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Δ–Springer varieties and Hall–Littlewood polynomials

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Abstract

The Δ-Springer varieties are a generalization of Springer fibers introduced by Levinson, Woo and the author that have connections to the Delta Conjecture from algebraic combinatorics. We prove a positive Hall–Littlewood expansion formula for the graded Frobenius characteristic of the cohomology ring of a Δ-Springer variety. We do this by interpreting the Frobenius characteristic in terms of counting points over a finite field $\mathbb{F}_q$ and partitioning the Δ-Springer variety into copies of Springer fibers crossed with affine spaces. As a special case, our proof method gives a geometric meaning to a formula of Haglund, Rhoades and Shimozono for the Hall–Littlewood expansion of the symmetric function in the Delta Conjecture at $t = 0$.

Contents

1. Introduction

In [10], Levinson, Woo and the author introduced the Δ-Springer varieties $Y_{n,\lambda,s}$ as a generalization of Springer fibers that give a compact geometric realization of the Delta Conjecture at $t = 0$. Precisely, they showed that the symmetric function $\Delta'_{e_{k-1}} e_n|_{t=0}$ corresponds under the graded Frobenius characteristic map to the cohomology ring of a certain Δ-Springer variety, up to a minor twist. In this article, we

give a proof of the expansion, originally announced in [9], of the graded Frobenius characteristic of the cohomology ring of a Δ-Springer variety as a positive sum of modified Hall–Littlewood symmetric functions. As a special case, our proof gives a geometric explanation of the Hall–Littlewood expansion for $\Delta'_{e_{k-1}} e_n|_{t=0}$ given by Haglund, Rhoades and Shimozono [12].

1.1. Background and context

Springer fibers $\mathcal{B}_\lambda$ are a family of subvarieties of the complete flag variety indexed by partitions λ that have remarkable connections to representation theory. Notably, Springer [22, 23] constructed an action of the symmetric group S_n on the cohomology ring of a Springer fiber and showed that the top nonzero cohomology group is an irreducible representation of S_n and that all finite-dimensional irreducible representations of S_n appear this way. Furthermore, the graded S_n-module structure of the cohomology ring is well studied. Under the Frobenius characteristic map that sends an S_n-module to a symmetric function, the cohomology ring of a Springer fiber corresponds to the Hall–Littlewood symmetric function [15],

$$\widetilde{H}_\lambda(\mathbf{x}; q) = \mathrm{Frob}(H^*(\mathcal{B}_\lambda; \mathbb{Q}); q).$$

Alternatively, the modified Hall–Littlewood symmetric functions can be obtained by counting $\mathbb{F}_q$-points. Letting $\mathrm{Stein}^\mu_\lambda$ be the Steinberg variety of partial flags of type μ that are preserved by a fixed nilpotent matrix of Jordan type λ, it is well known that for all prime powers q,

$$\widetilde{H}_\lambda(\mathbf{x}; q) = \sum_{\mu \vdash n} |\mathrm{Stein}^\mu_\lambda(\mathbb{F}_q)| m_\mu(\mathbf{x}), \tag{1.1}$$

where m_μ is the monomial symmetric function and $\mathrm{Stein}^\mu_\lambda(\mathbb{F}_q)$ stands for the set of $\mathbb{F}_q$ points of $\mathrm{Stein}^\mu_\lambda$.

Similar interpretations have been given for the modified Macdonald polynomials $\widetilde{H}_\lambda(\mathbf{x}; q, t)$, which are a generalization of modified Hall-Littlewood symmetric functions with coefficients in two parameters q and t. Haiman proved that $\widetilde{H}_\lambda(\mathbf{x}; q, t)$ is the bi-graded Frobenius characteristic of the fiber of a certain tautological bundle on the Hilbert scheme of points in the plane [13, 14]. Mellit realized $\widetilde{H}_\lambda(\mathbf{x}; q, t)$ as a weighted count of points in an affine Springer fiber [18].

In a related line of work, the Delta Conjecture of Haglund, Remmel and Wilson [11] gives two combinatorial formulas for a symmetric function $\Delta'_{e_{k-1}} e_n$, where $\Delta'_{e_{k-1}}$ is a certain eigenoperator that diagonalizes the modified Macdonald polynomial basis. There has been recent progress on realizing this symmetric function naturally as the Frobenius characteristic of an S_n-module. Haglund, Rhoades and Shimozono [12] found a graded ring $R_{n,k}$ whose graded Frobenius characteristic is $\mathrm{rev}_q \circ \omega(\Delta'_{e_{k-1}} e_n|_{t=0})$, where ω is the involution that swaps elementary symmetric functions e_n with complete homogeneous symmetric functions h_n, and rev_q reverses the coefficients as a polynomial in q. Pawlowski and Rhoades subsequently defined the space of spanning line configurations, which is a smooth noncompact variety whose cohomology is $R_{n,k}$. Additionally, Zabrocki [24] has conjectured that the graded Frobenius characteristic of the superspace coinvariant ring in the $n - k$ fermionic degree is $\Delta'_{e_{k-1}} e_n$, and Bergeron [1] has conjectured that $\Delta'_{e_{k-1}} e_n$ can be obtained via skewing operators. See [20] for a survey of these and related results.

In [9], the author generalized the ring $R_{n,k}$ to a family of quotient rings $R_{n,\lambda,s}$ that also include Tanisaki's presentation of the cohomology ring of a Springer fiber as a special case. Each ring $R_{n,\lambda,s}$ has the structure of a graded S_n-module whose graded Frobenius characteristic has several combinatorial formulas that extend previously known formulas for $\Delta'_{e_{k-1}} e_n|_{t=0}$ and $\widetilde{H}_\lambda(\mathbf{x}; q)$. In particular, there are two monomial symmetric function expansions in terms of *inversions* and *diagonal inversions*, respectively, of labeled diagrams called *partial row-decreasing fillings*. We recall the relevant notation and statistics in Section 2.

Theorem 1.1 [9]. *We have*

$$\mathrm{Frob}(R_{n,\lambda,s};q) = \sum_{\mu \vdash n} \sum_{\varphi \in \mathrm{PRD}_{n,\lambda,s}^\mu} q^{\mathrm{inv}(\varphi)} m_\mu(\mathbf{x}) = \sum_{\mu \vdash n} \sum_{\varphi \in \mathrm{PRD}_{n,\lambda,s}^\mu} q^{\mathrm{dinv}(\varphi)} m_\mu(\mathbf{x}).$$

Levinson, Woo and the author [10] then constructed a compact variety $Y_{n,\lambda,s}$, the Δ-Springer variety, whose cohomology ring is $R_{n,\lambda,s}$. In the special case of $\lambda = (1^k)$ and $s = k$, the variety $Y_{n,(1^k),k}$ gives a compact geometric realization of the Delta Conjecture symmetric function at $t = 0$, since $H^*(Y_{n,(1^k),k}) \cong R_{n,k}$. It was also shown that $Y_{n,\lambda,s}$ has many of the same geometric and representation-theoretic properties as Springer fibers, including a characterization of the S_n-module structure of the top cohomology group.

Theorem 1.2 [10]. *The variety $Y_{n,\lambda,s}$ is equidimensional of complex dimension*

$$d = n(\lambda) + (s-1)(n-k),$$

where $n(\lambda) = \sum_i \binom{\lambda'_i}{2}$. For $s > \ell(\lambda)$, we have an isomorphism of S_n-modules

$$H^{2d}(Y_{n,\lambda,s}; \mathbb{Q}) \cong \mathrm{Ind} \uparrow_{S_k \times S_{n-k}}^{S_n} (S^\lambda),$$

where S^λ is considered as a $S_k \times S_{n-k}$-module in which S_{n-k} acts trivially.

1.2. Results of this paper

Our main theorem is a positive expansion of the graded Frobenius characteristic of $H^*(Y_{n,\lambda,s}; \mathbb{Q})$ (equivalently of $R_{n,\lambda,s}$) into modified Hall–Littlewood symmetric functions, a result that was announced in [9].

Theorem 1.3. *We have*

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s}; \mathbb{Q}); q) = \mathrm{rev}_q \left[\sum_{\substack{\nu \in \mathrm{Par}(n,s), \\ \nu \supseteq \lambda}} q^{n(\nu/\lambda)} \prod_{i \geq 0} \begin{bmatrix} \nu'_i - \lambda'_{i+1} \\ \nu'_i - \nu'_{i+1} \end{bmatrix}_q H_\nu(\mathbf{x}; q) \right] \tag{1.2}$$

$$= \sum_{\substack{\nu \in \mathrm{Par}(n,s), \\ \nu \supseteq \lambda}} q^{\sum_i (s-\nu'_i)(\nu'_{i+1} - \lambda'_{i+1})} \prod_{i \geq 0} \begin{bmatrix} \nu'_i - \lambda'_{i+1} \\ \nu'_i - \nu'_{i+1} \end{bmatrix}_q \widetilde{H}_\nu(\mathbf{x}; q), \tag{1.3}$$

where $\nu'_0 := s$.

A special case of this formula has been used to verify the construction of *higher Specht bases* for some infinite subfamilies of the rings $R_{n,\lambda,s}$ [8].

The outline of the proof of Theorem 1.3 is as follows. We start with the fact that the left-hand side of (1.2) is given by the inversion formula in Theorem 1.1. We then show that $Y_{n,\lambda,s}^\mu$, the projection of $Y_{n,\lambda,s}$ down to a partial flag variety, has an affine paving such that the dimension of a cell is computed by the inversion statistic inv. We use this to show that the $m_\mu(\mathbf{x})$ coefficient of the inversion formula in Theorem 1.1 can be computed by counting $\mathbb{F}_q$ points of $Y_{n,\lambda,s}^\mu$. We then show that the $\mathbb{F}_q$ points of $Y_{n,\lambda,s}^\mu$ can alternatively be computed by partitioning $Y_{n,\lambda,s}^\mu$ into a disjoint union of copies of affine spaces crossed with Steinberg varieties.

When $\lambda = (1^k)$ and $s = k$, the right-hand side of (1.2) is the right-hand side of [12, Theorem 6.14], the formula for $\mathrm{Frob}(R_{n,k}; q)$ proven by Haglund, Rhoades and Shimozono. Since $R_{n,k} \cong H^*(Y_{n,(1^k),k}; \mathbb{Q})$ (up to doubling the degree), this can be seen as a solution to Problem 9.9 in [19], but with the space of spanning line configurations $X_{n,k}$ replaced by the Δ-Springer variety $Y_{n,(1^k),k}$.

2. Background

2.1. Compositions and partitions

Let us recall basic notions about symmetric functions and their connections to S_n-modules. A **composition** α of n of length s is a tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_s = n$. We write $|\alpha| = n$ for the size of α. Let $\mathrm{Comp}(n, s)$ be the set of compositions of n with length s. We say α is a **strong composition** if $\alpha_i > 0$ for all $i \leq s$. Given α a strong composition of n, let

$$\alpha[i] := \{\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + 2, \ldots, \alpha_1 + \cdots + \alpha_i\}$$

be the *i*th block of α.

A **partition** is a composition λ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0$. The **length** of λ, written $\ell(\lambda)$, is the number of positive parts of λ. We sometimes write $\lambda \vdash n$ to mean that λ is a partition of n. The **conjugate** partition λ' is the partition of n whose *i*th entry λ'_i is the number of indices j such that $\lambda_j \geq i$. Occasionally, we write (a^b) to mean the partition $(a, a, \ldots, a)$ with b many a's. We write (λ, a) to mean the partition obtained by appending a to the end of λ. Let $\mathrm{Par}(n, s)$ be the set of partitions λ of n such that $\ell(\lambda) \leq s$, where we identify partitions up to adding trailing 0s.

Given $\alpha \in \mathrm{Comp}(n, s)$ and $\lambda \in \mathrm{Par}(k, s)$ for some $k \leq n$, we write $\lambda \subseteq \alpha$ if $\lambda_i \leq \alpha_i$ for all $i \leq s$. A **coinversion** of α is a pair (i, j) such that $1 \leq i < j \leq s$ and $\alpha_i < \alpha_j$. Let $\mathrm{coinv}(\alpha)$ be the number of coinversions of α. For example, for $n = 10$, $k = 4$ and $s = 5$, we have $\lambda = (2, 1, 1, 0, 0) \in \mathrm{Par}(4, 5)$, and $\alpha = (2, 3, 1, 0, 4) \in \mathrm{Comp}(10, 5)$ such that $\lambda \subseteq \alpha$. It can be checked that $\mathrm{coinv}(\alpha) = 5$.

2.2. Symmetric functions and q-analogues

A **symmetric function** f is a formal power series in the variables $\mathbf{x} := \{x_1, x_2, \ldots\}$ that is invariant under any permutation of the variables. Let $m_\lambda(\mathbf{x})$, $h_\lambda(\mathbf{x})$ and $s_\lambda(\mathbf{x})$ be the usual **monomial**, **complete homogeneous** and **Schur symmetric functions**, each of which forms a basis of the ring of symmetric functions as λ ranges over all partitions of n. See [17] for their definitions.

To each finite-dimensional representation of S_n over $\mathbb{Q}$, we associate a symmetric function via the Frobenius characteristic map as follows. For $\lambda \vdash n$, let S^λ be the Specht module, which is the irreducible representation of S_n indexed by λ. Given a finite-dimensional vector space V over $\mathbb{Q}$ that has the structure of an S_n-module, it decomposes as a direct sum of Specht modules, $V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus a_\lambda}$, where a_λ is the multiplicity of S^λ in V. The **Frobenius characteristic** of V is then defined to be

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} a_\lambda s_\lambda(\mathbf{x}).$$

Given a graded vector space $V = \bigoplus_{i \geq 0} V^{(i)}$ where $V^{(i)}$ is finite dimensional, its **Hilbert series** is the generating function

$$\mathrm{Hilb}(V; q) := \sum_{i \geq 0} \dim(V^{(i)}) q^i.$$

If each $V^{(i)}$ also has the structure of a finite-dimensional S_n-module, its **graded Frobenius characteristic** is

$$\mathrm{Frob}(V; q) := \sum_{i \geq 0} \mathrm{Frob}(V^{(i)}) q^i.$$

The ring $\Lambda_{\mathbb{Q}(q)}$ has a basis given by the **Hall–Littlewood symmetric functions** $P_\mu(\mathbf{x}; q)$, which have the property that

$$s_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} K_{\lambda, \mu}(q) P_\mu(\mathbf{x}; q),$$

where $K_{\lambda,\mu}(q)$ is the Kostka–Foulkes polynomial; see [17] for more details. The **dual Hall–Littlewood symmetric functions** are defined by

$$H_\mu(\mathbf{x};q) = \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) s_\lambda(\mathbf{x}).$$

These symmetric functions are sometimes alternatively denoted by $Q'_\mu(\mathbf{x};q) = H_\mu(\mathbf{x};q)$. Given a polynomial $f(q) = a_0 + a_1 q + \cdots + a_m q^m$ with symmetric function coefficients such that $a_m \neq 0$, the q-**reversal** of f is $\mathrm{rev}_q(f) = a_m + a_{m-1}q + \cdots + a_1 q^{m-1} + a_0 q^m$. In the case of Hall-Littlewood symmetric functions, the degree of $H_\mu(\mathbf{x};q)$ as a polynomial in q is $n(\mu) := \sum_i \binom{\mu'_i}{2}$. The reversal of $H_\mu(\mathbf{x};q)$ is the **modified (dual) Hall–Littlewood symmetric function**, written

$$\widetilde{H}_\mu(\mathbf{x};q) = \mathrm{rev}_q(H_\mu(\mathbf{x};q)) = q^{n(\mu)} H_\mu(\mathbf{x};q^{-1}).$$

We use the following standard q-analogues of integers, factorials and binomial coefficients.

$$[n]_q := 1 + q + \cdots + q^{n-1}, \tag{2.1}$$

$$[n]_q! := [n]_q [n-1]_q \cdots [2]_q [1]_q, \tag{2.2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{2.3}$$

2.3. Flag varieties and Schubert cells

Given a field $\mathbb{F}$, a **partial flag** in $\mathbb{F}^K$ is a nested sequence of vector subspaces of $\mathbb{F}^K$,

$$V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_m). \tag{2.4}$$

Given a strong composition $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = \mu_1 + \cdots + \mu_m \leq K$, define the **partial flag variety** to be the set of partial flags of $\mathbb{F}^K$ such that the dimensions of the successive quotients V_i/V_{i-1} are recorded by the parts of μ,

$$\mathcal{B}^\mu(\mathbb{F}^K) := \{V_\bullet = (V_1 \subset \cdots \subset V_m) \mid V_i \subseteq \mathbb{F}^K, \dim(V_i/V_{i-1}) = \mu_i \text{ for } i \leq m\}. \tag{2.5}$$

Here, by convention, we define $V_0 := 0$. (Note that we have switched to using μ as a strong composition here, since the symbol α will be reserved later to denote a different composition.) We occasionally write $\mathcal{B}^\mu := \mathcal{B}^\mu(\mathbb{F}^K)$ when $|\mu| = K$ and the field $\mathbb{F}$ is clear from context.

The partial flag variety is realized as a projective algebraic variety as G/P^μ, where $G = GL_K$ and P^μ is the parabolic subgroup of block upper triangular matrices with blocks of sizes $\mu_1, \mu_2, \ldots, \mu_m, K - |\mu|$. We remark that although $\mathcal{B}^\mu(\mathbb{F}^K)$ is isomorphic to $\mathcal{B}^{\mu,K-|\mu|}$, it will be notationally convenient for us to distinguish the two spaces. In the case when $K = n$ and $\mu = (1^n)$, $\mathcal{B}^{(1^n)}$ is the **complete flag variety**.

Returning to the general case when $|\mu| \leq K$, let $n := |\mu|$ and let $[n] := \{1, 2, \ldots, n\}$. Given an injective map $w : [n] \to [K]$, we say that w is μ-**increasing** if for each $i \leq \ell(\mu)$, w is increasing on the ith block of μ; that is, $w_j < w_k$ for all elements $j < k$ of $\mu[i]$. Given any w, let $\mathrm{sort}_\mu(w)$ be the unique injective μ-increasing function such that the images of the set $\mu[i]$ under w and $\mathrm{sort}_\mu(w)$ are the same for each i.

Define $f_1, f_2, \ldots, f_K$ to be the standard ordered basis of $\mathbb{F}^K$. Given a μ-increasing injective map $w : [n] \to [K]$, let the **coordinate flag** $F_\bullet^{(w)} \in \mathcal{B}^\mu(\mathbb{F}^K)$ be defined by setting $F_p^{(w)} = \mathrm{span}\{f_{w(1)}, \ldots, f_{w(\mu_1 + \cdots + \mu_p)}\}$ for all p such that $1 \leq p \leq \ell(\mu)$. Now define the **Schubert cell** C_w to be the P^μ orbit of $F_\bullet^{(w)}$. When $\mathbb{F} = \mathbb{C}$, the Schubert cells are the cells of an affine paving (in fact, a CW-complex) of the partial flag variety.

There is another description of the Schubert cells that will be useful for us. Given a vector $v = \sum_i a_i f_i \in \mathbb{F}^K$, we say its **leading term** is the term $a_i f_i$ with i maximal such that $a_i \neq 0$. Given any

$V_\bullet \in C_w \subseteq \mathcal{B}^\mu(\mathbb{F}^K)$, for each i such that $1 \leq i \leq \ell(\mu)$, there exist unique vectors $v_1, \ldots, v_n \in \mathbb{F}^K$ such that for $p \in \mu[i]$, we have $v_p \in V_i \setminus V_{i-1}$, v_p has leading term $f_{w(p)}$, and

$$v_p = f_{w(p)} + \sum_{h=1}^{w(p)-1} \beta_{w(p),h} f_h, \tag{2.6}$$

for some $\beta_{w(p),h}$ such that $\beta_{w(p),h} = 0$ if $h \in \{w(1), \ldots, w(p-1)\}$. Note $V_i = \text{span}\{v_1, \ldots, v_{\mu_1 + \cdots + \mu_i}\}$. The $\beta_{w(p),h}$ that are not required to be 0 can be taken as algebraically independent coordinates on C_w. We will say that (2.6) is the **standard coordinate representation** of $V_\bullet$ and that the $\beta_{w(p),h}$ are the **standard coordinates** of the Schubert cell. Equivalently, if $V_\bullet$ is represented as a matrix whose first $\mu_1 + \cdots + \mu_i$ many columns span V_i, then the $\beta_{w(p),h}$ are the matrix entries after column reducing.

Note that we have a projection map

$$\pi^\mu : \mathcal{B}^{(1^n)}(\mathbb{F}^K) \to \mathcal{B}^\mu(\mathbb{F}^K)$$

defined by sending $V_\bullet$ to $(V_{\mu_1}, V_{\mu_1 + \mu_2}, \ldots, V_n)$. Given w a μ-increasing injective function, it is evident from the coordinate description of the Schubert cell $C_w \subseteq \mathcal{B}^{(1^n)}(\mathbb{F}^K)$ that C_w is mapped isomorphically under π^μ onto the corresponding Schubert cell $C_w \subseteq \mathcal{B}^\mu(\mathbb{F}^K)$ (which justifies our abuse of notation by not decorating C_w with μ).

2.4. Springer fibers and Steinberg varieties

Springer fibers are subvarieties of the complete flag variety studied by Springer [22, 23]. Given a nilpotent $n \times n$ matrix N over $\mathbb{F}$, the Springer fiber associated to it is

$$\mathcal{B}_N(\mathbb{F}) := \{V_\bullet \in \mathcal{B}^{(1^n)} \mid NV_i \subseteq V_i \text{ for } i \leq n\}$$

with the reduced induced scheme structure. Given two nilpotent matrices whose Jordan block sizes are recorded by the partition $\lambda \vdash n$, their associated Springer fibers are isomorphic, so we simply denote the Springer fiber by $\mathcal{B}_\lambda := \mathcal{B}_N$, where λ is the Jordan type of N. Springer showed that although the symmetric group S_n does not act directly on $\mathcal{B}_\lambda$, it does act on the cohomology ring $H^*(\mathcal{B}_\lambda(\mathbb{C}); \mathbb{Q})$, and used this action to geometrically construct the irreducible representations of S_n. The connection between Springer fibers and symmetric functions is summed up in the following elegant formula [6, 15]:

$$\widetilde{H}_\lambda(\mathbf{x}; q) = \text{Frob}(H^*(\mathcal{B}_\lambda(\mathbb{C}); \mathbb{Q}); q). \tag{2.7}$$

Here, and throughout the paper, we define the grading on cohomology so that H^{2k} is in degree k. This is possible since the odd degree cohomology groups of all of the varieties mentioned in this paper are 0.

It can be shown by induction that the Springer fiber has an alternative definition as a variety where the conditions $NV_i \subseteq V_i$ are replaced with the conditions $NV_i \subseteq V_{i-1}$ for all i, since N is nilpotent and we are considering complete flags. For partial flags of type μ a composition of K, these conditions give rise to two different varieties, which we refer to as the **Steinberg** and the **Spaltenstein varieties**, defined respectively as

$$\text{Stein}_\lambda^\mu(\mathbb{F}) := \{V_\bullet \in \mathcal{B}^\mu \mid NV_i \subseteq V_i \text{ for } 1 \leq i \leq \ell(\mu)\},$$
$$\text{Spalt}_\lambda^\mu(\mathbb{F}) := \{V_\bullet \in \mathcal{B}^\mu \mid NV_i \subseteq V_{i-1} \text{ for } 1 \leq i \leq \ell(\mu)\}.$$

We note that Spalt_λ^μ is denoted by $\mathcal{B}_\mu^\lambda$ in [10]. By the construction of the Springer action, there is an isomorphism of graded vector spaces [2, 4]

$$H^*(\text{Stein}_\lambda^\mu(\mathbb{C}); \mathbb{Q}) \cong H^*(\mathcal{B}_\lambda(\mathbb{C}); \mathbb{Q})^{S_\mu}, \tag{2.8}$$

where the superscript S_μ denotes taking the fixed subspace under the action of the Young subgroup $S_\mu := S_{\mu[1]} \times \cdots \times S_{\mu[\ell(\mu)]} \subseteq S_n$ permuting the elements in the sets $\mu[1], \ldots, \mu[\ell(\mu)]$ independently. Similarly, there is an identification of the cohomology of $\mathrm{Spalt}_\lambda^\mu(\mathbb{C})$ with the S_μ-anti-invariants of the cohomology of the Springer fiber, up to a grading shift.

For X a complex variety, an **affine paving** of X is a filtration of X by closed subvarieties

$$\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m = X$$

such that for all i, $X_i \setminus X_{i-1} \cong \bigsqcup_j C_{i,j}$, a disjoint union where $C_{i,j} \cong \mathbb{C}^{a_{i,j}}$ for some $a_{i,j}$. If a compact complex variety X has an affine paving, then the dimensions of the even degree cohomology groups can be computed by counting dimensions of cells,

$$\dim_{\mathbb{Q}} H^{2d}(X; \mathbb{Q}) = \#\{(i,j) \mid \dim_{\mathbb{C}}(C_{i,j}) = d\},$$

and the odd degree cohomology groups are 0.

It is well known [4, 5, 21] that both $\mathrm{Stein}_\lambda^\mu(\mathbb{C})$ and $\mathrm{Spalt}_\lambda^\mu(\mathbb{C})$ have affine pavings. Hence, their odd degree cohomology groups are 0.

2.5. Δ-*Springer varieties and affine pavings*

We now recall the definition of the Δ-Springer variety $Y_{n,\lambda,s}$ and the affine pavings of $Y_{n,\lambda,s}$ constructed in [10].

Fix an integer $n \geq 0$, a partition λ of size $|\lambda| = k \leq n$ and an integer $s \geq \ell(\lambda)$. Define $\Lambda := \Lambda(n, \lambda, s) = (\lambda_1 + n - k, \lambda_2 + n - k, \ldots, \lambda_s + n - k)$, where $\lambda_i := 0$ for $i > \ell(\lambda)$, and define $K := |\Lambda| = k + (n - k)s$. Given $\mathbb{F}$ a field and a nilpotent matrix N_Λ over $\mathbb{F}$ of Jordan type Λ, the Δ-**Springer variety** over $\mathbb{F}$ is defined to be

$$Y_{n,\lambda,s}(\mathbb{F}) := \{V_\bullet \in \mathcal{B}^{(1^n)}(\mathbb{F}^K) \mid N_\Lambda V_i \subseteq V_i \text{ for all } i, N_\Lambda^{n-k}\mathbb{F}^K \subseteq V_n\}.$$

When $n = |\lambda| = k$ and s is arbitrary, the Δ-Springer variety specializes to $\mathcal{B}_\lambda$, the usual Springer fiber.

The Δ-Springer variety can equivalently be defined as the projection of a certain Spaltenstein variety. Letting $(1^n, (s-1)^{n-k}) = (1^n, s-1, s-1, \ldots, s-1)$, where $s-1$ is repeated $n-k$ many times, and $\pi : \mathcal{B}^{(1^n,(s-1)^{n-k})} \to \mathcal{B}^{(1^n)}(\mathbb{F}^K)$ be the projection map that forgets all but the first n parts of the flag, we have [10, Lemma 5.8]

$$Y_{n,\lambda,s} = \pi\left(\mathrm{Spalt}_\Lambda^{(1^n,(s-1)^{n-k})}\right). \tag{2.9}$$
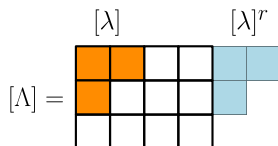
To briefly illustrate why (2.9) holds, given $V_\bullet \in \mathrm{Spalt}_\Lambda^{(1^n,(s-1)^{n-k})}$, then by definition $N_\Lambda V_i \subseteq V_{i-1}$ for all i and $V_{2n-k} = \mathbb{C}^K$. Combining these conditions for the $n-k$ many indices $i = n+1, \ldots, 2n-k$ gives $N_\Lambda^{n-k}\mathbb{C}^K = N_\Lambda^{n-k}V_{2n-k} \subseteq V_n$, which is the same condition as in $Y_{n,\lambda,s}$. Additionally, the conditions $N_\Lambda V_i \subseteq V_{i-1}$ for $i = 1, \ldots, n$ are equivalent to the conditions $N_\Lambda V_i \subseteq V_i$ for $i = 1, \ldots, n$ since N_Λ is nilpotent so the action of N_Λ on the 1-dimensional space V_i/V_{i-1} must be 0.

We denote by $[\Lambda]$ the Young diagram of Λ according to English convention, which formally is the set

$$[\Lambda] = \{(i,j) \mid 1 \leq i \leq \ell(\Lambda), 1 \leq j \leq \Lambda_i\},$$

where (i,j) is the cell in the ith row from the top and the jth column from the left. There are two copies of $[\lambda]$ inside of $[\Lambda]$ that we will consider, which are respectively left justified and right justified inside of $[\Lambda]$,

$$[\lambda] := \{(i,j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\},$$
$$[\lambda]^r := \{(i,j) \mid 1 \leq i \leq \ell(\lambda), \Lambda_i - \lambda_i + 1 \leq j \leq \Lambda_i\}.$$

Figure 1. *An example of* $[\Lambda]$ *for* $n = 7$, $\lambda = (2, 1)$, $s = 3$, *with the two copies of the Young diagram of* λ *shaded.*
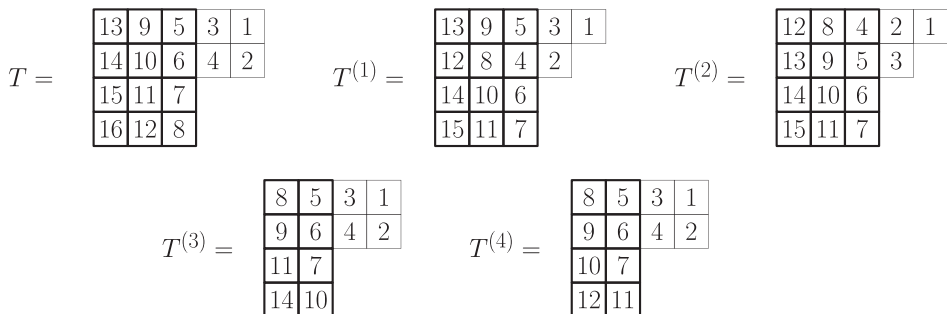
Figure 2. *The reading order filling* T *of* $[\Lambda(7, (2, 2), 4)]$, *and the associated labelings* $T^{(i)}$.

The affine pavings of $Y_{n,\lambda,s} = Y_{n,\lambda,s}(\mathbb{C})$ are defined in terms of specific choices of the nilpotent matrix N_Λ. Recall $f_1, \ldots, f_K \in \mathbb{C}^K$ is the standard basis, and let $[K] := \{1, 2, \ldots, K\}$. Given a bijection $T : [\Lambda] \to [K]$, define two nilpotent matrices N_T and N_T^t by

$$N_T(f_{T(i,j)}) := \begin{cases} 0 & \text{if } j = \Lambda_i \\ f_{T(i,j+1)} & \text{otherwise.} \end{cases} \tag{2.10}$$

$$N_T^t(f_{T(i,j)}) := \begin{cases} 0 & \text{if } j = 1 \\ f_{T(i,j-1)} & \text{otherwise.} \end{cases} \tag{2.11}$$

For example, for T as in Figure 2, we have $N_T f_5 = f_3$ and $N_T f_1 = 0$, whereas $N_T^t f_3 = f_5$ and $N_T^t f_{16} = 0$. Observe that both N_T and N_T^t have Jordan type Λ by construction and that N_T^t is simply the transpose of the matrix N_T. We define Y_{N_T} and $Y_{N_T^t}$ to be the Δ-Springer varieties for N_T and N_T^t, respectively, where we are abusing notation by suppressing the data of n, λ and s.

Definition 2.1. We say that T is (n, λ, s)**-Schubert compatible** if

○ T restricts to a bijection between $[\lambda]^r$ and $[k]$.
○ T is decreasing along each row from left to right.
○ For all $(i, j) \in [\lambda]^r$, the label $T(i, j)$ is greater than all labels in column $j + 1$.
○ For $i' < i$, we have $T(i', \Lambda_{i'}) < T(i, \Lambda_i)$.
○ Whenever $T(a, b) > T(c, d)$ for $b, d > 1$, then $T(a, b - 1) > T(c, d - 1)$.

When n, λ and s are clear from context, we simply say T is **Schubert compatible**.

Definition 2.2. The **reading order** of $[\Lambda]$ is the sequence of cells obtained by reading down each column of $[\Lambda]$, ordering the columns from right to left. The **reading order filling** of $[\Lambda]$ is the unique bijection $T : [\Lambda] \to [K]$ such that $T(i, j) = \ell$ if and only if (i, j) is the ℓth cell in reading order.

It is noted in [10] that the reading order filling is Schubert compatible. See the filling T in Figure 2 for an example of the reading order filling.

When T is Schubert compatible, the intersection of a Schubert cell with Y_{N_T} is either empty or a copy of affine space, and the nonempty intersections are the cells of an affine paving of Y_{N_T}. In fact, these cells have a recursive structure which we state next.

Let T be Schubert compatible. For $1 \leq i \leq s$, define the **flattening function** $\mathrm{fl}_T^{(i)}$ and filling $T^{(i)}$ as follows. If $i \leq \ell(\lambda)$, then $\mathrm{fl}_T^{(i)}$ is the unique order-preserving function with the following domain and codomain:

$$\mathrm{fl}_T^{(i)} : [K] \setminus \{T(i, \Lambda_i)\} \rightarrow [K-1].$$

Let $T^{(i)}$ be the filling obtained by deleting the cell (i, Λ_i), applying $\mathrm{fl}_T^{(i)}$ to the label in each remaining cell and reordering the rows so that the labels in the right-most cells increase from top to bottom. If $i > \ell(\lambda)$, then $\mathrm{fl}_T^{(i)}$ is the unique order-preserving function

$$\mathrm{fl}_T^{(i)} : [K] \setminus (\{T(i, \Lambda_i)\} \cup \{T(i', 1) \mid i' \neq i\}) \rightarrow [K-s].$$

We define $T^{(i)}$ in the same way as the previous case, except we also delete the cells $(i', 1)$ for $i' \neq i$ and shift those rows to the left by one unit before applying $\mathrm{fl}_T^{(i)}$ and reordering the rows. See Figure 2 for examples.

Given an injective map $w : [n] \rightarrow [K]$, we say w is **admissible** with respect to T if the image of w contains $[k]$, and whenever $T(a, b) = w(i)$ for some i, then either $b = \Lambda_a$ or $T(a, b+1) = w(i')$ for some $i' < i$. It is proven in [10].

We have the following recursion for the cells $C_w \cap Y_{N_T}$.

Lemma 2.3 [10]. *Let T be Schubert compatible. The intersection $C_w \cap Y_{N_T}$ is nonempty if and only if w is admissible with respect to T. If $C_w \cap Y_{N_T}$ is nonempty, then there is an isomorphism*

$$C_w \cap Y_{N_T} \cong \mathbb{C}^{i-1} \times (C_{\mathrm{fl}_T^{(i)}(w)} \cap Y_{N_{T^{(i)}}}), \tag{2.12}$$

where i is such that $w(1) = T(i, \Lambda_i)$.

Remark 2.4. The proof of Lemma 2.3 given in [10] can be used without change to show that for any field $\mathbb{F}$, there is a bijection

$$C_w \cap Y_{N_T}(\mathbb{F}) \cong \mathbb{F}^{i-1} \times (C_{\mathrm{fl}_T^{(i)}(w)} \cap Y_{N_{T^{(i)}}}(\mathbb{F})). \tag{2.13}$$

We use the symbol $\cong$ throughout the paper to mean a bijection between sets, but all bijections below are easily seen to be isomorphisms of complex algebraic varieties in the case when $\mathbb{F} = \mathbb{C}$. Furthermore, the term 'isomorphism' should be translated as 'bijection' in the general setting and as 'isomorphism of complex algebraic varieties' in the case $\mathbb{F} = \mathbb{C}$.

A **partial row-decreasing filling (PRD)** of $[\Lambda]$ is a filling of a subset of size n of the cells of $[\Lambda] = [\Lambda(n, \lambda, s)]$ with positive integers such that the filled cells are right justified in each row, the labeling weakly decreases along each row, and each cell of $[\lambda]^r$ is filled. A PRD is **standard** if the filling uses the labels $1, 2, \ldots, n$ without repeats.

Given w admissible with respect to T, let $\mathrm{PRD}_T(w)$ be the standard partial row-decreasing filling of $[\Lambda]$ such that, for $1 \leq i \leq n$, if $w(i) = T(a, b)$, then the cell (a, b) of $[\Lambda]$ is labeled by i. It can be checked that the map PRD_T gives a bijection between admissible w and standard partial row-decreasing fillings. Hence, standard partial row-decreasing fillings of $[\Lambda]$ also index the cells of Y_{N_T}. See Figure 3 for an example of $\mathrm{PRD}_T(w)$ for T the reading order filling of $[\Lambda(7, (2, 2), 4)]$ and $w = 2713594$ (where we have listed out the images of $1, 2, \ldots, 7$ in order).

2.6. Monomial symmetric function formulas

In [10], it is shown that the map on cohomology induced by inclusion of varieties

$$H^*(\mathcal{B}^{(1^n)}(\mathbb{C}^K); \mathbb{Q}) \twoheadrightarrow H^*(Y_{n, \lambda, s}(\mathbb{C}); \mathbb{Q})$$

$$\mathrm{PRD}_T(2713594) = \quad \begin{array}{|c|c|c|c|} \hline & 6 & 5 & 4 & 3 \\ \hline & & & 7 & 1 \\ \hline & & 2 & & \\ \hline & & & & \\ \hline \end{array}$$

Figure 3. *For T as in Figure 2, the partial row-decreasing filling* $\mathrm{PRD}_T(w)$ *associated to the admissible function* $w = 2713594$.

is surjective. There is a well-defined S_n action on $H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q})$ which is the unique one that makes the above map S_n-equivariant. Thus, $H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q})$ has the structure of a graded S_n-module. In [9], several formulas for the graded Frobenius characteristic of $H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q})$ are given, which we recall next.

For $\mu \vdash n$, let $\mathrm{PRD}^\mu_{n,\lambda,s}$ be the set of partial row-decreasing fillings of $\Lambda = \Lambda(n,\lambda,s)$ with μ_i many i's. Given such a labeling $\varphi \in \mathrm{PRD}^\mu_{n,\lambda,s}$, let $\varphi_{i,j}$ be the label in cell (i,j). Given a pair of cells $((i,j), (p,q))$ of $[\Lambda]$, we say they are an **attacking pair** if either $j = q$ and $i < p$, or if $j = q+1$ and $i > p$.

Definition 2.5. Given $\varphi \in \mathrm{PRD}^\mu_{n,\lambda,s}$, a **diagonal inversion** of φ is an attacking pair $((i,j), (p,q))$ of cells of $[\Lambda]$ such that one of the following hold:

(D1) (i,j) and (p,q) are filled such that $\varphi_{i,j} > \varphi_{p,q}$,
(D2) (i,j) is not filled and (p,q) is filled.

Let $\mathrm{dinv}(\varphi)$ be the number of diagonal inversions of φ.

For $\varphi = \mathrm{PRD}_T(2713594)$ as in Figure 3, φ has three diagonal inversions of type (D1); $((1,5), (2,5)), ((2,4), (1,3))$, and $((1,3), (3,3))$; and three of type (D2): $((2,3), (3,3)), ((4,3), (1,2))$, and $((2,3), (1,2))$, so $\mathrm{dinv}(\varphi) = 6$.

We have the following restatement of Theorem 1.1.

Theorem 2.6 [10]. *We have*

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q}); q) = \sum_{\mu \vdash n} \sum_{\varphi \in \mathrm{PRD}^\mu_{n,\lambda,s}} q^{\mathrm{dinv}(\varphi)} m_\mu(\mathbf{x}).$$

Remark 2.7. In Corollary 3.7, we show that the statistic dinv counts the dimensions of the cells of Y_{N_T} when T is the reading order of $[\Lambda]$. There is another inversion statistic inv defined in [9, 10] that also gives a monomial expansion of the graded Frobenius characteristic of $H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q})$. We do not define it here because it does not immediately come from a Schubert compatible filling of the cells of $[\Lambda]$. However, it may still be possible to show it counts dimensions of cells of Y_{N_T} for some choice of filling T.

3. Counting $\mathbb{F}_q$-points of projected Δ-Springer varieties

In this section, we analyze the projections of the Δ-Springer variety to other partial flag varieties. We then show that the graded Frobenius characteristic of $H^*(Y_{n,\lambda,s}(\mathbb{C});\mathbb{Q})$ can be written in terms of counting $\mathbb{F}_q$ points of these projected varieties.

Fix n, λ, s, and $k = |\lambda|$ as in Subsection 2.5, let $\Lambda := \Lambda(n,\lambda,s)$ and let $K := |\Lambda| = k + (n-k)s = n + (n-k)(s-1)$.

Definition 3.1. Let μ be a strong composition of n and let N_Λ is a nilpotent matrix of type Λ. Define the **projected Δ-Springer variety** to be

$$Y^\mu_{n,\lambda,s}(\mathbb{F}) := \{V_\bullet \in \mathcal{B}^\mu(\mathbb{F}^K) \mid N_\Lambda V_i \subseteq V_i \text{ for } i \leq \ell(\mu), \text{ and } N^{n-k}_\Lambda \mathbb{F}^K \subseteq V_{\ell(\mu)}\}. \tag{3.1}$$

Lemma 3.2. *We have*

$$Y_{n,\lambda,s}^{\mu} = \pi^{\mu}\left(\text{Spalt}_{\Lambda}^{(1^n, (s-1)^{n-k})}\right),$$

where $\pi^{\mu} : \mathcal{B}^{(1^n, (s-1)^{n-k})} \to \mathcal{B}^{\mu}(\mathbb{F}^K)$ is the usual projection map of partial flag varieties.

Proof. Let $\rho^{\mu} : \mathcal{B}^{(1^n)}(\mathbb{F}^K) \to \mathcal{B}^{\mu}(\mathbb{F}^K)$ be the usual projection map, which factors through π^{μ} (i.e., $\pi^{\mu} = \rho^{\mu} \circ \pi$), where π is the projection in (2.9). By (2.9), it is immediate that $\pi^{\mu}(\text{Spalt}_{\Lambda}^{(1^n, (s-1)^{n-k})}) = \rho^{\mu}(Y_{n,\lambda,s})$, which is contained in the right-hand side of (3.1). For the other containment, it is necessary to show that any $V_{\bullet} \in \mathcal{B}^{\mu}(\mathbb{F}^K)$ on the right-hand side can be extended to an element of $Y_{n,\lambda,s}$. This follows from the fact that

$$(\rho^{\mu})^{-1}(V_{\bullet}) \cap Y_{n,\lambda,s} \cong \mathcal{B}_{N_1} \times \cdots \times \mathcal{B}_{N_{\ell}},$$

where N_i is the nilpotent operator induced by N_{Λ} on V_i/V_{i-1}. Indeed, since each of the Springer fibers in the product is nonempty, then $V_{\bullet}$ can be extended to an element $W_{\bullet} \in Y_{n,\lambda,s}$ so that $\rho^{\mu}(W_{\bullet}) = V_{\bullet}$. Thus, we have equality of the two sets. $\qquad\square$

For $T : [\Lambda] \to [K]$ a bijection, we denote by $Y_{N_T}^{\mu}$ the projected Δ-Springer variety $Y_{n,\lambda,s}^{\mu}$ for the specific choice of nilpotent N_T, and similarly for N_T^t.

Lemma 3.3. *Let T be Schubert compatible and w admissible with respect to T. The projection $\rho^{\mu} : Y_{N_T} \to Y_{N_T}^{\mu}$ maps $C_w \cap Y_{N_T}$ to $C_{\text{sort}_{\mu}(w)} \cap Y_{N_T}^{\mu}$. When w is μ-increasing, $C_w \cap Y_{N_T}$ maps isomorphically onto $C_w \cap Y_{N_T}^{\mu}$. Thus, when $\mathbb{F} = \mathbb{C}$, the subspaces $C_w \cap Y_{n,\lambda,s}^{\mu}(\mathbb{C})$ for w admissible and μ-increasing are the cells of an affine paving of $Y_{n,\lambda,s}^{\mu}(\mathbb{C})$.*

Proof. The first part of the lemma is immediate from the fact that ρ^{μ} maps C_w to $C_{\text{sort}_{\mu}(w)}$ and the definition of $Y_{N_T}^{\mu}$. Let w be μ-increasing. Since $\rho^{\mu} : \mathcal{B}^{(1^n)}(\mathbb{F}^K) \to \mathcal{B}^{\mu}(\mathbb{F}^K)$ maps C_w isomorphically onto $C_w \subseteq \mathcal{B}^{\mu}(\mathbb{F}^K)$, then the restriction of ρ^{μ} to $C_w \cap Y_{N_T}$ maps $C_w \cap Y_{N_T}$ isomorphically onto its image, so it suffices to show that $C_w \cap Y_{N_T}$ maps surjectively onto $C_w \cap Y_{N_T}^{\mu}$.

Let $V_{\bullet} \in C_w \cap Y_{N_T}^{\mu}$. For each i, let $v_1, \ldots, v_n$ be the vectors such that for $p \in \mu[i]$, we have $v_p \in V_i \setminus V_{i-1}$, v_p has leading term $f_{w(p)}$, and

$$v_p = f_{w(p)} + \sum_{h=1}^{w(p)-1} \beta_{w(p),h} f_h \tag{3.2}$$

for some $\beta_{w(p),h}$ such that $\beta_{w(p),h} = 0$ if $h \in \{w(1), \ldots, w(p-1)\}$. We claim that the partial flag $W_{\bullet} \in \mathcal{B}^{(1^n)}(\mathbb{F}^K)$ defined by $W_i = \text{span}\{v_1, \ldots, v_i\}$ for $i \leq n$ is in Y_{N_T}. Indeed, since $N_T V_i \subseteq V_i$, then $N_T W_{\mu_i} \subseteq W_{\mu_i}$ for all i. For $\mu_1 + \cdots + \mu_{i-1} + 1 \leq p \leq \mu_1 + \cdots + \mu_i$, then $N_T v_p \in W_{\mu_i}$. Let (a, b) be the coordinates of the label $w(p)$ in T, so $w(p) = T(a, b)$. Since T is Schubert compatible, either $N_T v_p = 0$ or the leading term of $N_T v_p$ is $N_T f_{w(p)} = f_{T(a,b+1)}$. In the latter case, since w is admissible with respect to T, then $T(a, b+1) = w(p')$ for some $p' < p$. Since w is μ-increasing, we have $w(p') \notin \{p+1, \ldots, \mu_i\}$, so the expansion of $N_T v_p$ into the v_i vectors cannot have any terms with nonzero coefficient on $v_{p+1}, \ldots, v_{\mu_i}$ (otherwise the leading term would not be $f_{w(p')}$); hence, $N_T v_p \in W_p$. Therefore, we have the containment $N_T W_p \subseteq W_p$ for all p.

Since $N_T^{n-k} \mathbb{F}^{|\Lambda|} \subseteq V_{\ell(\mu)} = W_n$, then we have $W_{\bullet} \in Y_{n,\lambda,s}$ and by construction, $\rho^{\mu}(W_{\bullet}) = V_{\bullet}$, so ρ^{μ} maps $C_w \cap Y_{n,\lambda,s}$ surjectively, hence isomorphically, onto $C_w \cap Y_{n,\lambda,s}^{\mu}$.

When $\mathbb{F} = \mathbb{C}$, since $C_w \cap Y_{n,\lambda,s}$ is isomorphic to an affine space, then $C_w \cap Y_{n,\lambda,s}^{\mu}$ is as well. Since the C_w are cells of an affine paving of $\mathcal{B}^{\mu}(\mathbb{C}^K)$, then the intersections $C_w \cap Y_{n,\lambda,s}^{\mu}(\mathbb{C})$ for μ-increasing w are the cells of an affine paving of $Y_{n,\lambda,s}^{\mu}(\mathbb{C})$. $\qquad\square$

Recall that for T Schubert compatible and w admissible with respect to T, $\mathrm{PRD}_T(w)$ is defined to be the row-decreasing filling of $[\Lambda]$ such that if $w(i) = T(p, q)$, then the cell (p, q) is labeled by i. Equivalently, if we let $\mathrm{PRD}_T(w)^{-1}(a)$ be the position of the label a in $\mathrm{PRD}_T(w)$, we have

$$w(a) = T(\mathrm{PRD}_T(w)^{-1}(a)). \tag{3.3}$$

Definition 3.4. For T Schubert compatible and w admissible with respect to T, an **inversion of** w **with respect to** T is a pair (c, i) with $1 \le c \le n$, $1 \le i \le s$ such that one of the following conditions holds:

(IT1) There exists a label $\ell > c$ of $\mathrm{PRD}_T(w)$ in row i such that $w(\ell) < w(c)$,
(IT2) The condition (IT1) does not hold, and there exists an empty cell (i, j) of $\mathrm{PRD}_T(w)$ in row i such that $T(i, j) < w(c)$.

Define $\mathrm{inv}_T(w)$ to be the number of such pairs.

For example, letting T be as in Figure 2 (reading order for $\Lambda(7, (2, 2), 4)$) and $w = 2713594$ as in Figure 3, then the inversions of w with respect to T of type (IT1) are $(c, i) = (1, 1), (2, 1)$ and $(5, 2)$, and those of type (IT2) are $(c, i) = (2, 2), (6, 2)$ and $(6, 4)$. Thus, $\mathrm{inv}_T(w) = 6$.

Lemma 3.5. *Let T be a Schubert-compatible filling of $[\Lambda]$. If w is admissible with respect to T and μ-increasing, then*

$$C_w \cap Y_{N_T}^\mu \cong \mathbb{F}^{\mathrm{inv}_T(w)}.$$

Proof. By Lemma 3.3, it suffices to prove the result for the case $\mu = (1^n)$. We proceed by induction on n. In the base case when $n = 0$ (and s is arbitrary), then $\lambda = \emptyset$, and the unique admissible w is the empty function $w = \emptyset$. Furthermore, $C_\emptyset \cap Y_{N_T} = Y_{N_T}$ is a single point and $\mathrm{inv}_T(w) = 0$. Thus, the base case holds.

Let $n \ge 1$ and T and w be arbitrary with $w(1) = T(i, \Lambda_i)$. To complete the induction, by Lemma 2.3, it suffices to show

$$\mathrm{inv}_T(w) - \mathrm{inv}_{T^{(i)}}\left(\mathrm{fl}_T^{(i)}(w)\right) = i - 1. \tag{3.4}$$

First, we claim that the left-hand side of (3.4) is the number of inversions of w with respect to T that are of the form $(1, i')$. Indeed, there is an injective map from the inversions of $\mathrm{fl}_T^{(i)}(w)$ with respect to $T^{(i)}$ to the inversions of w with respect to T by sending (c, r) to $(c + 1, r')$, where r' is the row of T corresponding to row r of $T^{(i)}$ in the row-sorting step of forming $T^{(i)}$. The inversions of w that are not in the image of this map are those of the form $(1, i')$, together with inversions of w of the form (c, r) with $c > 1$ of type (IT2) that involve an empty cell $(p, 1)$ that was deleted from T to form $T^{(i)}$ in the case when $i > \ell(\lambda)$. However, observe that $[\Lambda]/[\lambda]^r$ has $n - k$ cells in each row, and the number of labels in $[\Lambda]/[\lambda]^r$ is $n - k$. Therefore, since both 1 and c are not in row p, both entries $(p, 1)$ and $(p, 2)$ are empty, and $T(p, 2) < T(p, 1) < w(c)$, so (c, p) is in the image of the injective map described above. Thus, the left-hand side of (3.4) only counts inversions of w of the form $(1, i')$.

Second, we claim that $(1, i')$ is an inversion if and only if $i' < i$. Indeed, if $i' < i$, then $T(i', \Lambda_{i'}) < T(i, \Lambda_i) = w(1)$ since T is Schubert compatible. If $(i', \Lambda_{i'})$ is filled in $\mathrm{PRD}_T(w)$, let its label be ℓ. Since w is injective, then $\ell > 1$. Furthermore, $w(\ell) = T(\mathrm{PRD}_T(w)^{-1}(\ell)) = T(i', \Lambda_{i'}) < w(1)$; thus, $(1, i')$ is an inversion of type (IT1). Otherwise, $(i', \Lambda_{i'})$ is not filled in $\mathrm{PRD}_T(w)$, so since $T(i', \Lambda_{i'}) < w(1)$, then $(1, i')$ is an inversion of type (IT2).

Suppose $(1, i')$ is an inversion for some $i' \ge i$. Then in either of the cases (IT1) or (IT2), there exists a cell (i', j') in row i' such that $T(i', j') < w(1) = T(i, \Lambda_i)$ (where $T(i', j') = w(\ell)$ in the (IT1) case). By Schubert compatibility, we have $T(i, \Lambda_i) \le T(i', \Lambda_{i'}) \le T(i', j')$, a contradiction. Therefore, $(1, i')$ is an inversion if and only if $i' < i$, so the number of inversions counted by the left-hand side of (3.4) is $i - 1$. The inductive step is thus complete. $\square$

Lemma 3.6. *Let T be the reading order filling of* $[\Lambda]$. *For all w admissible with respect to T, we have* $\operatorname{inv}_T(w) = \operatorname{dinv}(\operatorname{PRD}_T(w))$.

Proof. Let $\varphi = \operatorname{PRD}_T(w)$. We define a map from diagonal inversions of φ to inversions of w with respect to T as follows. For each diagonal inversion $((i,j),(p,q))$, let its corresponding inversion of w be $(\varphi_{p,q}, i)$. The pair $(\varphi_{p,q}, i)$ is indeed an inversion. If $((i,j),(p,q))$ is a type (D1) inversion, then $(\varphi_{p,q}, i)$ is a type (IT1) inversion, where $\ell = \varphi_{i,j}$. If $((i,j),(p,q))$ is type (D2), then $(\varphi_{p,q}, i)$ is type (IT1) or type (IT2) depending on whether there exists a label $\ell > \varphi_{p,q}$ in row i of φ or not, respectively. We claim that the map defined above is a bijection between diagonal inversions of φ and inversions of w with respect to T.

To show bijectivity, we give the inverse bijection, as follows. Given (c, i) an inversion of w, let (p, q) be the coordinates of the label c in φ. If (c, i) is of type (IT1), then there exists a label $\ell > c$ in row i of φ such that $w(\ell) < w(c)$. Since $w(\ell) < w(c)$, then either $i < p$ and ℓ is in a cell above and weakly to the right of c, or $i > p$ and ℓ is in a cell below and strictly to the right of c. In the first case, (c, i) corresponds to the diagonal inversion $((i,q),(p,q))$. Indeed, the cell (i,q) is either empty in φ, so $((i,q),(p,q))$ is a type (D2) diagonal inversion, or the cell (i,q) is labeled and $\varphi_{i,q} \geq \ell > c$ by the row-decreasing property of φ so $((i,q),(p,q))$ is a type (D1) diagonal inversion. In the second case when $i > p$, then (c, i) corresponds to the diagonal inversion $((i,q+1),(p,q))$. Indeed, the cell $(i,q+1)$ is either empty in φ, so $((i,q+1),(p,q))$ is a type (D2) diagonal inversion, or the cell $(i,q+1)$ is labeled and $\varphi_{i,q+1} \geq \ell > c$ so $((i,q+1),(p,q))$ is a type (D1) diagonal inversion.

Finally, if (c, i) is of type (IT2), then there exists an empty cell (i,j) in φ such that $T(i,j) < w(c)$. Then either $i < p$ and $((i,q),(p,q))$ is a diagonal inversion of type (D2), or $i > p$ and $((i,q+1),(p,q))$ is a diagonal inversion of type (D2). It can then be checked that this is the inverse map, and hence, the number of inversions of w with respect to T is equal to the number of diagonal inversions of φ. □

Combining Lemma 3.6 with Lemma 3.5, we see that dinv has geometric meaning: it counts the dimensions of cells in Y_{N_T}.

Corollary 3.7. *For T the reading order filling of* $[\Lambda]$ *and w that is μ-increasing and admissible with respect to T, we have*

$$C_w \cap Y_{N_T}^\mu \cong \mathbb{F}^{\operatorname{dinv}(\operatorname{PRD}_T(w))}. \tag{3.5}$$

We are now able to prove the main theorem of this section.

Theorem 3.8. *For all prime powers q, we have*

$$\operatorname{Frob}(H^*(Y_{n,\lambda,s}(\mathbb{C}); \mathbb{Q}); q) = \sum_{\mu \vdash n} |Y_{n,\lambda,s}^\mu(\mathbb{F}_q)| \, m_\mu(\mathbf{x}).$$

Proof. Let T be the reading order filling of $[\Lambda]$ and let $\mu \vdash n$. We claim that

$$\sum_{\varphi \in \operatorname{PRD}_{n,\lambda,s}^\mu} q^{\operatorname{dinv}(\varphi)} = \sum_{\substack{w \text{ admissible wrt } T, \\ \mu\text{-increasing}}} q^{\operatorname{dinv}(\operatorname{PRD}_T(w))}. \tag{3.6}$$

Indeed, given w admissible with respect to T that is μ-increasing, define $\varphi \in \operatorname{PRD}_{n,\lambda,s}^\mu$ by replacing the labels $\mu_1 + \cdots + \mu_{i-1} + 1, \mu_1 + \cdots + \mu_{i-1} + 2, \ldots, \mu_1 + \cdots + \mu_i$ in $\operatorname{PRD}_T(w)$ with i for each $i \leq \ell(\mu)$. Since w is μ-increasing, then the labels $\mu_1 + \cdots + \mu_{i-1} + 1, \mu_1 + \cdots + \mu_{i-1} + 2, \ldots, \mu_1 + \cdots + \mu_i$ in $\operatorname{PRD}_T(w)$ are increasing in reading order, so there are no diagonal inversions among these labels. Therefore, it follows from the definition of dinv that $\operatorname{dinv}(\varphi) = \operatorname{dinv}(\operatorname{PRD}_T(w))$.

Using (3.6), we have

$$\text{Frob}(H^*(Y_{n,\lambda,s}(\mathbb{C}); \mathbb{Q}); q) = \sum_{\mu \vdash n} \sum_{\varphi \in \text{PRD}^\mu_{n,\lambda,s}} q^{\text{dinv}(\varphi)} m_\mu(\mathbf{x}) \tag{3.7}$$

$$= \sum_{\substack{\mu \vdash n \\ w \text{ admissible wrt } T, \\ \mu\text{-increasing}}} \sum |C_w \cap Y^\mu_{N_T}(\mathbb{F}_q)| m_\mu(\mathbf{x}) \tag{3.8}$$

$$= \sum_{\mu \vdash n} |Y^\mu_{N_T}(\mathbb{F}_q)| m_\mu(\mathbf{x}). \tag{3.9}$$

The first equality is Theorem 2.6, the second follows from Corollary 3.7 and the fact that PRD_T restricts to a bijection between the set of w that are admissible and μ-increasing and $\text{PRD}^\mu_{n,\lambda,s}$, and the third equality follows from the fact that the intersections $C_w \cap Y^\mu_{N_T}$ for w that are admissible with respect to T and μ-increasing partition the space $Y^\mu_{N_T}$ by Lemma 3.3. $\qquad\square$

As a corollary, we obtain the following generalization of Borho and Macpherson's result (2.8).

Corollary 3.9. *For all $\mu \vdash n$, we have an isomorphism of graded vector spaces*

$$H^*(Y^\mu_{n,\lambda,s}(\mathbb{C}); \mathbb{Q}) \cong H^*(Y_{n,\lambda,s}(\mathbb{C}); \mathbb{Q})^{S_\mu}. \tag{3.10}$$

Proof. By Frobenius reciprocity (see, for example, [3]), the $m_\mu(\mathbf{x})$ coefficient of the graded Frobenius characteristic of a graded S_n-module is the Hilbert series of the S_μ-invariant subspace. By Theorem 3.8, applying this fact to the cohomology of $Y_{n,\lambda,s}(\mathbb{C})$ yields

$$|Y^\mu_{n,\lambda,s}(\mathbb{F}_q)| = \text{Hilb}(H^*(Y_{n,\lambda,s}(\mathbb{C}); \mathbb{Q})^{S_\mu}; q) \tag{3.11}$$

for all prime powers q. By the last part of Lemma 3.3,

$$\text{Hilb}(H^*(Y^\mu_{n,\lambda,s}(\mathbb{C}); \mathbb{Q}); q) = \sum_{\substack{w \text{ admissible wrt } T, \\ \mu\text{-increasing}}} q^{\dim_\mathbb{C}(C_w \cap Y^\mu_{N_T}(\mathbb{C}))}.$$

By (2.13), it can be checked by induction that $|C_w \cap Y^\mu_{N_T}(\mathbb{F}_q)| = q^{\dim_\mathbb{C}(C_w \cap Y^\mu_{N_T}(\mathbb{C}))}$ for each w admissible and μ-increasing. Since the $C_w \cap Y^\mu_{N_T}(\mathbb{F}_q)$ partition $\mathbb{F}_q$, we have

$$|Y^\mu_{n,\lambda,s}(\mathbb{F}_q)| = \text{Hilb}(H^*(Y^\mu_{n,\lambda,s}(\mathbb{C}); \mathbb{Q}); q). \tag{3.12}$$

Combining (3.11) and (3.12), the two spaces have the same Hilbert series and are hence isomorphic as graded vector spaces. $\qquad\square$

Remark 3.10. We note that a second proof of Corollary 3.9 has been given by Gillespie and the author [7] by realizing $Y_{n,\lambda,s}$ as a special case of a variety studied by Borho and MacPherson in their study of partial resolutions of nilpotent orbit closures [2].

4. A Springer fiber decomposition of $Y_{n,\lambda,s}$

In this section, we decompose the Δ-Springer variety $Y^\mu_{n,\lambda,s}$ into subspaces that are isomorphic to Steinberg varieties crossed with affine spaces, so, in particular, $Y_{n,\lambda,s}$ decomposes into copies of Springer fibers crossed with affine spaces. We then use this to prove our expansion of the graded Frobenius characteristic of the cohomology ring in terms of Hall-Littlewood polynomials in Section 5.

Figure 4. *The reverse reading order filling T for* $[\Lambda(7,(2,2),4)]$, *with* $[\lambda] = [(2,2)]$ *shaded in orange. The diagram of* $\alpha = (2,4,0,1) \in \mathrm{Comp}(7,4)$ *is the union of the orange and blue cells.*

Definition 4.1. The **reverse reading order** of $[\Lambda]$ is the sequence of cells obtained by reading down each column of $[\Lambda]$, ordering the columns from left to right. The **reverse reading order filling** of $[\Lambda]$ is the unique bijection $T : [\Lambda] \to [K]$ such that $T(i,j) = \ell$ if and only if (i,j) is the ℓth cell in reverse reading order.

Note that in reverse reading order, only the order of the columns are reversed (not the order of the rows). See Figure 4 for the reverse reading order filling of $[\Lambda(7,(2,2),4)]$.

Throughout this section, we let T be the reverse reading order filling of $[\Lambda]$. For notational convenience, we use the simplified notation $N := N_T$ and $N^t := N_T^t$.

Given $\alpha \in \mathrm{Comp}(n,s)$ such that $\lambda \subseteq \alpha$, let $[\alpha]$ be the subdiagram of $[\Lambda]$ defined by

$$[\alpha] := \{(i,j) : 1 \le i \le s, \ 1 \le j \le \alpha_i\}.$$

Note that the inequality $\alpha_i \le \Lambda_i = (n-k) + \lambda_i$ is guaranteed for all i by the fact that α has size n and contains λ (which has size k), so $[\alpha]$ is indeed a subdiagram of $[\Lambda]$.

Let $\mathbb{F}^\alpha \subseteq \mathbb{F}^K$, which is the subspace spanned by f_i for i a label of T contained in $[\alpha]$. Similarly, let $\mathbb{F}^\lambda := (N^t)^{n-k} \mathbb{F}^K$ be the subspace spanned by f_i for i a label of T contained in $[\lambda]$.

Let $w : [n] \to [K]$ be the unique (n)-increasing injective function whose image is the set of labels of T in $[\alpha]$, and let C_w be the corresponding Schubert cell in the Grassmannian $\mathcal{B}^{(n,K-n)} = \mathrm{Gr}(n, \mathbb{F}^K)$. Define

$$Z_\alpha^\mu := \{V_\bullet \in Y_{N^t}^\mu \mid V_n = \mathbb{F}^\alpha\}, \tag{4.1}$$

$$\widehat{Z}_\alpha^\mu := \{V_\bullet \in Y_{N^t}^\mu \mid V_n \in C_w\}. \tag{4.2}$$

Note that since N^t restricts to a nilpotent matrix on the subspace $\mathbb{F}^\alpha$ with Jordan type $\mathrm{sort}(\alpha)$, then $Z_\alpha^\mu \cong \mathrm{Stein}_{\mathrm{sort}(\alpha)}^\mu$.

The following is the main theorem of this section. In the case $\mu = (1^n)$, it says that the subspaces $\widehat{Z}_\alpha$, which partition $Y_{n,\lambda,s}$, are isomorphic to a Springer fiber crossed with an affine space.

Theorem 4.2. *We have*

$$\widehat{Z}_\alpha^\mu \cong \mathbb{F}^\ell \times Z_\alpha^\mu,$$

where $\ell = \sum_i (s - \alpha_i')(\alpha_{i+1}' - \lambda_{i+1}') + \mathrm{coinv}(\alpha)$ *(where* coinv *was defined at the end of Subsection 2.1) and* α_i' *is the number of cells of* $[\alpha]$ *in the ith column where* $\alpha_0' := s$.

In order to prove Theorem 4.2, we need several definitions and lemmata.

Definition 4.3. A pair (i,j) with $i > j$ is a **free pair** for α if the cell with label i of T is in $[\alpha]/[\lambda]$ and the cell with label j of T is the leftmost cell of $[\Lambda] \setminus [\alpha]$ in j's own row.

Remark 4.4. Let w be the unique admissible (n)-increasing injective function whose image is the set of labels of T in $[\alpha]$, as above. Free pairs (i,j) are defined to correspond to a subset of standard coordinates $\beta_{i,j}$ of C_w. Note that $\beta_{i,j}$ is undefined unless i is in the image of w, and that if i is in the image of w, then $\beta_{i,j} = 0$ if j is a label in $[\alpha]$.

Example 4.5. For $n = 7$, $\lambda = (2, 2)$, $s = 4$ and $\alpha = (2, 4, 0, 1)$ as depicted in Figure 4, the free pairs are $(14, 9)$, $(14, 8)$, $(14, 3)$, $(10, 9)$, $(10, 8)$, $(10, 3)$ and $(4, 3)$.

Lemma 4.6. *The number of free pairs of α is*

$$\sum_i (s - \alpha_i')(\alpha_{i+1}' - \lambda_{i+1}') + \operatorname{coinv}(\alpha). \tag{4.3}$$

Proof. Let (i, j) be a free pair for α. If i and j are not in the same column, in which case j must be in a column to the left of i, then associate (i, j) to the pair (i, j'), where j' is the label in the same row as j and in the column immediately to the left of the column of i. This correspondence sets up a bijection between free pairs in which i and j are not in the same column and pairs (i, j') where i is a label in $[\alpha] \setminus [\lambda]$ and j' is a label in $[\Lambda] \setminus [\alpha]$ in the column immediately to the left of i. Counting these latter pairs by the column containing i, we have the sum

$$\sum_p (s - \alpha_p')(\alpha_{p+1}' - \lambda_{p+1}'). \tag{4.4}$$

If i and j are in the same column, then associate the free pair (i, j) to the coinversion (r, r') of α, where r is the row of j and r' is the row of i. Indeed, (r, r') is a coinversion since $r < r'$ by the fact that $i > j$ and the definition of reverse reading order, and $\alpha_r < \alpha_{r'}$ since i is in $[\alpha]$ and j is not. This correspondence is a bijection between free pairs with i and j in the same column and coinversions of α. Thus, the total number of free pairs is given by (4.3). □

Example 4.7. For α as in Example 4.5, the free pairs $(10, 9)$ and $(4, 3)$ contribute to $\operatorname{coinv}(\alpha)$ in (4.3). The terms of (4.4) are: $1 \cdot 0$ for $p = 1$, $2 \cdot 1$ for $p = 2$ corresponding to pairs $(10, 3)$ and $(10, 8)$, $3 \cdot 1$ for $p = 3$ corresponding to pairs $(14, 9)$, $(14, 3)$ and $(14, 8)$, and $1 \cdot 0$ for $p = 4$.

Definition 4.8. For (i, j) a free pair with $i > j$, let $U_{i,j}(t)$ be the $K \times K$ matrix such that

$$U_{i,j}(t)(N^m f_i) = N^m(f_i + t f_j) \qquad \text{for } m \geq 0 \text{ such that } N^m f_i \neq 0, \tag{4.5}$$
$$U_{i,j}(t)((N^t)^m f_i) = (N^t)^m(f_i + t f_j) \qquad \text{for } m \geq 0, \tag{4.6}$$

and $U_{i,j}(t) f_\ell = f_\ell$ for all labels ℓ of T that are not in the same row as i.

Similarly, let $\widehat{U}_{i,j}(t)$ be the matrix such that

$$\widehat{U}_{i,j}(t)(N^m f_i) = N^m(f_i + t f_j) \qquad \text{for } m \geq 0 \text{ such that } N^m f_i \neq 0 \tag{4.7}$$

and $\widehat{U}_{i,j}(t) f_\ell = f_\ell$ for all labels ℓ of T that are either not in the same row as i or not weakly to the right of i.

Example 4.9. Continuing our running Example 4.5, the operator $U_{14,9}(t)$ is defined by

- $U_{14,9}(t) f_{14} = f_{14} + t f_9$,
- $U_{14,9}(t) f_{16} = f_{16} + t f_{13}$,
- $U_{14,9}(t) f_{10} = f_{10} + t f_5$,
- $U_{14,9}(t) f_6 = f_6 + t f_1$,
- $U_{14,9}(t) f_2 = f_2 + 0$,

and $U_{14,9}(t) f_\ell = f_\ell$ for all $\ell \notin \{2, 6, 10, 14, 16\}$. However,

- $\widehat{U}_{14,9}(t) f_{14} = f_{14} + t f_9$,
- $\widehat{U}_{14,9}(t) f_{16} = f_{16} + t f_{13}$,

and $\widehat{U}_{14,9}(t) f_\ell = f_\ell$ for all $\ell \notin \{14, 16\}$.

Lemma 4.10. *The matrices $U_{i,j}(t)$ and $\widehat{U}_{i,j}(t)$ are unipotent upper triangular.*

Proof. By construction, $U_{i,j}(t)$ has 1s along the diagonal, so it suffices to check it is upper triangular. Thus, it suffices to check that for all m, the vector $N^m f_i$ has index greater than the vector $N^m f_j$ (if both are nonzero), and the vector $(N^t)^m f_i$ has index greater than the vector $(N^t)^m f_j$ (if both are nonzero). Both of these claims follow immediately from the definition of free pairs and reverse reading order. The same reasoning applies to $\widehat{U}_{i,j}(t)$. $\qquad\square$

Lemma 4.11. *Let* $(i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell)$ *be the free pairs for α listed so that $i_1 \geq i_2 \geq \cdots \geq i_\ell$. Then for all $p \leq \ell$,*

$$U_{i_p, j_p}(t_p) \, U_{i_{p-1}, j_{p-1}}(t_{p-1}) \cdots U_{i_1, j_1}(t_1) \, \mathbb{F}^\alpha = \widehat{U}_{i_p, j_p}(t_p) \, \widehat{U}_{i_{p-1}, j_{p-1}}(t_{p-1}) \cdots \widehat{U}_{i_1, j_1}(t_1) \, \mathbb{F}^\alpha. \qquad (4.8)$$

Proof. We proceed by induction on p. The base case $p = 0$ is trivial, so suppose that for some $p \geq 0$, (4.8) holds. It suffices to show that

$$U_{i_{p+1}, j_{p+1}}(t_{p+1})\widehat{U}_{i_p, j_p}(t_p) \cdots \widehat{U}_{i_1, j_1}(t_1)\mathbb{F}^\alpha = \widehat{U}_{i_{p+1}, j_{p+1}}(t_{p+1})\widehat{U}_{i_p, j_p}(t_p) \cdots \widehat{U}_{i_1, j_1}(t_1)\mathbb{F}^\alpha. \qquad (4.9)$$

Let $v \in \widehat{U}_{i_p, j_p}(t_p) \cdots \widehat{U}_{i_1, j_1}(t_1)\mathbb{F}^\alpha$. Since the image of the operator $U_{i_{p+1}, j_{p+1}}(t_{p+1}) - \widehat{U}_{i_{p+1}, j_{p+1}}(t_{p+1})$ is contained in the span of the vectors $(N^t)^m f_{j_{p+1}}$ for $m > 0$ (in fact, they are equal when $t_{p+1} \neq 0$), then

$$U_{i_{p+1}, j_{p+1}}(t_{p+1})v - \widehat{U}_{i_{p+1}, j_{p+1}}(t_{p+1})v \qquad (4.10)$$

is a linear combination of the vectors $(N^t)^m f_{j_{p+1}}$ for $m > 0$. For a fixed $m > 0$, let $f_{j'} = (N^t)^m f_{j_{p+1}}$. Then j' is in a column strictly to the left of i_q for all $q \leq p + 1$. Since the operators $\widehat{U}_{i_q, j_q}(t_q)$ for $q \leq p + 1$ fix $f_{j'}$, and since $f_{j'} \in \mathbb{F}^\alpha$, then

$$f_{j'} \in \widehat{U}_{i_p, j_p}(t_p) \cdots \widehat{U}_{i_1, j_1}(t_1)\mathbb{F}^\alpha.$$

Furthermore, since j' is not in the same row as i_{p+1}, then $U_{i_{p+1}, j_{p+1}}(t_{p+1})$ and $\widehat{U}_{i_{p+1}, j_{p+1}}(t_{p+1})$ fix $f_{j'}$, so $f_{j'}$ is in both sides of (4.9). Therefore, (4.10) is in the intersection of the left-hand side and right-hand side of (4.9). Since $U_{i_{p+1}, j_{p+1}}(t_{p+1})v$ and $\widehat{U}_{i_{p+1}, j_{p+1}}(t_{p+1})v$ are arbitrary elements of the left- and right-hand sides of (4.9), respectively, then the two sets are equal. The induction is thus complete. $\qquad\square$

Example 4.12. For our running example and $(i_1, j_1) = (14, 9)$, then $U_{14,9}(t) - \widehat{U}_{14,9}(t)$ sends f_{10} to $t f_5$ and f_6 to $t f_1$ (and everything else to 0) but both $t f_5$ and $t f_1$ are in $\mathbb{F}^\alpha$ and fixed by both $U_{14,9}(t)$ and $\widehat{U}_{14,9}(t)$, so $U_{14,9}(t)\mathbb{F}^\alpha = \widehat{U}_{14,9}(t)\mathbb{F}^\alpha = \text{span}\{f_{14} + t f_9, f_{10}, f_6, f_5, f_4, f_2, f_1\}$.

Lemma 4.13. *Let* $(i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell)$ *be the free pairs for α, listed so that $i_1 \geq i_2 \geq \cdots \geq i_\ell$. Then we have a well-defined map*

$$\mathbb{F}^\ell \times Z_\alpha^\mu \to \widehat{Z}_\alpha^\mu$$

defined by sending $(\vec{t}, V_\bullet)$ to $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2)U_{i_1, j_1}(t_1)V_\bullet$.

Proof. Letting $V_\bullet \in Z_\alpha^\mu$, it suffices to show that $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2)U_{i_1, j_1}(t_1)V_\bullet \in \widehat{Z}_\alpha^\mu$. Since each $U_{i_p, j_p}(t_p)$ is unipotent upper triangular by Lemma 4.10 and $V_n = \mathbb{F}^\alpha \in C_w$, then

$$U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2)U_{i_1, j_1}(t_1)V_n \in C_w.$$

By construction, we have $U_{i_p, j_p}(t_p)N^t = N^t U_{i_p, j_p}(t_p)$ for all $p \leq \ell$, so N^t preserves each part of the partial flag $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2)U_{i_1, j_1}(t_1)V_\bullet$.

Finally, we must check that $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2)U_{i_1, j_1}(t_1)V_n \supseteq \mathbb{F}^\lambda$. Indeed, this follows from the fact that $V_n = \mathbb{F}^\alpha \supseteq \mathbb{F}^\lambda$, Lemma 4.11, and the fact that each $\widehat{U}_{i_p, j_p}(t_p)$ fixes $\mathbb{F}^\lambda$. Thus, the map is well defined. $\qquad\square$

Example 4.14. For our running example with $n = 7$, $\lambda = (2,2)$, $s = 4$ and $\alpha = (2,4,0,1)$, one (of many) partial flags in $Z_\alpha^{(1^7)}$ is the one represented by the $K \times n$ matrix A below (we do not need the full $K \times K$ matrix because the span of the first n columns completely determines the type $(1^n, K-n)$ partial flag). The reader can check that the matrix representative of the partial flag

$$U_{4,3}(t_7)U_{10,3}(t_6)U_{10,8}(t_5)U_{10,9}(t_4)U_{14,3}(t_3)U_{14,8}(t_2)U_{14,9}(t_1)$$

is the matrix B below.

$$A = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} t_4 & t_1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_6 & t_3 & 0 & 0 & t_7 \\ 0 & t_5 & t_2 & 0 & 0 & 6 & 1 \\ 0 & t_4 & t_1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_6 & 0 & 0 & 0 \\ 0 & 0 & t_5 & t_2 & 0 & 0 & 0 \\ 0 & 0 & t_4 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We leave it to the reader to check that B represents a partial flag $\widehat{Z}_\alpha^{(1^7)}$.

Lemma 4.15. *Let $V_\bullet \in Z_\alpha^\mu$ and let $(i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell)$ be the free pairs for α listed so that $i_1 \geq i_2 \geq \cdots \geq i_\ell$. For all $p \leq \ell$, the standard coordinate β_{i_p, j_p} of $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_1, j_1}(t_1)V_n \in C_w$ is t_p.*

Proof. By construction of $\widehat{U}_{i,j}(t)$, the standard coordinates of $\widehat{U}_{i,j}(t)V$ are obtained by applying $\widehat{U}_{i,j}(t)$ directly to each v_p and collecting terms (in terms of matrix representatives, $\widehat{U}_{i,j}(t)$ sends column-reduced matrices to column-reduced matrices); hence, the standard coordinates of $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_1, j_1}(t_1)V_n = \widehat{U}_{i_\ell, j_\ell}(t_\ell) \cdots \widehat{U}_{i_1, j_1}(t_1)V_n$ are obtained by applying these operators to the standard coordinates of V_n. Observe the following:

1. The β_{i_p, j_p} coordinate of V_n is 0 since $V_n = \mathbb{F}^\alpha$.
2. The operator $\widehat{U}_{i_p, j_p}(t_p)$ is the only $\widehat{U}_{i_q, j_q}(t_q)$ operator that changes the β_{i_p, j_p} coordinate. Otherwise, there must exist some q and $m > 0$ such that $N^m f_{i_q} = f_{i_p}$ and $N^m f_{j_q} = f_{j_p}$. But this would imply that the label j_p is strictly to the right of the label j_q in T, contradicting the fact that j_p is the leftmost label in $[\Lambda]/[\lambda]$ in j_p's row.
3. $\widehat{U}_{i_p, j_p}(t_p) f_{i_p} = f_{i_p} + t_p f_{j_p}$.

Combining 1–3 above, the β_{i_p, j_p} coordinate after applying all of these operators is t_p. $\square$

Lemma 4.16. *Let $V \in C_w$ such that $N^t V \subseteq V$ and let (i, j) be a pair of labels of T such that $i > j$, i is in $[\alpha]$, and j is in $[\Lambda]/[\alpha]$ (for example, (i, j) is a free pair of α). Suppose that for all labels $r < i$ of T in $[\alpha]$, we have $f_r \in V$. Then for any $m > 0$, if $N^m f_i = f_{i'}$ and $N^m f_j = f_{j'}$, then the $\beta_{i,j}$ and $\beta_{i',j'}$ coordinates of V are equal.*

Proof. Letting v_q be the vectors for the standard coordinate representation of V and letting $a_\ell = \beta_{i,\ell}$ and $b_{\ell'} = \beta_{i',\ell'}$ for notational convenience, we have

$$v_{w^{-1}(i)} = f_i + a_j f_j + \sum_{j \neq \ell < i} a_\ell f_\ell,$$

$$v_{w^{-1}(i')} = f_{i'} + b_{j'} f_{j'} + \sum_{j' \neq \ell' < i'} b_{\ell'} f_{\ell'},$$

$$(N^t)^m v_{w^{-1}(i')} = f_i + b_{j'} f_j + \sum_{j' \neq \ell' < i'} b_{\ell'}((N^t)^m f_{\ell'}).$$

Since all of the above vectors are in V, then

$$v_{w^{-1}(i)} - (N^t)^m v_{w^{-1}(i')} = (a_j - b_{j'}) f_j + \sum_{j \neq \ell < i} a_\ell f_\ell - \sum_{j' \neq \ell' < i'} b_{\ell'}((N^t)^m f_{\ell'}) \in V. \qquad (4.11)$$

Observe that the f_j coefficient of this vector is exactly $a_j - b_{j'}$, since the two sums contain no f_j terms.

Suppose by way of contradiction that $a_j \neq b_{j'}$. Then the vector (4.11) is nonzero, and since it is in V, its leading term must be a scalar multiple of f_p for some p a label in $[\alpha]$. Since $a_\ell = \beta_{i,\ell} = 0$ for all $\ell \in \{w(1), w(2), \ldots, i\}$, then no nonzero terms in the first sum correspond to ℓ a label in $[\alpha]$. Therefore, the leading term must be $b_{\ell'}((N^t)^m f_{\ell'})$ for some ℓ' in the second sum. Let ℓ be such that $(N^t)^m f_\ell = f_\ell$, and note that $\ell > j$ since it is the leading term. However, since $\ell' < i'$ by assumption, then $\ell < i$ (a property of reverse reading order). Since we necessarily have that ℓ is a label in $[\alpha]$, then by the hypothesis in the statement of the lemma, we have $f_\ell \in V$. Therefore, we can eliminate the $b_{\ell'} f_\ell = b_{\ell'}((N^t)^m f_{\ell'})$ term from (4.11). After repeating this process of eliminating terms from (4.11), we get a nonzero vector in V whose leading term does not correspond to a label in $[\alpha]$, a contradiction. Therefore, we must have $a_j = b_{j'}$, which completes the proof. $\qquad \square$

Lemma 4.17. *Let $(i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell)$ be the free pairs for α, listed so that $i_1 \geq i_2 \geq \cdots \geq i_\ell$. Then the map*

$$\mathbb{F}^\ell \times Z_\alpha^\mu \to \widehat{Z}_\alpha^\mu$$

defined by sending $(\vec{t}, V_\bullet)$ to $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2) U_{i_1, j_1}(t_1) V_\bullet$ is an isomorphism.

Proof. First, observe that the map is well defined by Lemma 4.13. Second, by Lemma 4.15, the β_{i_m, j_m} coordinate of $U_{i_\ell, j_\ell}(t_\ell) \cdots U_{i_2, j_2}(t_2) U_{i_1, j_1}(t_1) V_\bullet$ is t_m for each m. Therefore, any two pairs that map to the same partial flag must have the same $\vec{t}$ vector, and since the operators $U_{i,j}(t)$ are invertible, the map is injective.

Finally, we show that the map is surjective. Let $V_\bullet \in \widehat{Z}_\alpha^\mu$ and let $t_m = \beta_{i_m, j_m}$ be the standard coordinate of V_n. We claim that $U_{i_1, j_1}^{-1}(t_1) U_{i_2, j_2}^{-1}(t_2) \cdots U_{i_\ell, j_\ell}^{-1}(t_\ell) V_\bullet \in Z_\alpha^\mu$. Observe that $U_{i,j}^{-1}(t) = U_{i,j}(-t)$. Thus, it is clear that this flag is in $\widehat{Z}_\alpha^\mu$, so it suffices to prove that

$$U_{i_1, j_1}(-t_\ell) U_{i_2, j_2}(-t_2) \cdots U_{i_\ell, j_\ell}(-t_\ell) V_n = \mathbb{F}^\alpha.$$

This follows from the next claim.

Claim: Letting $\beta_{*,*}$ be the standard coordinates of $U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell) V_n$, for all $1 \leq p \leq q \leq \ell$, we have $\beta_{i_q, j_q} = 0$, and for all pairs (i_q, j) with $j < i_q$ that are not free pairs, we have $\beta_{i_q, j} = 0$.

We prove the claim by reverse induction on p. In the base case when $p = q = \ell$, then for all $r < i_\ell$ such that r is a label of T in $[\alpha]$, then $f_r \in V_n$, so we have $U_{i_\ell, j_\ell}(-t_\ell) V_n = \widehat{U}_{i_\ell, j_\ell}(-t_\ell) V_n$. Since the standard coordinates of $\widehat{U}_{i_\ell, j_\ell}(-t_\ell) V_n$ are obtained by applying $\widehat{U}_{i_\ell, j_\ell}(-t_\ell)$ to the standard coordinate representation of V_n, then $\widehat{U}_{i_\ell, j_\ell}(-t_\ell)$ eliminates the β_{i_ℓ, j_ℓ} coordinate by construction, so the first part of the claim holds in the base case.

Let (i_ℓ, j) be a pair with $j < i_\ell$ that is not a free pair. If j is a label in $[\alpha]$, then $\beta_{i_\ell, j} = 0$ by definition, so assume j is a label of T in $[\Lambda]/[\alpha]$. Let i' be the label in the cell immediately to the left of i_ℓ and let j' be the label of the cell immediately to the left of j (both of which must exist since (i_ℓ, j) is not a free pair. Notice that if i_ℓ were in the first column, then j would also have to be in the first column). Since, as we mentioned above, $f_r \in V_n$ for all $r < i_\ell$ such that r is in $[\alpha]$, then $f_r \in \widehat{U}_{i_\ell, j_\ell}(-t_\ell)V_n = U_{i_\ell, j_\ell}(-t_\ell)V_n$ for $r < i'$. By Lemma 4.16, we have $\beta_{i_\ell, j} = \beta_{i', j'}$. Furthermore, by the minimality of i_ℓ, we have that i' is in $[\lambda]$ (otherwise, (i', j') would be a free pair, contradicting the minimality of i_ℓ). Thus, $\beta_{i', j'} = 0$ and so also $\beta_{i_\ell, j} = 0$, which completes the base case.

In the inductive step, suppose the statement of the claim holds for p. Then by the inductive hypothesis, for all $r < i_{p-1}$ such that r is a label of T in $[\alpha]$ (note, this may exclude i_p if $i_p = i_{p-1}$), then all $\beta_{r,j}$ coordinates of $U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_n$ are 0, so $f_r \in U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_n$. Therefore,

$$U_{i_{p-1}, j_{p-1}}(-t_{p-1})U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_n$$
$$= \widehat{U}_{i_{p-1}, j_{p-1}}(-t_{p-1})U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_n.$$

By construction, $\widehat{U}_{i_{p-1}, j_{p-1}}(-t_{p-1})$ eliminates the $\beta_{i_{p-1}, j_{p-1}}$ coordinate and does not change the coordinates guaranteed to be 0 in the inductive hypothesis. Given a pair (i_{p-1}, j) such that j is a label of T not in $[\alpha]$, then there exist m and $q > p - 1$ such that $(N^t)^m f_{i_{p-1}} = f_{i_q}$ and $(N^t)^m f_{j_{p-1}} = f_{j_q}$. Note this implies $N^m f_{i_q} = f_{i_{p-1}}$ and $N^m f_{j_q} = f_{j_{p-1}}$. Since

$$f_r \in U_{i_{p-1}, j_{p-1}}(-t_{p-1})U_{i_p, j_p}(-t_p) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_n$$

for all $r \leq i_q$ such that r is a label in $[\alpha]$, then by Lemma 4.16, the β_{i_q, j_q} and $\beta_{i_{p-1}, j}$ coordinates of this subspace are equal and thus are both 0. This completes the induction and the proof of the claim.

By the claim, we have for $p = 1$ that $U_{i_1, j_1}(-t_1)U_{i_2, j_2}(-t_2) \cdots U_{i_\ell, j_\ell}(-t_\ell)V_\bullet \in Z_\alpha^\mu$, so the map in the statement of the lemma is surjective. In the case when $\mathbb{F} = \mathbb{C}$, since the inverse is a regular map, it is an isomorphism of varieties. $\qquad \square$

Proof of Theorem 4.2. The isomorphism $\widehat{Z}_\alpha^\mu \cong \mathbb{F}^\ell \times Z_\alpha^\mu$ follows by combining Lemma 4.6 and Lemma 4.17. $\qquad \square$

5. Proof of Theorem 1.3

In this section, we use the results from Sections 3 and 4 to prove the main theorem, Theorem 1.3. Throughout this section, we utilize the notation $v'_0 := s$ for $v \in \text{Par}(n, s)$. Recall our convention that we consider $H^{2k}(Y_{n,\lambda,s}; \mathbb{Q})$ to be the degree k component of the cohomology ring.

Lemma 5.1. *Let $v \in \text{Par}(n, s)$ such that $\lambda \subseteq v$. Then*

$$\sum_{\substack{\alpha \in \text{Comp}(n,s), \\ \text{sort}(\alpha) = v, \\ \alpha \supseteq \lambda}} q^{\text{coinv}(\alpha)} = \prod_{i \geq 0} \begin{bmatrix} v'_i - \lambda'_{i+1} \\ v'_i - v'_{i+1} \end{bmatrix}_q. \tag{5.1}$$

Proof. We proceed by induction on $m = v_1$. In the base case $m = v_1 = 1$, then $\lambda = (1^k)$ and $v = (1^n)$. In this case, each α in the summation has parts 0 and 1 such that $\alpha_1 = \cdots = \alpha_k = 1$. Deleting the first k parts from α, we get a weak composition consisting of 0s and 1s of size $n - k$ and length $s - k$ (and the same number of coinversions as α). Thus, the left-hand side of (5.1) can be rewritten as

$$\sum_{\alpha \in \text{Comp}(n-k, s-k)} q^{\text{coinv}(\alpha)} = \begin{bmatrix} s - k \\ n - k \end{bmatrix}_q = \begin{bmatrix} s - k \\ s - n \end{bmatrix}_q = \begin{bmatrix} v'_0 - \lambda'_1 \\ v'_0 - v'_1 \end{bmatrix}_q,$$

which equals the right-hand side, so the base case holds.

Let $m > 1$ and assume by way of induction that the identity holds for all v with $v_1 = m - 1$, and let $v \in \mathrm{Par}(n, s)$ such that $\lambda \subseteq v$. Define $\overline{v}$ and $\overline{\lambda}$ by removing all cells in column m of the Young diagrams of v and λ, respectively. Then $\alpha \in \mathrm{Comp}(n, s)$ such that $\lambda \subseteq \alpha$ and $\mathrm{sort}(\alpha) = v$ is obtained uniquely by the following process:

- Start with $\overline{\alpha} \in \mathrm{Comp}(n - v'_m, s)$ such that $\overline{\lambda} \subseteq \overline{\alpha}$ and $\mathrm{sort}(\overline{\alpha}) = \overline{v}$,
- Append one cell to the first λ'_m rows of $[\overline{\alpha}]$,
- Append one cell to $(v'_m - \lambda'_m)$ many of the remaining rows of $[\overline{\alpha}]$ that have length $m - 1$.

There are $(v'_{m-1} - \lambda'_m)$ many rows of $[\overline{\alpha}]$ with length $m - 1$ (apart from the first λ'_m rows). Furthermore, the difference $\mathrm{coinv}(\alpha) - \mathrm{coinv}(\overline{\alpha})$ is the number of pairs $i < j$ such that $\alpha_i = m - 1$ and $\alpha_j = m$. For a fixed $\overline{\alpha}$, the total contribution to the left-hand side of (5.1) is thus

$$q^{\mathrm{coinv}(\overline{\alpha})} \begin{bmatrix} v'_{m-1} - \lambda'_m \\ v'_m - \lambda'_m \end{bmatrix}_q = q^{\mathrm{coinv}(\overline{\alpha})} \begin{bmatrix} v'_{m-1} - \lambda'_m \\ v'_{m-1} - v'_m \end{bmatrix}_q,$$

which is $q^{\mathrm{coinv}(\overline{\alpha})}$ times the $i = m - 1$ factor on the right-hand side of (5.1). An application of the inductive hypothesis then completes the proof. $\square$

Theorem 5.2. *We have*

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s}; \mathbb{Q}); q) = \sum_{\substack{v \in \mathrm{Par}(n,s), \\ v \supseteq \lambda}} q^{\sum_i (s - v'_i)(v'_{i+1} - \lambda'_{i+1})} \prod_{i \geq 0} \begin{bmatrix} v'_i - \lambda'_{i+1} \\ v'_i - v'_{i+1} \end{bmatrix}_q \widetilde{H}_v(\mathbf{x}; q). \tag{5.2}$$

Proof. Combining (1.1), Theorem 3.8, Theorem 4.2 and Lemma 5.1, we have for all prime powers q the following string of identities.

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s}; \mathbb{Q}); q) \tag{5.3}$$

$$= \sum_{\mu \vdash n} |Y^\mu_{n,\lambda,s}(\mathbb{F}_q)| m_\mu(\mathbf{x}) \tag{5.4}$$

$$= \sum_{\mu \vdash n} \sum_{\substack{\alpha \in \mathrm{Comp}(n,s), \\ \alpha \supseteq \lambda}} |\widehat{Z}^\mu_\alpha(\mathbb{F}_q)| m_\mu(\mathbf{x}) \tag{5.5}$$

$$= \sum_{\mu \vdash n} \sum_{\substack{\alpha \in \mathrm{Comp}(n,s), \\ \alpha \supseteq \lambda}} q^{\sum_i (s - \alpha'_i)(\alpha'_{i+1} - \lambda'_{i+1}) + \mathrm{coinv}(\alpha)} |Z^\mu_\alpha(\mathbb{F}_q)| m_\mu(\mathbf{x}) \tag{5.6}$$

$$= \sum_{\substack{\alpha \in \mathrm{Comp}(n,s), \\ \alpha \supseteq \lambda}} q^{\sum_i (s - \alpha'_i)(\alpha'_{i+1} - \lambda'_{i+1}) + \mathrm{coinv}(\alpha)} \left(\sum_{\mu \vdash n} |Z^\mu_\alpha(\mathbb{F}_q)| m_\mu(\mathbf{x}) \right) \tag{5.7}$$

$$= \sum_{\substack{\alpha \in \mathrm{Comp}(n,s), \\ \alpha \supseteq \lambda}} q^{\sum_i (s - \alpha'_i)(\alpha'_{i+1} - \lambda'_{i+1}) + \mathrm{coinv}(\alpha)} \widetilde{H}_{\mathrm{sort}(\alpha)}(\mathbf{x}; q) \tag{5.8}$$

$$= \sum_{\substack{v \in \mathrm{Par}(n,s), \\ v \supseteq \lambda}} q^{\sum_i (s - v'_i)(v'_{i+1} - \lambda'_{i+1})} \sum_{\substack{\alpha \in \mathrm{Comp}(n,s), \\ \mathrm{sort}(\alpha) = v, \\ \alpha \supseteq \lambda}} q^{\mathrm{coinv}(\alpha)} \widetilde{H}_v(\mathbf{x}; q) \tag{5.9}$$

$$= \sum_{\substack{v \in \mathrm{Par}(n,s), \\ v \supseteq \lambda}} q^{\sum_i (s - v'_i)(v'_{i+1} - \lambda'_{i+1})} \prod_{i \geq 0} \begin{bmatrix} v'_i - \lambda'_{i+1} \\ v'_i - v'_{i+1} \end{bmatrix}_q \widetilde{H}_v(\mathbf{x}; q). \tag{5.10}$$

In (5.9), we have used the fact that if $\mathrm{sort}(\alpha) = v$, then $\alpha'_i = v'_i$ for all i. Since the identities hold for infinitely many values of q, the final identity holds for q a formal parameter. $\square$

Corollary 5.3. *We have*

$$\mathrm{Frob}(H^*(Y_{n,\lambda,s};\mathbb{Q});q) = \mathrm{rev}_q\left[\sum_{\substack{\nu\in\mathrm{Par}(n,s),\\ \nu\supseteq\lambda}} q^{n(\nu/\lambda)}\prod_{i\geq 0}\begin{bmatrix}\nu_i' - \lambda_{i+1}'\\ \nu_i' - \nu_{i+1}'\end{bmatrix}_q H_\nu(\mathbf{x};q)\right], \tag{5.11}$$

where $\nu_0' := s$, $\lambda_0' := s$, *and* $n(\nu/\lambda) := \sum_i \binom{\nu_i'-\lambda_i'}{2}$.

Proof. By [10, Theorem 1.2], the degree of $\mathrm{Frob}(H^*(Y_{n,\lambda,s};\mathbb{Q});q)$ is $n(\lambda)+(s-1)(n-k)$. By Theorem 5.2, it suffices to show that rev_q applied to the right-hand side of (5.2) is equal to the expression in the right-hand side of (5.11) inside of rev_q. In other words, it suffices to prove that

$$q^{n(\lambda)+(s-1)(n-k)}\sum_{\substack{\nu\in\mathrm{Par}(n,s),\\ \nu\supseteq\lambda}} q^{-\sum_i(s-\nu_i')(\nu_{i+1}'-\lambda_{i+1}')}\prod_{i\geq 0}\begin{bmatrix}\nu_i' - \lambda_{i+1}'\\ \nu_i' - \nu_{i+1}'\end{bmatrix}_{q^{-1}}\widetilde{H}_\nu(\mathbf{x};q^{-1}) = \tag{5.12}$$

$$\sum_{\substack{\nu\in\mathrm{Par}(n,s),\\ \nu\supseteq\lambda}} q^{n(\nu/\lambda)}\prod_{i\geq 0}\begin{bmatrix}\nu_i' - \lambda_{i+1}'\\ \nu_i' - \nu_{i+1}'\end{bmatrix}_q H_\nu(\mathbf{x};q). \tag{5.13}$$

Since $\widetilde{H}_\nu(\mathbf{x};q^{-1}) = q^{-n(\nu)}H_\nu(\mathbf{x};q)$ and $\begin{bmatrix}a\\b\end{bmatrix}_{q^{-1}} = q^{-b(a-b)}\begin{bmatrix}a\\b\end{bmatrix}_q$, we are reduced to proving that

$$n(\nu/\lambda) = n(\lambda) + (s-1)(n-k) - \sum_i(s-\nu_i')(\nu_{i+1}'-\lambda_{i+1}') - \sum_i(\nu_i'-\nu_{i+1}')(\nu_{i+1}'-\lambda_{i+1}') - n(\nu), \tag{5.14}$$

which follows by a straightforward calculation using the facts that $\sum_i \nu_i' = n$ and $\sum_i \lambda_i' = k$. Indeed, combining terms in the right-hand side of (5.14), we have

$$\frac{1}{2}\left(\sum_i \lambda_i'(\lambda_i'-1) - 2(s-1)(n-k) - \sum_i 2(s-\nu_{i+1}')(\nu_{i+1}'-\lambda_{i+1}') - \sum_i \nu_i'(\nu_i'-1)\right)$$

$$= \frac{1}{2}\left(-2(s-1)(n-k) + \sum_i\left(2s(\nu_{i+1}'-\lambda_{i+1}') + \nu_i' - \lambda_i' + (\lambda_i')^2 + (\nu_i')^2 - 2\nu_{i+1}'\lambda_{i+1}'\right)\right)$$

$$= \frac{1}{2}\left(-2(s-1)(n-k) + 2sn - 2sk + 2\sum_i(\nu_i'-\lambda_i') + \sum_i\left((\lambda_i')^2 + (\nu_i')^2 - 2\nu_i'\lambda_i' - \nu_i' - \lambda_i'\right)\right)$$

$$= \frac{1}{2}\left(0 + \sum_i(\nu_i'-\lambda_i')(\nu_i'-\lambda_i'-1)\right) = n(\nu/\lambda).$$

The proof is thus complete. $\qquad\qquad\square$

6. Further directions

We list a few open problems and possible further connections with geometry and combinatorics:

○ Counting partial flags also has connections with the q-Burge correspondence by work of Karp and Thomas [16]. Is there a generalization of the q-Burge correspondence that corresponds to counting pairs of points of the Δ-Springer varieties?

○ The proof of Theorem 1.3 given in this article is geometric, relying on counting the points of the Δ-Springer variety over $\mathbb{F}_q$ using different ways to partition the space. A more direct combinatorial proof should be possible starting from Theorem 1.1. Such a combinatorial proof would find a

weight-preserving bijection that collects terms of this formula into sums that can be identified as monomial expansions of Hall-Littlewood symmetric functions.

○ Formulas for Macdonald polynomials, which are two-parameter generalizations of Hall-Littlewood symmetric functions, have been obtained by Mellit [18] as weighted sums over points of affine Springer fibers. It may be possible to define generalizations of Δ-Springer varieties in the setting of affine flag varieties. One can then ask whether weighted point counts over these varieties yield symmetric functions that can simultaneously be defined using Macdonald eigenoperators. In particular, since $\Delta'_{e_{k-1}} e_n|_{t=0}$ is, up to a minor twist, the graded Frobenius characteristic of the cohomology of $Y_{n,(1^k),k}$, one might hope that extending the definition of $Y_{n,(1^k),k}$ to the affine setting and taking a weighted point count can recover the full symmetric function $\Delta'_{e_{k-1}} e_n$.

Competing interest. The authors have no competing interest to declare.

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