cP}, t) = \Pi(F^{ind}, 0) - \Pi(F^{cP}, 0) = \mu(q-\lambda)$$

where μ denotes the mean of G. It is easy to verify that (39) is satisfied.

The case of interest is $t > 0$. Then $\Pi(I, t) = 0$, so that

$$\Pi(F^{ind}, t) - \Pi(F^{cP}, t) = q\Pi(G, t) - \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \Pi(G^{*k}, t).$$

From (38) it follows that for any $t > 0$

$$q\Pi(G, t) - \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} [(k-1)\,\Pi(G, 0) + \Pi(G, t)] \leq \Pi(F^{ind}, t) - \Pi(F^{cP}, t)$$

$$\leq q\Pi(G, t) - \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} k\Pi(G, t).$$

Since $\Pi(G, 0) = \mu$, this can be rewritten as

$$(q-1+e^{-\lambda})\,\Pi(G, t)-(\lambda-1+e^{-\lambda})\,\mu \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq (q-\lambda)\,\Pi(G, t)$$

Taking into account that $0 \leq \Pi(G, t) \leq \mu$ for $t > 0$, this implies

$$[(e^{-\lambda}-p)^{-}+1-\lambda-e^{-\lambda}]\,\mu \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq (q-\lambda)^{+}\,\mu$$

which proves the theorem.

Q.E.D.

Remark that if $\lambda_i \leq q_i$ for $i = 1, 2, \ldots, n$, the compound Poisson approximation is always on the safe side, in the sense that $\Pi(F^{ind}, t) \leq \Pi(F^{cP}, t)$ for all t.

As in the previous sections, the error bounds (39) will be further analysed for some special choices of the λ_i.

Case 1: If $\lambda_i = q_i$, $i = 1, 2, \ldots, n$, one has for all retentions t

$$(40)\qquad -\frac{1}{2}\sum_{i=1}^{n} \mu_i q_i^2 < \sum_{i=1}^{n} \mu_i(p_i-e^{-q_i}) \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq 0\,.$$

The upper bound is given in BÜHLMANN et al. (1977). The lower bound was derived by GERBER (1984) in the special case of deterministic claim amounts. For the general case of stochastic claim amounts he proved that $-\sum_{i=1}^{n} \mu_i q_i^2$ is a lower bound, but believed that this result could be improved. This is indeed the case, as shown here.

Case 2: For the choice $\lambda_i = -\ln p_i$, $i = 1, 2, \ldots, n$, one gets for all t

$$(41)\qquad -\frac{1}{2}\sum_{i=1}^{n} \mu_i(\ln p_i)^2 < \sum_{i=1}^{n} \mu_i(q_i+\ln p_i) \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq 0\,.$$

Case 3: Kornya's first order approximation is obtained by setting $\lambda_i = q_i/p_i$, $i = 1, 2, \ldots, n$. Then, one has for all t

$$(42)\qquad -\frac{1}{2}\sum_{i=1}^{n} \mu_i\, q_i^2/p_i \leq \Pi(F^{ind}, t)-\Pi(F^{cP}, t) \leq 0\,.$$

To round off the analysis of the bounds (39), consider

$$g_L(\lambda) = 1-\lambda-e^{-\lambda}+(e^{-\lambda}-p)^{-}$$
$$g_U(\lambda) = (q-\lambda)^{+}$$

and

$$(43)\qquad g(\lambda) = g_U(\lambda)-g_L(\lambda)\,.$$

Since $g_L(\lambda)$ is an ever decreasing function of λ and $g_U(\lambda) = 0$ for $\lambda \geq q$, the function $g(\lambda)$ takes on its minimum at a value $\lambda^* \leq q$. In case $\lambda \leq q$, $g(\lambda)$ becomes equal to $e^{-\lambda} - p$ which is a decreasing function of λ. Hence, $g(\lambda)$ is minimized for

$$\lambda^* = q. \tag{44}$$

Comparison with (20) shows that the Poisson parameters which minimize the error bounds depend on the measure used to define the difference between the exact and the approximate model.

6. COMPARISON WITH HIPP'S BOUNDS

The lower and upper bounds given in the Theorems 1, 2 and 3 increase linearly with the number of policies and so it is to be expected that they will be too pessimistic for large portfolios. This is inherent to the way in which these bounds are derived, i.e. the use of the Lemmas 2, 4 and 6.

Alternative bounds for the error in calculating the distribution function and stop-loss premiums were derived by HIPP (1985) in case of the classical compound Poisson approximation with $\lambda_i = q_i$, $i = 1, 2, \ldots, n$. His method consists in applying concentration functions. The concentration function $C(F, r)$ of a distribution function F on an interval of length $r > 0$ is defined by

$$C(F, r) = \sup_x \, [F(x+r) - F(x)]. \tag{45}$$

An updated version of Hipp's results is given in the risk theory book by HIPP and MICHEL (1990), where the following bounds are derived.

Theorem 4 (Hipp): Consider the compound Poisson approximation with $\lambda_i = q_i$, $i = 1, 2, \ldots, n$. Denote, for each i, by μ_i and $\mu_i^{(2)}$ respectively the first and second moment about the origin of the conditional claim amount distribution G_i. Then, one has for all s

$$|F^{ind}(s) - F^{cP}(s)| \leq \frac{\pi^2}{4} \sum_{i=1}^{n} \frac{q_i^2}{p_i} \, C(\bar{F}, \mu_i). \tag{46}$$

and for all t

$$-\frac{\pi^2}{4} \sum_{i=1}^{n} \frac{q_i^2}{p_i} \left(\mu_i + \frac{\mu_i^{(2)}}{2\mu_i} \right) C(\bar{F}, \mu_i) \leq \Pi(F^{ind}, t) - \Pi(F^{cP}, t) \leq 0 \tag{47}$$

where $\bar{F}$ is the compound Poisson distribution with Poisson parameter

$$\bar{\lambda} = \frac{1}{2} \sum_{i=1}^{n} p_i q_i$$

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and claim amount distribution

$$\bar{G} = \frac{1}{2\bar{\lambda}} \sum_{i=1}^{n} p_i q_i G_i .$$

To apply these bounds the concentration functions $C(\bar{F}, \mu_i)$ must be evaluated. In case the G_i are arithmetic distributions the numbers $C(\bar{F}, \mu_i)$ can be computed numerically, but for this another application of Panjers's recursive algorithm is needed. However, from a practical point of view, it seems unreasonable to spend much effort in calculating the theoretical bounds exactly. After all, the main advantage of a compound Poisson approximation is that the necessary calculations can be done in a minimal time. When the computing time is not a major constraint preference should be given to other, more accurate, approximation methods, as proposed in the recent literature.

The demand for making the bounds (46) and (47) easy to handle, necessitates to dispose of a quickly computable estimate for the $C(\bar{F}, \mu_i)$. Hipp has mentioned several upper bounds for concentration functions, but most of them are hard to compute. Further, Hipp's work contains no indication which of these bounds should be used in a given application. This is indeed a difficult problem, since the best choice depends on the form of the claim amount distribution $\bar{G}$.

In order to get an idea of the magnitude of Hipp's error bounds the following general and simple bound for $C(\bar{F}, \mu_i)$ can be used

(48) $$C(\bar{F}, \mu_i) \leq \mu_i' \, C(\bar{F}, 1) \leq \mu_i' (2 e \bar{\lambda})^{-1/2}$$

where μ_i' denotes the smallest integer greater than or equal to μ_i.

Since $C(\bar{F}, \mu_i)$ is of order $\bar{\lambda}^{-1/2}$, Hipp's method leads to error bounds of order $\sqrt{n}$, whereas the Theorems 1, 2 and 3 yield only bounds of order n. This indicates that (46) and (47) are asymptotically better than the corresponding formulas (14) and (40). A further discussion of asymptotic results can be found in KUON, RADTKE and REICH (1991).

The preceeding considerations show that the error bounds resulting from both methods complement each other. The bounds given in the Theorem 1, 2 and 3 are easy to calculate and of practical interest for small and moderate portfolios. For large portfolios Hipp's bounds are sharper, but they are much more complicated to compute.

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