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A differentiation in locally convex spaces

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The theory of Γ-finite linear operators developed by Robert T. Moore is used to construct a differential calculus in locally convex spaces. This note contains the fundamental theory up to the implicit function theorem.

This is the first part of a series of notes in which we shall construct a differential calculus in locally convex Hausdorff spaces. The aim is to show that it is possible to generalize the Banach space calculus to locally convex spaces without losing its simplicity and power.

As we have explained in [5, Introduction], there have been several difficulties in constructing such a calculus. Some definitions did not imply continuity. For the definitions which imply continuity, the chain rules of higher order did not hold. As we have shown in [6], [7], and [8], the differentiability of the inverse map always required a complicated treatment, and, above all, it has been impossible to generalize the inverse mapping theorem or the implicit function theorem in Banach spaces with their simple forms retained.

All these difficulties are due to the fact that the derivatives have merely been assumed to be continuous and linear. Unlike the case of Banach spaces, the set $L(E)$ of all continuous linear maps on a locally convex space E with any one of the usual topologies is not suitable for constructing the calculus. The difficulty about the chain rules is due to the fact that the composition in $L(E)$ is not continuous unless E is normable. (See [5, Appendix 2].) The difficulty about the inverse maps

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is due to the fact that the inverse operation in the set of all invertible maps in $L(E)$ is not continuous. In order to construct a simple and effective calculus on E, we need a theory of linear maps on E which is free from these difficulties.

As far as we know, there are three candidates. The first is the theory developed by Marinescu in [2] and other papers, where the main idea is to assume the existence of a relation between the sets of continuous semi-norms on the spaces E and F depending on the given map $u : E \to F$. Using this idea, he has obtained a form of the implicit function theorem. The second is the theory of completely bounded maps developed in [1], which was used to prove an inverse mapping theorem in [5]. The third is the theory of Γ-finite maps by Moore developed in [3] and other papers. Although the method used in the definition of the Γ-finiteness is a special case of Marinescu's idea, it provides us with a reasonably simple and remarkably versatile tool. From the viewpoint that the calculus is essentially a tool, it is desirable that it stands on a simple theory of linear maps.

Our differentiation is based mainly on Moore's theory. We shall have to make only one change, because Moore has considered only self-maps, whereas we need the Γ-finite maps from one space into another in order to define, for example, the higher differentiability.

In the following, we shall always assume that E, F, and G denote locally convex Hausdorff spaces over the real number field R. The real numbers will be denoted by Greek letters. X and Y will always stand for open subsets in E and F respectively.

1. Calibrations

A *calibration* for E is a set of continuous semi-norms on E which induces the topology of E. The set $P(E)$ of all continuous semi-norms on E is obviously a calibration for E. If Γ is a calibration for E, then, by the definition, for any $p \in P(E)$, there exist $p_i \in \Gamma$ $(1 \le i \le n)$ such that

$$p(x) \le \max\left(p_1(x), \ldots, p_n(x)\right) \quad \text{for all } x \in E .$$

The basic idea of the theory developed in this note is to choose a suitable

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calibration depending on the map under consideration.

Let Γ be a calibration for the product space $E \times F$. For $p \in \Gamma$, we put

$$p_E(x) = p(x, 0) \quad \text{and} \quad p_F(y) = p(0, y) ,$$

which will be called the *E-component* and the *F-component of* p respectively. We also put

$$\Gamma_E = \{p_E : p \in \Gamma\} \quad \text{and} \quad \Gamma_F = \{p_F : p \in \Gamma\} ,$$

which will be called the *E-component* and the *F-component of* Γ respectively. It is obvious that Γ_E is a calibration for E and Γ_F is a calibration for F. Moreover, each $p_1 \in \Gamma_E$ is related to some $p_2 \in \Gamma_F$ by the fact that there exists $p \in \Gamma$ such that $p_1 = p_E$ and $p_2 = p_F$.

Conversely, suppose that we have calibrations Γ_1 and Γ_2 for E and F respectively. Furthermore, suppose that there is a relation ρ in $\Gamma_1 \times \Gamma_2$ such that its domain is Γ_1 and its range is Γ_2. If $(p_1, p_2) \in \Gamma_1 \times \Gamma_2$ is ρ-related, we define a continuous semi-norm $[p_1, p_2]$ on $E \times F$ by

$$[p_1, p_2](x, y) = p_1(x) + p_2(y) \quad \text{for} \quad (x, y) \in E \times F .$$

Then the set

$$\Gamma = \{[p_1, p_2] : (p_1, p_2) \in \Gamma_1 \times \Gamma_2 \text{ and } \rho\text{-related}\}$$

is a calibration for $E \times F$ such that $\Gamma_1 = \Gamma_E$ and $\Gamma_2 = \Gamma_F$.

It helps to simplify the calculation if the relation between p and (p_E, p_F) is clearly indicated. We shall say that Γ *is a calibration for* (E, F) if Γ is a calibration for $E \times F$ and

$$p(x, y) = p_E(x) + p_F(y) \quad \text{if} \quad p \in \Gamma \quad \text{and} \quad (x, y) \in E \times F .$$

This convention is made only for the sake of convenience. Instead of taking the sum, we may take $\max\{p_E(x), p_F(y)\}$ without causing any change

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in the statement of theorems.

We shall denote by $\Gamma = (\Gamma_1, \Gamma_2)$ the fact that Γ is a calibration for (E, F) and its components are Γ_1 and Γ_2.

Let Γ_1 and Γ_2 be calibrations for (E, F) and (F, G) respectively. For $p \in \Gamma_1$ and $q \in \Gamma_2$, define a continuous semi-norm $p \circ q$ on $E \times G$ by

$$(p \circ q)(x, z) = p_E(x) + q_G(z) \quad \text{for} \quad (x, z) \in E \times G .$$

We put

$$\Gamma_1 \circ \Gamma_2 = \{p \circ q : (p, q) \in \Gamma_1 \times \Gamma_2 \text{ and } p_F = q_F\} ,$$

and, if $\Gamma_1 \circ \Gamma_2$ is a calibration for (E, G), we shall say that Γ_1 and Γ_2 are *composable* and $\Gamma_1 \circ \Gamma_2$ will be called the *composition of* Γ_1 and Γ_2. It is obvious that, if Γ_1 and Γ_2 are composable and $p \circ q \in \Gamma_1 \circ \Gamma_2$, then

$$(p \circ q)_E = p_E \quad \text{and} \quad (p \circ q)_G = q_G .$$

Finally, we set up two rules.

(1). *When $E = F$, then any calibration Γ for (E, F) shall always satisfy the following condition: $p_E = p_F$ for every $p \in \Gamma$. In this case we denote its components by the same symbol Γ.*

(2). *When F is a normed space, we shall always assume that Γ_F consists of the single element that is the norm of F. In this case, again, we denote Γ_E by the same symbol Γ. The same rule applies to the case when E is a normed space.*

The second rule implies, in particular, that all calibrations for (E, F) and (F, G) are composable if F is a normed space.

2. Γ-limits

Let Γ be a calibration for (E, F), $f : X \to F$, and $a \in X$. If, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

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$$p_F\big(f(x)-b\big) < \varepsilon \quad \text{if} \quad x \in X \ , \quad p_E(x-a) < \delta \ , \ \text{and} \quad p \in \Gamma \ ,$$

then, we say that b is the Γ-*limit of* f *as* $x \to a$ and denote this fact by

$$\Gamma - \lim_{x\to a} f(x) = b \ .$$

Obviously, the Γ-limit is unique if it exists.

(2.1). *Let* $f_i : X \to F$ $(i = 1, 2)$. *If*

$$\Gamma - \lim_{x\to a} f_i(x) = b_i \quad (i = 1, 2) \ ,$$

then

$$\Gamma - \lim_{x\to a} \big(\alpha f_1+\beta f_2\big)(x) = \alpha b_1 + \beta b_2 \ .$$

(2.2). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *Let* $f : X \to F$ *and* $g : Y \to G$, *where* $f(X) \subset Y$. *Then, if*

$$\Gamma_1 - \lim_{x\to a} f(x) = b \quad \text{and} \quad \Gamma_2 - \lim_{y\to b} g(y) = c \ ,$$

then

$$\Gamma - \lim_{x\to a} (g \circ f)(x) = c \ .$$

3. The Γ-continuity

Let Γ be a calibration for (E, F) .

A map $f : X \to F$ is said to be Γ-*continuous at* $a \in X$ if

$$\Gamma - \lim_{x\to a} f(x) = f(a) \ .$$

Obviously, the Γ-continuous maps are continuous.

(3.1). *A linear map* $u : E \to F$ *is* Γ-*continuous at a point if and only if there exists* $\alpha > 0$ *such that*

$$p_F\big(u(x)\big) \le \alpha p_E(x) \quad \text{if} \quad x \in E \quad \text{and} \quad p \in \Gamma \ .$$

Proof. If u is Γ-continuous at $a \in E$, there exists $\delta > 0$ such that

$$p_F\big(u(a+x)-u(a)\big) < 1 \quad \text{if} \quad p_E(x) < \delta \quad \text{and} \quad p \in \Gamma ,$$

or

$$p_F\big(u(x)\big) < 1 \quad \text{if} \quad p_E(x) < \delta \quad \text{and} \quad p \in \Gamma .$$

Then, for $\alpha = (\frac{1}{2}\delta)^{-1}$, we have the required inequality.

The converse is obvious.

This fact shows that the Γ-continuity coincides, for linear maps, with the Γ-finiteness introduced by [3], where the set of Γ-finite linear maps of E into E was denoted by $F_\Gamma(E)$. We shall use the same notation: $F_\Gamma(E, F)$ will denote the set of all Γ-continuous linear maps of E into F .

By (2.1), $F_\Gamma(E, F)$ is a normed space with the norm:

$$\|u\|_\Gamma = \sup\{p_F\big(u(x)\big) : p_E(x) \le 1 \quad \text{and} \quad p \in \Gamma\} .$$

Therefore,

$$p_F\big(u(x)\big) \le \|u\|_\Gamma p_E(x) \quad \text{if} \quad u \in F_\Gamma(E, F) \quad \text{and} \quad p \in \Gamma .$$

The following fact was observed by [3] in the case when $E = F$.

(3.2). *If* F *is sequentially complete,* $F_\Gamma(E, F)$ *is a Banach space.*

The following fact was also observed by [3] in the case when $E = F$. We shall add a proof to show how to choose a calibration which is suitable to the given map.

(3.3). *If* $u : E \to F$ *is a completely bounded linear map, then, there exists a calibration* Γ *for* (E, F) *such that* $u \in F_\Gamma(E, F)$.

Proof. A linear map $u : E \to F$ is completely bounded if and only if there exists an absolutely convex neighbourhood U of zero in E such that $u(U)$ is a bounded subset of F . Let p_0 be the continuous semi-norm corresponding to U . Then, for any $q \in P(F)$, there exists $\lambda_q > 0$ such that

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$$q(u(x)) \le \lambda_q p_0(x) \quad \text{for all} \quad x \in E .$$

Let Γ be the set of all continuous semi-norms on $E \times F$ which are in the following form: $p(x) + q(y)$, where $p \ge \lambda_q p_0$.

(3.4). *Let* $E = E_1 \times E_2$ *and* $u : E_1 \times E_2 \to F$ *be a bilinear map. Let* Γ *be a calibration for* (E, F) *such that* $\Gamma_E = (\Gamma_{E_1}, \Gamma_{E_2})$. *Then* u *is* Γ*-continuous at* $(0, 0)$ *if and only if there exists* $\alpha > 0$ *such that*

$$p_F(u(x_1, x_2)) \le \alpha p_{E_1}(x_1) p_{E_2}(x_2) \quad \textit{for all} \quad (x_1, x_2) \in E_1 \times E_2 .$$

Proof. Assume that u is Γ-continuous at $(0, 0)$. Then, there exists $\delta > 0$ such that

$$p_F(u(x_1, x_2)) < 1 \quad \text{if} \quad p_E(x_1, x_2) < \delta \quad \text{and} \quad p \in \Gamma .$$

We prove that the inequality holds for $\alpha = (\frac{1}{3}\delta)^2$.

If $p_{E_1}(x_1) = 0$, then, for every $x_2 \in E_2$, there exists $\beta > 0$ such that

$$p_E(\xi x_1, \beta x_2) < \delta \quad \text{for all} \quad \xi > 0 .$$

Hence, $p_F(u(\xi x_1, \beta x_2)) < 1$ for all $\xi > 0$, which implies that $p_F(u(x_1, x_2)) = 0$.

In the same way, we see that $p_{E_2}(x_2) = 0$ implies $p_F(u(x_1, x_2)) = 0$.

Finally, assume that $\lambda_1^{-1} = p_{E_1}(x_1) \ne 0$ and $\lambda_2^{-1} = p_{E_2}(x_2) \ne 0$. Then,

$$p_E(\tfrac{1}{3}\delta\lambda_1 x_1, \tfrac{1}{3}\delta\lambda_2 x_2) < \delta ,$$

from which it follows that

$$p_F(u(x_1, x_2)) < (\tfrac{1}{3}\delta)^2 p_{E_1}(x_1) p_{E_2}(x_2) .$$

Conversely, assume that the inequality holds and $\varepsilon > 0$. Then, if $p(x_1, x_2) < \delta$, where $\delta < \min\left(1, \frac{2}{\alpha}\varepsilon\right)$, then

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$$\alpha p_{E_1}(x_1)p_{E_2}(x_2) \le \frac{\alpha}{2}\left\{p_{E_1}(x_1)^2+p_{E_2}(x_2)^2\right\} \le \frac{\alpha}{2}\left(p_{E_1}(x_1)+p_{E_2}(x_2)\right)$$
$$= \frac{\alpha}{2} p_E(x_1, x_2) < \varepsilon .$$

Hence, u is Γ-continuous at $(0, 0)$.

A calibration Γ for (E, F) determines the calibration $(\Gamma_E, \|\cdot\|_\Gamma)$ for $(E, F_\Gamma(E, F))$. For the sake of convenience, we shall denote this calibration by the same symbol Γ. Hence, $u \in F_\Gamma(E, F_\Gamma(E, F))$ means that there exists $\alpha > 0$ such that

$$\|u(x)\|_\Gamma \le \alpha p_E(x) \text{ for all } x \in E \text{ and } p \in \Gamma .$$

(3.5). $u \in F_\Gamma(E, F_\Gamma(E, F))$ *if and only if there exists* $\alpha > 0$ *such that*

$$p_F(u(x)(y)) \le \alpha p_E(x)p_E(y) \text{ if } x, y \in E \text{ and } p \in \Gamma .$$

Proof. If $u \in F_\Gamma(E, F_\Gamma(E, F))$, then, since $u(x) \in F_\Gamma(E, F)$,

$$p_F(u(x)(y)) \le \|u(x)\|_\Gamma p_E(y) \text{ if } p \in \Gamma ,$$

from which it follows that

$$p_F(u(x)(y)) \le \|u\|_{\Gamma_0} p_E(x)p_E(y) ,$$

where Γ_0 is the calibration $(\Gamma_E, \|\cdot\|_\Gamma)$ for $(E, F_\Gamma(E, F))$.

The converse is obvious.

This fact has an abvious extension to the case where the number of E is $k \ge 2$. Let us denote the set $F_\Gamma(E, \ldots, E, F_\Gamma(E, F), \ldots)$ by $F_\Gamma(E^k, F)$. When $u \in F_\Gamma(E^k, F)$, $u(x_1)$ is an element of $F_\Gamma(E^{k-1}, F)$ and $u(x_1)(x_2)$ is an element of $F_\Gamma(E^{k-2}, F)$. We shall denote $u(x_1)(x_2) \ldots (x_k)$ by $u(x_1, \ldots, x_k)$. Then, we can easily show that $u \in F_\Gamma(E^k, F)$ if and only if there exists $\alpha > 0$ such that

$$p_F(u(x_1, \ldots, x_k)) \le \alpha p_E(x_1) \ldots p_E(x_k) \text{ if } p \in \Gamma .$$

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For $u \in \mathcal{F}_\Gamma(E^k, F)$, we put

$$\|u\|_\Gamma = \sup\{p_F(u(x_1, \ldots, x_k)) : p_E(x_i) \le 1 \text{ and } p \in \Gamma\} .$$

(3.6). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *Then, if* $u \in \mathcal{F}_{\Gamma_1}(E, F)$ *and* $v \in \mathcal{F}_{\Gamma_2}(F, G)$, *then* $v \circ u \in \mathcal{F}_\Gamma(E, G)$ *and* $\|v\circ u\|_\Gamma \le \|u\|_{\Gamma_1}\|v\|_{\Gamma_2}$.

From this and (3.2), we have a fact, observed by [3], that, if E is sequentially complete, $\mathcal{F}_\Gamma(E, E)$ is a Banach algebra with the unit. Hence, if $u \in \mathcal{F}_\Gamma(E, E)$ and $\|u\|_\Gamma < 1$, then $1 - u$ has the Γ-continuous inverse which is expressed as the series of C. Neumann. Later, we shall need one of its consequences in the following form. We denote by $G_\Gamma(E, F)$ the set of all Γ-isomorphisms of E onto F; that is, the set of all Γ-continuous linear isomorphisms whose inverses are Γ^{-1}-continuous, where $\Gamma^{-1} = (\Gamma_F, \Gamma_E)$.

(3.7). *Let* F *be sequentially complete. If* $u \in G_\Gamma(E, F)$, $v \in \mathcal{F}_\Gamma(E, F)$, *and* $\|v\|_\Gamma < \frac{1}{2}\|u^{-1}\|^{-1}_{\Gamma^{-1}}$, *then*

(1) $u + v \in G_\Gamma(E, F)$,

(2) $\|(u+v)^{-1}-u^{-1}\|_{\Gamma^{-1}} \le 2\|u^{-1}\|^2_{\Gamma^{-1}}\|v\|_\Gamma$,

(3) $\|(u+v)^{-1}-u^{-1}+u^{-1}\circ v\circ u^{-1}\|_{\Gamma^{-1}} \le 2\|u^{-1}\|^3_{\Gamma^{-1}}\|v\|^2_\Gamma$.

An immediate consequence of (3.7) is the following fact.

(3.8). *Suppose that* F *is sequentially complete and* $u : X \to G_\Gamma(E, F)$. *Then,* $\Gamma - \lim_{x\to a} u(x) = u(a)$ *implies*

$$\Gamma^{-1} - \lim_{x\to a} u(x)^{-1} = u(a)^{-1} .$$

Another consequence of (3.6) is the Γ-continuity of the composition map.

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(3.9). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *Then, for the maps*

$$u : X \to F_{\Gamma_1}(E, F) \quad \text{and} \quad v : X \to F_{\Gamma_2}(F, G) ,$$

if

$$\Gamma_1 - \lim_{x \to a} u(x) = u_0 \quad \text{and} \quad \Gamma_2 - \lim_{x \to a} v(x) = v_0 ,$$

then

$$\Gamma - \lim_{x \to a} v(x) \circ u(x) = v_0 \circ u_0 .$$

The projection is always Γ-continuous for some Γ .

(3.10). *Let* $\pi : E \times F \to E$ *be the projection. Then, for any calibration* Γ *for* (E, F) , $\pi \in F_{\Gamma_0}(E \times F, E)$ *for* $\Gamma_0 = (\Gamma, \Gamma_E)$.

From (3.9) and (3.1) we have the following fact.

(3.11). *Suppose that* $E = E_1 \times E_2$ *and* Γ_E *is a calibration for* (E_1, E_2) . *Let* $u : X \to F_{\Gamma_1}(E_1, F)$ *for* $\Gamma_1 = (\Gamma_{E_1}, \Gamma_F)$ *and* $\pi : E \to E_1$ *be the projection. Then*

$$\Gamma - \lim_{x \to a} u(x) \circ \pi = u_0 \circ \pi$$

if and only if

$$\Gamma_1 - \lim_{\pi(x) \to \pi(a)} u(x) = u_0 .$$

4. The Γ-differentiability

Let Γ be a calibration for (E, F) .

A map $f : X \to F$ is said to be Γ-*differentiable at* $a \in X$ if there exists $u \in F_\Gamma(E, F)$ such that the following condition is satisfied: *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$p_F\big(f(a+x)-f(a)-u(x)\big) < \varepsilon p_E(x) \quad \text{if} \quad a + x \in X , \quad p_E(x) < \delta , \text{ and } \quad p \in \Gamma .$$

It is easy to see that such u is unique if it exists; we denote it by $f'(a)$ and call it the Γ-*derivative of* f *at* a .

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We shall use the following notations:

$$r_u(f, a, x) = f(a+x) - f(a) - u(x) ,$$

and

$$r(f, a, x) = f(a+x) - f(a) - f'(a)(x) .$$

It is obvious that, if f and g are Γ-differentiable at a, then $f + g$ and αf are Γ-differentiable at a. Hence, the set $\mathcal{D}_\Gamma(X, F)$ of all maps of X into F which are Γ-differentiable at every point of X is a linear space.

In [5], the following definition was given: a map $f : X \to F$ is said to be *Fréchet differentiable at* $a \in X$ if there exists a continuous linear map $u : E \to F$ such that

$$\varepsilon^{-1} r_u(f, a, \varepsilon x) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

uniformly on each bounded set; that is,

$$\lim_{\varepsilon \to 0} \sup_{x \in B} p\left(\varepsilon^{-1} r_u(f, a, \varepsilon x)\right) = 0$$

for any bounded subset B and for any $p \in P(F)$. The properties of this differentiation have been investigated in [5] in detail. The following fact makes it possible to use those results in [5].

(4.1). *If* $f : X \to F$ *is* Γ*-differentiable at* $a \in X$ *, it is Fréchet differentiable at* a *with the same derivative.*

Proof. We need to show that, if $\varepsilon_n \to 0$ and $\{x_n\}$ is a bounded sequence, then

$$\lim_{n \to \infty} p_F\left(\varepsilon_n^{-1} r(f, a, \varepsilon_n x_n)\right) = 0 \quad \text{for each} \quad p \in \Gamma .$$

Let $\varepsilon > 0$, and take $\delta > 0$ in the definition of the Γ-differentiability. Then, for each $p \in \Gamma$, since $\varepsilon_n x_n \to 0$, there exists n_0 such that $a + \varepsilon_n x_n \in X$ and $p_E(\varepsilon_n x_n) < \delta$. Hence,

$$p_F\left(\varepsilon_n^{-1} r(f, a, \varepsilon_n x_n)\right) < \varepsilon p_E(x_n) ,$$

which ends the proof.

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Although the Fréchet differentiability did not imply continuity, the Γ-differentiability does.

(4.2). *If* $f : X \to F$ *is* Γ*-differentiable at* $a \in X$ *, then, for any* $\varepsilon > 0$ *, there exists* $\delta > 0$ *such that*

$$p_F\big(f(a+x)-f(a)\big) \le \big(\|f'(a)\|_\Gamma+\varepsilon\big)p_E(x)$$

if $a + x \in X$ *and* $p_E(x) < \delta$ *. Hence,* f *is* Γ*-continuous at* a *.*

The proof is obvious.

(4.3). *If* $u \in F_\Gamma(E, F)$ *, then* $u \in \mathcal{D}_\Gamma(E, F)$ *and* $u'(x) = u$ *for every* $x \in E$ *.*

(4.4). *Let* $E = E_1 \times E_2$ *and* Γ *be a calibration for* (E, F) *such that* Γ_E *is a calibration for* $\big(E_1, E_2\big)$ *. Then every bilinear map* $u : E_1 \times E_2 \to F$ *which is* Γ*-continuous at* $(0, 0)$ *is* Γ*-differentiable at every point, and*

$$u'(a, b)(x, y) = u(a, y) + u(x, b) .$$

Proof. First, we observe that the linear map $v : E \to F$ defined by

$$v(x, y) = u(a, y) + u(x, b)$$

is Γ-continuous at $(0, 0)$, because, by (3.5), we can take $\alpha > 0$ such that

$$p_F\big(u(x, y)\big) \le \alpha p_{E_1}(x)p_{E_2}(y) \quad \text{if} \quad p \in \Gamma ,$$

so that

$$\begin{aligned} p_F\big(v(x, y)\big) &\le p_F\big(u(a, y)\big) + p_F\big(u(x, b)\big) \\ &\le \alpha p_{E_1}(a)p_{E_2}(y) + \alpha p_{E_1}(x)p_{E_2}(b) \\ &\le \alpha p_E(a, b)p_E(x, y) . \end{aligned}$$

Now, let $\varepsilon > 0$, and take $\delta > 0$ such that $\delta < \varepsilon/\alpha$. Then, if $p \in \Gamma$ and $p_E(x, y) < \delta$,

$$\begin{aligned} p_F\big(r_v\big(u, (a, b), (x, y)\big)\big) &= p_F\big(u(x, y)\big) \le \alpha p_{E_1}(x)p_{E_2}(y) \\ &\le \alpha p_E(x, y)^2 < \alpha\delta p_E(x, y) < \varepsilon p_E(x, y) , \end{aligned}$$

which ends the proof.

(4.5). *Assume that* $F = F_1 \times F_2$ *and* Γ_F *is a calibration for* (F_1, F_2) . *Then a map* $f : X \to F$ *, defined by*

$$f(x) = \left(f_1(x), f_2(x)\right) ,$$

is Γ*-differentiable at* $a \in X$ *if and only if* f_i *is* Γ_i*-differentiable at* a *for each* i *, where* $\Gamma_i = \left\{\Gamma_E, \Gamma_{F_i}\right\}$. *If this is the case,*

$$f'(a)(x) = \left(f_1'(a)(x), f_2'(a)(x)\right) .$$

Proof. Let $\pi_i : F \to F_i$ be the projections. Then, $f_i = \pi_i \circ f$. If f is Γ-differentiable at a , then, by (3.10), $u_i = \pi_i \circ f'(a) \in \mathcal{F}_{\Gamma_i}\left(E, F_i\right)$, and u_i is the Γ_i-derivative of f_i at a for each i , because

$$(*) \qquad p_F\left(r(f, a, x)\right) = \sum_{i=1}^{2} p_{F_i}\left(r_{u_i}\left(f_i, a, x\right)\right) .$$

Conversely, if f_i is Γ_i-differentiable at a for each i , define $u : E \to F$ by

$$u(x) = \left(f_1'(a)(x), f_2'(a)(x)\right) .$$

Then, $u \in \mathcal{F}_\Gamma(E, F)$ and it is the Γ-derivative of f at a because of a similar equality as (*).

Next we prove the first order chain rule.

(4.6). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *If* $f : X \to F$ *is* Γ_1*-differentiable at* $a \in X$ *and* $g : Y \to G$ *is* Γ_2*-differentiable at* $b = f(a)$ *, where* $f(X) \subset Y$ *, then* $g \circ f$ *is* Γ*-differentiable at* a *and* $(g \circ f)'(a) = g'(b) \circ f'(a)$.

Proof. We put $u = g'(b) \circ f'(a)$; then, by (3.6), $u \in \mathcal{F}_\Gamma(E, G)$, and

$$r_u(g\circ f,\ a,\ x) = g'(b)\big(r(f,\ a,\ x)\big) + r\big(g,\ b,\ f(a+x)-f(a)\big) .$$

Now let $\varepsilon > 0$, and take $\delta_1 > 0$ such that

$$p_F\big(r(f,\ a,\ x)\big) < \varepsilon p_E(x) \quad \text{if} \quad a + x \in X\ ,\quad p_E(x) < \delta_1\ , \text{ and } \ p \in \Gamma_1\ ,$$

and

$$q_G\big(r(g,\ b,\ y)\big) < \varepsilon q_F(y) \quad \text{if} \quad b + y \in Y\ ,\quad q_F(y) < \delta_1\ , \text{ and } \ q \in \Gamma_2\ .$$

By (4.2), there exists $\delta > 0$ such that $\big(\|f'(a)\|_{\Gamma_1}+\varepsilon\big)\delta < \delta_1$ and

$$p_F\big(f(a+x)-f(a)\big) < \delta_1 \quad \text{if} \quad a + x \in X\ ,\quad p_E(x) < \delta\ , \text{ and } \ p \in \Gamma_1\ .$$

Hence, for every $p \circ q \in \Gamma$, if $p_E(x) < \delta$ and $a + x \in X$, we have

$$\begin{aligned} q_G\big(r_u(g\circ f,\ a,\ x)\big) &\le q_G\big(g'(b)\big(r(f,\ a,\ x)\big)\big) + q_G\big(r\big(g,\ b,\ f(a+x)-f(a)\big)\big) \\ &\le \|g'(b)\|_{\Gamma_2} p_F\big(r(f,\ a,\ x)\big) + \varepsilon p_F\big(f(a+x)-f(a)\big) \\ &\le \varepsilon\big(\|g'(b)\|_{\Gamma_2}+\|f'(a)\|_{\Gamma_1}+\varepsilon\big)p_E(x)\ , \end{aligned}$$

which shows that $g \circ f$ is Γ-differentiable at a and $(g\circ f)'(a) = u$.

As we have shown in [*6*], [*7*], and [*8*], most of the existing differentiabilities behave very badly when the differentiability of the inverse map is involved. For the Γ-differentiability, we do not have such difficulty.

(4.7). *Let* $f : X \to F$ *be a bijection onto an open set* $f(X)$, Γ-*differentiable at* $a \in X$, *and* $f'(a)$ *be a* Γ-*isomorphism. Then the inverse map* g *of* f *is* Γ^{-1}-*differentiable at* $b = f(a)$ *if and only if* g *is* Γ^{-1}-*continuous at* b . *If this is the case,*

$$g'(b) = f'(a)^{-1} .$$

Proof. Assume that g is Γ^{-1}-continuous at b , and let $\varepsilon > 0$. We can assume that $\varepsilon < \|f'(a)^{-1}\|^{-1}_{\Gamma^{-1}}$. Then, there exists $\delta_1 > 0$ such that

$$p_F\big(r(f,\ a,\ x)\big) < \varepsilon p_E(x) \quad \text{if} \quad a + x \in X\ ,\quad p_E(x) < \delta_1\ , \text{ and } \ p \in \Gamma\ ,$$

and there exists $\delta > 0$ such that

$$p_E\big(g(b+y)-g(b)\big) < \delta_1 \quad \text{if} \quad b+y \in f(X)\ , \quad p_F(y) < \delta\ , \text{ and } \ p \in \Gamma\ .$$

Then if $b + y \in f(X)$, $q \in \Gamma^{-1}$, and $x = g(b+y) - g(b)$, then $a + x \in X$, and

$$\begin{aligned} q_E\big(g(b+y)-g(b)-f'(a)^{-1}(y)\big) & \\ &\le \|f'(a)^{-1}\|_{\Gamma^{-1}} q_F\big(r(f,\ a,\ x)\big) < \varepsilon\|f'(a)^{-1}\|_{\Gamma^{-1}} q_E(x) \\ &\le \varepsilon\|f'(a)^{-1}\|_{\Gamma^{-1}} q_F(y)/\Big(\|f'(a)^{-1}\|_{\Gamma^{-1}}^{-1}-\varepsilon\Big)\ , \end{aligned}$$

because, since $f'(a)^{-1}$ is Γ^{-1}-continuous,

$$\Big(\|f'(a)^{-1}\|_{\Gamma^{-1}}^{-1}-\varepsilon\Big) q_E(x) \le q_F\big(f(a+x)-f(a)\big)\ .$$

Hence g is Γ^{-1}-differentiable at b and $g'(b) = f'(a)^{-1}$.

5. The mean value theorem

Various forms of mean value theorems have been given in §1.3 of [5]. In particular, the following form follows from (1.3.3)2° there.

(5.1). *Let* $f \in \mathcal{D}_\Gamma(X,\ F)$. *Then for each* $p \in \Gamma$ *and each* $x \in E$ *such that* $a + \xi x \in X$ *if* $0 \le \xi \le 1$ *, there exists* $\theta \in (0,\ 1]$ *such that*

$$p_F\big(f(a+x)-f(a)\big) \le \|f'(a+\theta x)\|_\Gamma p_E(x)\ .$$

It follows immediately from (5.1) that a map $f : E \to F$ is constant if and only if $f \in \mathcal{D}_\Gamma(E,\ F)$ and $f'(x) = 0$ for all $x \in E$, and $f \in F_\Gamma(E,\ F)$ if and only if $f \in \mathcal{D}_\Gamma(E,\ F)$, $f(0) = 0$ and $f'(x)$ does not depend on x .

In [5, p. 9], we have defined that $f : X \to F$ is said to be *Gâteaux differentiable at* $a \in X$ if there exists a continuous linear map $u : E \to F$ such that

$$\lim_{\varepsilon\to0} \varepsilon^{-1} r_u(f,\ a,\ \varepsilon x) = 0$$

for each $x \in E$. If this is the case, we denote u by $f'(a)$.

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The Fréchet differentiability, and hence the Γ-differentiability, implies the Gâteaux differentiability.

(5.2). *Let* X *be convex and* $f : X \to F$ *be Gâteaux differentiable at every point of* X . *If* $f'(X) \subset F_\Gamma(E, F)$ *and* $f' : X \to F_\Gamma(E, F)$ *is* Γ*-continuous at* $a \in X$, *then* f *is* Γ*-differentiable at* a .

Proof. Let $\varepsilon > 0$, and take $\delta > 0$ which is determined by the Γ-continuity of $f' : X \to F_\Gamma(E, F)$. Then, if $a + x \in X$, $p_E(x) < \delta$, and $p \in \Gamma$, it follows from (5.1) that

$$p_F\big(f(a+x)-f(a)-f'(a)(x)\big) \le \|f'(a+\theta x)-f'(a)\|_\Gamma p_E(x) < \varepsilon p_E(x) ,$$

which means that $f'(a)$ is the Γ-derivative of f at a .

(5.3). *Let* X *be convex,* $f_n \in \mathcal{D}_\Gamma(X, F)$ *, and the following conditions be satisfied:*

(1) $\{f_n\}$ *converges to* $f : X \to F$ *uniformly;*

(2) $\{f_n'\}$ *converges to* $g : X \to F_\Gamma(E, F)$ *uniformly.*

Then $f \in \mathcal{D}_\Gamma(X, F)$ *and* $f'(x) = g(x)$ *for all* $x \in X$.

Proof. Let $\varepsilon > 0$. By (2) there exists n_0 such that

$$\|f_n'(x)-f_m'(x)\|_\Gamma < \tfrac{1}{3}\varepsilon \text{ if } m, n \ge n_0 \text{ and } x \in X .$$

Now let $a \in X$ and take $\delta > 0$ such that

$$p_F\big(r(f_{n_0}, a, x)\big) < \varepsilon p_E(x) \text{ if } a + x \in X , \ p_E(x) < \delta , \text{ and } p \in \Gamma .$$

By (5.1), if $p \in \Gamma$ and $m, n \ge n_0$,

$$\begin{aligned} p_F\big(f_n(a+x)-f_n(a)-f_m(a+x)+f_m(a)\big) &\le \|f_n'(a+\theta x)-f_m'(a+\theta x)\|_\Gamma p_E(x) \\ &< \tfrac{1}{3}\varepsilon p_E(x) \end{aligned}$$

if $a + x \in X$ and $p_E(x) < \delta$. Hence, by the assumption (1),

$$p_F\big(f(a+x)-f(a)-f_n(a+x)+f_n(a)\big) \le \tfrac{1}{3}\varepsilon p_E(x)$$

if $a + x \in X$, $n \ge n_0$, and $p \in \Gamma$. Therefore, if $a + x \in X$ and $p \in \Gamma$,

$$p_F\big(f(a+x)-f(a)-g(a)(x)\big) \le p_F\big(f(a+x)-f(a)-f_n(a+x)+f_n(a)\big)$$
$$+ p_F\big(r(f_{n_0}, a, x)\big) + p_F\big(f'_{n_0}(a)(x)-g(a)(x)\big) < \varepsilon p_E(x) ,$$

which means that $g(a)$ is the Γ-derivative of f at a.

6. The continuous Γ-differentiability

Let Γ be a calibration for (E, F).

A map $f : X \to F$ is said to be *continuously* Γ-*differentiable on* X if $f \in \mathcal{D}_\Gamma(X, F)$ and $f' : X \to F_\Gamma(E, F)$ is Γ-continuous. The set of all continuously Γ-differentiable maps of X into F will be denoted by $C_\Gamma(X, F)$, which is obviously a linear space.

It follows from (5.2) that, when X is convex, $f \in C_\Gamma(X, F)$ if and only if f is Gâteaux differentiable at every point of X with Γ-continuous derivatives and $f' : X \to F_\Gamma(E, F)$ is Γ-continuous.

(6.1). *Assume that* $F = F_1 \times F_2$ *and* Γ_F *is a calibration for* (F_1, F_2) *and* $f : X \to F$ *has the following form:*

$$f(x) = \big(f_1(x), f_2(x)\big) .$$

Then $f \in C_\Gamma(X, F)$ *if and only if* $f_i \in C_{\Gamma_i}(X, F_i)$, *where* $\Gamma_i = \big(\Gamma_E, \Gamma_{F_i}\big)$ *for each* i.

Proof. This follows from (4.5) and the relation

$$p_F\big(f'(a+x)(y)-f'(a)(y)\big) = \sum_{i=1}^{2} p_{F_i}\big(f_i'(a+x)(y)-f_i'(a)(y)\big) .$$

(6.2). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *Assume that* $f \in C_{\Gamma_1}(X, F)$ *and* $g \in C_{\Gamma_2}(Y, G)$, *where* $f(X) \subset Y$. *Then* $g \circ f \in C_\Gamma(X, G)$.

Proof. By (4.5) we only need to show that the map

$$(g \circ f)' : X \to F_\Gamma(E, G)$$

is Γ-continuous. However, this follows immediately from (2.2) and (3.9).

(6.3). *Let* F *be sequentially complete and let* $f : X \to F$ *be a bijection onto an open set* $Y = f(X)$. *Then, if* $f \in C_\Gamma(X, F)$ *and* $f'(X) \subset G_\Gamma(E, F)$, $f^{-1} \in C_{\Gamma^{-1}}(Y, E)$.

Proof. This is an immediate consequence of (3.8) and (4.7).

7. The partial Γ-differentiation

Suppose that Γ is a calibration for (E, F) , $E = E_1 \times E_2$, and Γ_E is a calibration for (E_1, E_2) . Suppose also that $X = X_1 \times X_2$ where X_i is an open subset of E_i for each i .

A map $f : X \to F$ is said to be *partially* Γ*-differentiable at* $(a_1, a_2) \in X$ *with respect to the first variable* if the partial map $f_{a_2} : X_1 \to F$, defined by

$$f_{a_2}(x) = f(x, a_2) ,$$

of X_1 into F is Γ_1-differentiable at a_1 , where $\Gamma_1 = \left(\Gamma_{E_1}, \Gamma_F\right)$. The derivative will be denoted by $\partial_1 f(a_1, a_2)$, which is an element of $F_{\Gamma_1}(E_1, F)$. In the same way the partial derivative $\partial_2 f(a_1, a_2)$ of f at (a_1, a_2) can be defined.

(7.1). *If* f *is* Γ*-differentiable at* $a = (a_1, a_2) \in X$, *then* $\partial_1 f(a_1, a_2)$ *and* $\partial_2 f(a_1, a_2)$ *exist and*

$$f'(a_1, a_2)(x_1, x_2) = \partial_1 f(a_1, a_2)(x_1) + \partial_2 f(a_1, a_2)(x_2) .$$

Proof. There exist linear maps $u_i : E_i \to F$ such that

$$f'(a_1, a_2)(x_1, x_2) = u_1(x_1) + u_2(x_2) .$$

Since

$$p_F(u_1(x_1)) = p_F(f'(a_1, a_2)(x_1, 0)) \le \|f'(a)\|_\Gamma p_{E_1}(x_1) ,$$

we have $u_1 \in F_{\Gamma_1}(E_1, F)$. Furthermore, since

$$r_{u_1}(f_{a_2}, a_1, x) = r(f, a, (x, 0)) ,$$

we have $u_1 = \partial_1 f(a_1, a_2)$. In the same way we have $u_2 = \partial_2 f(a_1, a_2)$.

(7.2). *Let* X *be convex. Then* $f \in C_\Gamma(X, F)$ *if and only if* $\partial_i f : X \to F_{\Gamma_i}(E_i, F)$ *exists and is* Γ*-continuous on* X *for each* i .

Proof. Assume that $f \in C_\Gamma(X, F)$. Then, by (7.1), $\partial_i f$ exist and, by (3.9), they are Γ-continuous.

Conversely, if $\partial_i f$ exist and are Γ-continuous, for $(a_1, a_2) \in X$, put

$$u(x_1, x_2) = \partial_1 f(a_1, a_2)(x_1) + \partial_2 f(a_1, a_2)(x_2) .$$

Then $u \in F_\Gamma(E, F)$ and, using the mean value theorem (5.1),

$$\begin{aligned} p_F(f(a_1+x_1, a_2+x_2)-f(a_1, a_2)-u(x_1, x_2)) \\ &\le p_F(f(a_1+x_1, a_2+x_2)-f(a_1+x_1, a_2)-\partial_2 f(a_1, a_2)(x_2)) \\ &\quad + p_F(f(a_1+x_1, a_2)-f(a_1, a_2)-\partial_1 f(a_1, a_2)(x_1)) \\ &\le \|\partial_2 f(a_1+x_1, a_2+\theta_2 x_2)-\partial_2 f(a_1, a_2)\|_{\Gamma_2} p_{E_2}(x_2) \\ &\quad + \|\partial_1 f(a_1+\theta_1 x_1, a_2)-\partial_1 f(a_1, a_2)\|_{\Gamma_1} p_{E_1}(x_1) . \end{aligned}$$

Hence it follows from the Γ-continuity of $\partial_i f$ that u is the Γ-derivative of f at (a_1, a_2) . The Γ-continuity of f' follows from (3.11).

8. The higher Γ-differentiability

Let Γ be a calibration for (E, F) .

A map $f : X \to F$ is said to be *twice* Γ*-differentiable at* $a \in X$ if $f \in \mathcal{D}_\Gamma(X, F)$ and the map $f' : X \to F_\Gamma(E, F)$ is Γ-differentiable at a . The set of all maps of X into F which are twice Γ-differentiable at every point of X is denoted by $\mathcal{D}^2_\Gamma(X, F)$. The second Γ-derivative of f at a will be denoted by $f^{(2)}(a)$, which is an element of $F_\Gamma(E^2, F)$.

Similarly we can define the k-times Γ-differentiability, the set

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$\mathcal{D}_\Gamma^k(X, F)$, and the k-th Γ-derivative $f^{(k)}(a)$, which is an element of $F_\Gamma(E^k, F)$.

A map $f : X \to F$ is said to be *k-times continuously Γ-differentiable on* X if $f \in \mathcal{D}_\Gamma^k(X, F)$ and the map $f^{(k)} : X \to F_\Gamma(E^k, F)$ is Γ-continuous on X . The set of all such maps is denoted by $C_\Gamma^k(X, F)$.

It is easy to see that the sets $\mathcal{D}_\Gamma^k(X, F)$ and $C_\Gamma^k(X, F)$ are linear spaces.

(8.1). *If* $f : X \to F$ *is k-times Γ-differentiable at* $a \in X$, *it is k-times Fréchet differentiable at* a *with the same derivative.*

From this and (1.8.2) of [5], it follows that $f^{(k)}(a)$ is, if it exists, a symmetric k-linear map.

Since $F_\Gamma(E, F)$ is a normed space, the following two facts belong to the normed space calculus and the proofs are omitted.

(8.2). *Let* Γ_1 *and* Γ_2 *be composable calibrations for* (E, F) *and* (F, G) *respectively and* $\Gamma = \Gamma_1 \circ \Gamma_2$. *Then the map*

$$\mathrm{comp} : F_{\Gamma_1}(E, F) \times F_{\Gamma_2}(F, G) \to F_\Gamma(E, G) ,$$

defined by

$$\mathrm{comp}(u, v) = v \circ u ,$$

is k-times differentiable at every point for every k . *In particular*

$$\mathrm{comp}'(u_0, v_0)(u, v) = \mathrm{comp}(u, v_0) + \mathrm{comp}(u_0, v) .$$

(8.3). *The map*

$$\mathrm{inv} : G_\Gamma(E, F) \to G_{\Gamma^{-1}}(F, E) ,$$

defined by $\mathrm{inv}(u) = u^{-1}$, *is k-times differentiable at every point for every* k .

Now we state the k-versions of (4.5), (4.6), and (4.7).

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(8.4). *Under the same assumptions as in* (4.5),

(1) f *is* k*-times* Γ*-differentiable at* a *if and only if* f_i *is* k*-times* Γ_i*-differentiable at* a *for each* i , *and*

$$f^{(k)}(a)(x_1, \ldots, x_k) = \left(f_1^{(k)}(a)(x_1, \ldots, x_k), f_2^{(k)}(a)(x_1, \ldots, x_k)\right) ;$$

(2) $f \in C_\Gamma^k(X, F)$ *if and only if* $f_i \in C_{\Gamma_i}^k(X, F_i)$ *for each* i .

Proof. In view of (4.5) we can start the proof by assuming that this statement holds up to $k - 1$ and f is k-times Γ-differentiable at a . Then, by (3.10),

$$u_i = \pi_i \circ f^{(k)}(a) \in F_\Gamma\left(E^k, F_i\right) ,$$

because the calibrations here are composable. Then, since by the assumption

$$p_F\left(r\left(f^{(k-1)}, a, x\right)(x_1, \ldots, x_{k-1})\right) = \\ = \sum_{i=1}^{2} p_{F_i}\left(r_{u_i}\left(f_i^{(k-1)}, a, x\right)(x_1, \ldots, x_{k-1})\right) ,$$

u_i is the Γ_i-derivative of $f^{(k-1)}$ at a for each i .

The converse can be proved in a similar way as in (6.1).

(8.5). *Under the same assumptions as in* (4.6),

(1) *if* f *is* k*-times* Γ_1*-differentiable at* a *and* g *is* k*-times* Γ_2*-differentiable at* $f(a)$, *then* $g \circ f$ *is* k*-times* Γ*-differentiable at* a ;

(2) *if* $f \in C_{\Gamma_1}^k(X, F)$ *and* $g \in C_{\Gamma_2}^k(Y, G)$, *then* $g \circ f \in C_\Gamma^k(X, G)$.

Proof. In view of (4.6) we can start the proof by assuming that the statement holds up to $k - 1$ and the assumptions are satisfied. Then

$$g' \circ f : X \to F_\Gamma(E, G)$$

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is $(k-1)$-times Γ-differentiable at a . Hence, by (8.4), the map of X into $F_{\Gamma_1}(E, F) \times F_{\Gamma_2}(F, G)$, defined by

$$x \mapsto \big(f'(x),\ g'(f(x))\big) ,$$

is $(k-1)$-times Γ-differentiable at a . Therefore, by (8.2), the map

$$(g \circ f)' : X \to F_{\Gamma}(E, G)$$

is $(k-1)$-times Γ-differentiable at a , which means that $g \circ f$ is k-times Γ-differentiable at a .

If we replace "Γ-differentiable at a " by "continuously Γ-differentiable on X ", then we have the proof for (2).

The expansion formula for $(g \circ f)^{(k)}$ for the Fréchet differentiation can be found in (1.8.3) of [5]; it also holds for the Γ-differentiation by (8.1).

(8.6). *Under the same assumptions as in* (4.7),

(1) *if* f *is* k*-times* Γ*-differentiable at* a *, then* f^{-1} *is* k*-times* Γ^{-1}*-differentiable at* $f(a)$ *;*

(2) *if* F *is sequentially complete,* $f \in C^k_{\Gamma}(X, F)$ *, and* $f'(X) \subset G_{\Gamma}(E, F)$ *, then* $f^{-1} \in C^k_{\Gamma^{-1}}\big(f(X), E\big)$.

Proof. Again we prove by induction. For $g = f^{-1}$,

$$g'\big(f(x)\big) = \operatorname{inv}\big(f'(x)\big) = (\operatorname{inv} \circ f')(x) .$$

Hence, by (8.3) and (8.5), g' is $(k-1)$-times Γ^{-1}-differentiable at $f(a)$, and hence g is k-times Γ^{-1}-differentiable at $f(a)$.

The proof of (2) follows from (6.3) and the same argument as above, where "Γ^{-1}-differentiable at a " is replaced by "continuously Γ^{-1}-differentiable on $f(X)$ ".

9. Implicit function theorems

The aim of this section is to present our version of the implicit

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function theorem by interpreting the proof for the case of Banach spaces into our language. We shall start with the inverse mapping theorem.

(9.1). *Let* E *or* F *be sequentially complete and* $f \in C^k_\Gamma(X, F)$. *If, for some* $a \in X$, $f'(a)$ *is a* Γ*-isomorphism, then* f *is a local* C^k_Γ*-diffeomorphism; that is, there exist open neighbourhoods* U *and* V *of* a *and* $f(a)$ *respectively such that* f *is a* Γ*-homeomorphism of* U *onto* V .

Proof. By considering the map

$$x \mapsto f'(a)^{-1}[f(a+x)-f(a)] ,$$

we can assume that $E = F$, $a = 0$, $f(0) = 0$, and $f'(0) = 1$ (the identity map on E). Furthermore, we can assume $\Gamma = (\Gamma, \Gamma)$.

Since $f' : X \to F_\Gamma(E, E)$ is Γ-continuous at zero, there exists $\delta > 0$ such that

$$\|f'(x)-1\|_\Gamma < \tfrac{1}{2} \text{ if } x \in X , \ p(x) < \delta \text{ and } p \in \Gamma .$$

We can choose a continuous semi-norm q on E such that $q(x) < \delta$ implies $x \in X$. Since Γ induces the topology of E ,

$$q \leq q_1 \cup q_2 \cup \dots \cup q_n$$

for some $q_i \in \Gamma$ $(1 \leq i \leq n)$. Let

$$U = \{x \in E : q_i(x) < \delta \quad (1 \leq i \leq n)\}$$

and $h = 1 - f$. Then, by the mean value theorem (5.1),

$$p\big(h(x)\big) < \tfrac{1}{2}p(x) \text{ if } x \in U \text{ and } p \in \Gamma .$$

Hence $x \in U$ implies $h(x) \in \frac{1}{2}U$; that is, h maps U into $\frac{1}{2}U$.

To prove that h is onto, let $y \in \frac{1}{2}U$, and consider the map

$$h_y(x) = y + h(x) .$$

Then the sequence $\{y_n\}$, defined by

$$y_1 = h_y(y) \text{ and } y_n = h_y(y_{n-1}) ,$$

is a Cauchy sequence. Let x_0 be the limit. Then, since h is continuous, we have $y = f(x_0)$. Moreover, since

$$p(y_n) \le \left(\sum_{j=0}^{n} 2^{-j}\right) p(y) \ ,$$

we have

$$q_i(x_0) \le 2q_i(y) < \delta \quad (1 \le i \le n) \ ,$$

which means that $x_0 \in U$.

Now let $x_i \in U$ $(i = 1, 2)$ and $p \in \Gamma$. Then, by (5.1),

$$\begin{aligned} p(x_1 - x_2) &\le p(f(x_1) - f(x_2)) + p(h(x_1) - h(x_2)) \\ &\le p(f(x_1) - f(x_2)) + \tfrac{1}{2}p(x_1 - x_2) \ , \end{aligned}$$

which implies

$$\tfrac{1}{2}p(x_1 - x_2) \le p(f(x_1) - f(x_2)) \ .$$

It follows from this inequality that f maps U onto $\frac{1}{2}U$ and f , when it is restricted to U , is injective. Hence, if we put

$$U_0 = f^{-1}(\tfrac{1}{2}U) \cap U \quad \text{and} \quad V_0 = \tfrac{1}{2}U \ ,$$

f is a bijection of U_0 onto V_0 .

Now by (6.3), $f^{-1} \in C_{\Gamma^{-1}}(V_0, U_0)$, which shows that $f : U_0 \to V_0$ is a C^k_Γ-diffeomorphism.

The implicit function theorem is deduced from the inverse mapping theorem; the proof is omitted.

(9.2). *Suppose that* $E = E_1 \times E_2$, F *is sequentially complete,* Γ *is a calibration for* (E, F) *, and* Γ_E *is a calibration for* (E_1, E_2) . *Let* $\Gamma_i = \left(\Gamma_{E_i}, \Gamma_F\right)$.

Assume that $f \in C^k_\Gamma(X, F)$, $f(a_1, a_2) = 0$, *and*

$\partial_2 f(a_1, a_2) \in G_{\Gamma_2}(E_2, F)$. *Then there is an open neighbourhood* $U \times V$ *of* (a_1, a_2) *in* X *and* $g \in C^k_{\Gamma_E}(U, V)$ *such that* $g(a_1) = a_2$ *and*

$$f^{-1}(0) \cap (U \times V) = \{(x, g(x)) : x \in U\} .$$

If this is the case,

$$g'(x) = -\left\{\partial_2 f(a_1, a_2)^{-1}\right\} \circ \partial_1 f(a_1, a_2) .$$

The split versions of the above theorem can also be proved under similar assumptions.

10. Remarks

1. As we have seen in the above discussions, once the Γ-differentiability of the map under consideration is established, the remainder of the proof consists of checking the suitability of the calibration and paraphrase of the proof of Banach space case. In other words, as far as the fundamental properties, such as developed above, are concerned, we have generalizations in simple forms, and the easiest way to find the suitable calibration is to use (5.2).

Naturally, finding a suitable calibration becomes much easier if F is a normed space, which includes the case of functionals. Let us denote by E_p the space E equipped with the topology defined by a single continuous semi-norm p on E , and let Γ_p be the set of all continuous semi-norms q such that $q \geq p$. Obviously Γ_p is a calibration for E .

(10.1). *Let* F *be a normed space. If* $f : X \to F$ *is differentiable at* $a \in X$ *in the ordinary sense as a map of* E_p *into* F *, then* f *is* Γ_p*-differentiable at* a .

Proof. Let $\varepsilon > 0$. Then, by the assumption, there exists $\delta > 0$ such that

$$\|r(f, a, x)\| < \varepsilon p(x) \quad \text{if} \quad a + x \in X \quad \text{and} \quad p(x) < \delta ,$$

where the derivative $f'(x)$ satisfies the following condition:

$$\|f'(a)(x)\| \leq \alpha p(x) \quad \text{for some} \quad \alpha > 0 \quad \text{and all} \quad x \in E .$$

It is obvious that these two inequalities are satisfied when p is replaced by any $q \in \Gamma_p$. This means that f is Γ_p-differentiable at a .

As an immediate consequence, we have a criterion for the Γ-differentiability of semi-norms.

(10.2). *A continuous semi-norm* p *on* E *is* Γ_p*-differentiable on* $E\backslash\{0\}$ *if and only if* p *is a differentiable semi-norm on* $E_p\backslash\{0\}$.

This fact will be a basis of the study of the Γ-smoothness of locally convex spaces, which will be treated in a subsequent note.

2. Since we have various forms of the inverse mapping theorem, we also have their consequences. For instance, if the Γ-Fredholm maps are suitably defined, Smale's version [4] of Sard's Theorem can be generalized to locally convex spaces.

3. Instead of the reals, we could take the complex numbers as the coefficient field. Then it will lead to the theory of Γ-analytic maps, the fundamental theory of which will be developed in the next note.

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