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Schubert presentation of the cohomology ring of flag manifolds G/T

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Abstract

Let G be a compact connected Lie group with a maximal torus T. In the context of Schubert calculus we present the integral cohomology $H^*(G/T)$ by a minimal system of generators and relations.

1. *Introduction*

Let G be a compact connected Lie group with Lie algebra $L(G)$ and exponential map $\exp : L(G) \to G$. For a non-zero vector $u \in L(G)$ the centralizer P of the one-parameter subgroup $\{\exp(tu) \in G \mid t \in \mathbb{R}\}$ on G is a *parabolic subgroup* of G. The homogeneous space G/P is canonically a projective variety, called a *flag manifold* of G [**25**, **26**]. If the vector u is non-singular the centralizer P is a maximal torus T on G, and the flag manifold G/T is also known as *the complete flag manifold* of G.

Schubert calculus [**31**] began with the intersection theory of the 19th century, together with its applications to enumerative geometry. Clarifying this calculus was an important problem of algebraic geometry [**6**, **19**, **24**, **32**]. van der Waerden and Weil, who secured the foundation of modern intersection theory [**36**], attributed the classical Schubert calculus to the determination of the integral cohomology rings $H^*(G/P)$ of flag manifolds G/P (see [**35**] and [**37**, p. 331]).

The cohomology of flag manifolds has now been well understood. The basis theorem of Chevalley [**2**, **7**, **8**] assures that the classical Schubert classes on a flag manifold G/P form an additive basis of the cohomology $H^*(G/P)$; an explicit formula for multiplying the basis elements was obtained by the present authors [**10**, **14**, **16**]. However, concerning many relevant studies [**9**, **11**, **13**, **15**, **18**, **19**, **25**, **26**], such a description of the ring $H^*(G/P)$ is not a practical one, since the number of Schubert classes on a flag manifold is normally very large, not to mention the number of *structure constants* required to expand the products of Schubert classes. It is natural to ask for a concise presentation of the ring $H^*(G/P)$, which is characterized as follows.

Given a set $\{x_1, \dots, x_k\}$ of k elements, let $\mathbb{Z}[x_1, \dots, x_k]$ be the ring of polynomials in $x_1, \dots, x_k$ with integer coefficients. For a subset $\{f_1, \dots, f_m\} \subset \mathbb{Z}[x_1, \dots, x_k]$, write $\langle f_1, \dots, f_m\rangle$ for the ideal generated by $f_1, \dots, f_m$.

Definition 1.1. A *Schubert presentation* of the integral cohomology ring of a flag manifold G/P is an isomorphism

$$H^*(G/P) = \mathbb{Z}[x_1, \dots, x_k]/\langle f_1, \dots, f_m\rangle, \tag{1.1}$$

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where $\{x_1, \ldots, x_k\}$ is a set of Schubert classes on G/P that generates the ring $H^*(G/P)$ multiplicatively, and where both the numbers k and m in (1.1) are minimal. $\square$

Prior to the use of Schubert classes as generators, the numbers k and m in Definition 1.1 can be seen to be invariants of the ring $H^*(G/P)$. Indeed, let $D(H^*(G/P)) \subset H^*(G/P)$ be the ideal of decomposable elements, and denote by $h(G,P)$ the cardinality of a basis for the quotient group $H^*(G/P)/D(H^*(G/P))$. Then $k = h(G,P) - 1$. Furthermore, if one changes the generating set in (1.1) to $x_1', \ldots, x_k'$, then each old generator x_i can be expressed as a polynomial g_i in the new ones $x_1', \ldots, x_k'$. The invariance of the number m is shown by the presentation

$$H^*(G/P) = \mathbb{Z}[x_1', \ldots, x_k']/\langle f_1', \ldots, f_m' \rangle,$$

where the polynomial f_j' is obtained from f_j by substituting the polynomial g_i for x_i, $1 \leqslant j \leqslant m$.

Among all the flag manifolds G/P associated to a Lie group G it is the complete flag manifold G/T that is of crucial importance. The inclusion $T \subset P \subset G$ of subgroups induces the fibration $P/T \hookrightarrow G/T \xrightarrow{\pi} G/P$ in which the induced map π^* embeds $H^*(G/P)$ as a subring of $H^*(G/T)$ (see Lemma 2.3). In this paper we establish a Schubert presentation for the cohomologies of all complete flag manifolds G/T.

Recall that all the 1-connected simple Lie groups consist of the three infinite families $SU(n+1), Sp(n)$, $\mathrm{Spin}(n+2)$, $n \geqslant 2$, of classical groups, as well as the five exceptional ones: G_2, F_4, E_6, E_7, E_8. It is also known that, for any compact connected Lie group G with a maximal torus T, one has a diffeomorphism $G/T = G_1/T_1 \times \ldots \times G_k/T_k$ with each G_i a 1-connected simple Lie group and with $T_i \subset G_i$ a maximal torus. Moreover, by the basis theorem of Chevalley (see Theorem 2.1), the integral cohomology $H^*(G/T)$ is torsion free. Therefore, the problem of finding a Schubert presentation of the ring $H^*(G/T)$ is reduced by the Künneth formula to the special cases where G is one of the 1-connected simple Lie groups. For this reason we can assume in the remaining part of the paper that the Lie groups under consideration are all simple. In addition, the cohomologies are over the ring $\mathbb{Z}$ of integers, unless otherwise stated.

For a Lie group G of rank n, let $\{\omega_1, \ldots, \omega_n\} \subset H^2(G/T)$ be a set of fundamental dominant weights of G [4], and set $m = h(G,T) - n - 1$. Our main result is the following theorem.

THEOREM 1.2. *There exists a set* $\{y_{d_1}, \ldots, y_{d_m}\}$ *of* m *Schubert classes on* G/T, *with* $1 < d_1 < \ldots < d_m$ *and* $\deg y_{d_j} = 2d_j$, *so that the inclusion* $\{\omega_1, \ldots, \omega_n, y_{d_1}, \ldots, y_{d_m}\} \in H^*(G/T)$ *induces an isomorphism*

$$H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_{d_1}, \ldots, y_{d_m}]/\langle e_i, f_j, g_j \rangle_{1 \leqslant i \leqslant k; 1 \leqslant j \leqslant m}, \tag{1.2}$$

where:

(i) $k = n - m$ *for all* $G \neq E_8$ *but* $k = n - m + 2$ *for* $G = E_8$;

(ii) $e_i \in \langle \omega_1, \ldots, \omega_n \rangle$, $1 \leqslant i \leqslant k$;

(iii) *the pair* (f_j, g_j) *of relations is related to the class* y_{d_j} *in the fashion*

$$f_j = p_j \cdot y_{d_j} + \alpha_j, \quad g_j = y_{d_j}^{k_j} + \beta_j, \quad 1 \leqslant j \leqslant m,$$

with $p_j \in \{2, 3, 5\}$ *and* $\alpha_j, \beta_j \in \langle \omega_1, \ldots, \omega_n \rangle$. $\square$

A set $S = \{y_{d_1}, \ldots, y_{d_m}\}$ of Schubert classes on G/T satisfying (1.2) will be called *a set of special Schubert classes* on G/T. In the course of showing Theorem 1.2, a set of special Schubert classes, as well as the corresponding system $\{e_i, f_j, g_j\}$ of polynomials, will be made explicit for each simple Lie group. Along the way an algebraic criterion for a set of Schubert classes on G/T to be special is given in Theorem 6.3.

Since the set $\{\omega_1,\ldots,\omega_n\}$ of fundamental weights is precisely the Schubert basis on the group $H^2(G/T)$ [**17**], the presentation (1.2) describes the ring $H^*(G/T)$ by certain Schubert classes on G/T. It is worthwhile to know whether it is indeed a Schubert presentation of the ring $H^*(G/T)$.

THEOREM 1.3. *If $G \neq E_8$, the formula (1.2) is a Schubert presentation of $H^*(G/T)$. If $G = E_8$, a Schubert presentation of the ring $H^*(E_8/T)$ is*

$$\mathbb{Z}[\omega_1,\ldots,\omega_8,y_{d_1},\ldots,y_{d_7}]/\langle e_i,f_j,g_t,\phi\rangle_{1\leqslant i\leqslant 3;1\leqslant j\leqslant 7,t=1,2,3,5}, \tag{1.3}$$

where:

(a) *the Schubert classes $y_{d_1},\ldots,y_{d_7}$ and the polynomials e_i,f_j,g_t are the same as those in (1.2) for the case of $G=E_8$;*

(b) $\phi = 2y_6^5 - y_{10}^3 + y_{15}^2 + \beta$ *with* $\beta \in \langle\omega_1,\ldots,\omega_8\rangle$. □

This paper is arranged as follows. Section 2 develops cohomology properties for fibrations in flag manifolds. Granted with the packages '*The Chow ring of Grassmannians*' and '*Giambelli polynomials*' compiled in [**17**, § 2.6], initial data facilitating our computation are generated in §§ 3 and 4. With these preparations the presentation (1.2) for the exceptional Lie groups is obtained in § 5. Finally, Theorems 1.2 and 1.3 are established in § 6.

Certain relations on the ring $H^*(G/T)$ may be seen as detailed. However, they are useful for encoding the topology of the corresponding Lie group G. Using the set $\{e_i,f_j,g_j\}$ of polynomials in (1.2), one can construct uniformly the integral cohomology of compact Lie groups [**11**, **15**], deduce explicit formulae for the generalized Weyl invariants of G in a characteristic p [**18**, Propositions 5.5–5.7], and determine the structure of the mod p cohomology $H^*(G;\mathbb{F}_p)$ as a module over the Steenrod algebra $\mathcal{A}_p$ [**18**].

2. *Fibrations in flag manifolds*

Let G be a Lie group with maximal torus T and Cartan subalgebra $L(T)$. Equip the Lie algebra $L(G)$ with an inner product $(,)$, so that the adjoint representation acts as isometries on $L(G)$. Assume that the rank of G is $n = \dim T$, and a system $\{\beta_1,\ldots,\beta_n\}$ of simple roots of G is so ordered as the vertices in the Dynkin diagram of G pictured in [**21**, p. 58]. Then the *Weyl group* of G is the subgroup $W \subset \mathrm{Aut}(L(T))$ generated by the reflections σ_i in the hyperplanes $L_i \subset L(T)$ perpendicular to the roots β_i, $1 \leqslant i \leqslant n$. By the relation $H^2(G/T)\otimes\mathbb{R} = L(T)$ due to Borel and Hirzebruch [**4**], the set $\{\omega_1,\ldots,\omega_n\} \subset H^2(G/T)$ of *fundamental dominant weights* of G can be regarded as the basis of the space $L(T)$ defined by the formulae

$$2(\beta_i,\omega_j)/(\beta_i,\beta_i) = \delta_{i,j}, \quad 1 \leqslant i,j \leqslant n.$$

For a parabolic subgroup P on G, let W and W' be the Weyl groups of G and P, respectively. In term of the length function $l : W \to \mathbb{Z}$ on W, the set $\overline{W}$ of left cosets of W' in W can be identified with the subset of W:

$$\overline{W} = \{w \in W \mid l(w_1) \geqslant l(w), w_1 \in wW'\};$$

see [**2**, 5.1]. It follows that every element $w \in \overline{W}$ admits a decomposition $w = \sigma_{i_1}\circ\ldots\circ\sigma_{i_r}$ with $1 \leqslant i_1,\ldots,i_r \leqslant n$ and $r = l(w)$. This decomposition is called *minimized*, written $w = \sigma[i_1,\ldots,i_r]$, if the relation $(i_1,\ldots,i_r) \leqslant (j_1,\ldots,j_r)$ holds for any $(j_1,\ldots,j_r)$ satisfying $w = \sigma_{j_1}\circ\ldots\circ\sigma_{j_r}$, where $\leqslant$ means the lexicographical order on multi-indices.

For an element $w \in \overline{W}$ with minimized decomposition $\sigma[i_1, \ldots, i_r]$, the *Schubert variety* X_w associated to w is the image of the composition

$$K_{i_1} \times \ldots \times K_{i_r} \to G \xrightarrow{p} G/P, \quad (k_1, \ldots, k_r) \longmapsto p(k_1 \ldots k_r),$$

where $K_i \subset G$ is the centralizer of $\exp(L_i)$ in G, p is the obvious quotient map, and where the product $\cdot$ takes place in G. In [7], Chevalley announced the following remarkable cellular decomposition on the flag manifold G/P:

$$G/P = \bigcup_{w \in \overline{W}} X_w, \quad \dim_{\mathbb{R}} X_w = 2l(w) \quad \text{(see also [2, 8])}. \tag{2.1}$$

The *Schubert class* $s_w \in H^{2l(w)}(G/P)$ corresponding to $w \in \overline{W}$ is defined to be the Kronecker dual of the fundamental classes $[X_w] \in H_{2l(w)}(G/P)$. Since only even-dimensional cells are involved in the decomposition (2.1), one has the next result, called the *basis theorem* of Schubert calculus.

THEOREM 2.1 (see [2, 7, 8]). *The set of Schubert classes $\{s_w \in H^*(G/P) \mid w \in \overline{W}\}$ constitutes a basis of the graded group $H^*(G/P)$.* □

For a subset $I \subseteq \{1, \ldots, n\}$, let P_I be the centralizer of the one-parameter subgroup $\alpha : \mathbb{R} \to G$, $\alpha(t) = \exp(t \sum_{i \in I} \omega_i)$ on G. Useful information on the geometry of the flag manifold G/P_I is given by the following lemma.

LEMMA 2.2 [14, 17]. *The centralizer of any one-parameter subgroup on G is conjugate to a subgroup P_I for some $I \subseteq \{1, \ldots, n\}$. Moreover:*
- (i) *P_I is a parabolic subgroup; its Dynkin diagram is obtained from that of G by deleting the vertices β_i with $i \in I$, and the edges adjoining them;*
- (ii) *the Weyl group W_I of P_I is the subgroup of W generated by $\sigma_j, j \notin I$;*
- (iii) *the Schubert basis of $H^*(G/P_I)$ is $\{s_w \mid w \in W/W_I\}$.*

Property (i) of Lemma 2.2 characterizes the subgroup P_I only up to its local type. A method for deciding the isomorphism type of P_I is given in [12].

For a proper subset $I \subset \{1, \ldots, n\}$, the inclusion $T \subset P_I \subset G$ of subgroups induces the fibration in flag manifolds

$$P_I/T \overset{i}{\hookrightarrow} G/T \xrightarrow{\pi} G/P_I. \tag{2.2}$$

The next result implies that the cohomologies of the fiber space P_I/T and the base space G/P_I are much simpler than that of the total space G/T.

LEMMA 2.3. *With respect to the inclusion $W_I \subset W$, the induced map i^* identifies the subset $\{s_w\}_{w \in W_I \subset W}$ of the Schubert basis of $H^*(G/T)$ with the Schubert basis $\{s_w\}_{w \in W_I}$ of $H^*(P_I/T)$.*

With respect to the inclusion $W/W_I \subset W$, the induced map π^* identifies the Schubert basis $\{s_w\}_{w \in W/W_I}$ of $H^*(G/P_I)$ with the subset $\{s_w\}_{w \in W/W_I}$ of the Schubert basis $\{s_w\}_{w \in W}$ of $H^*(G/T)$.

Proof. These come directly from the next two properties of Schubert varieties (see for example [17, § 2]). With respect to the cell decompositions (2.1) on the three flag manifolds P_I/T, G/T, and G/P_I, one has:
- (i) for each $w \in W_I \subset W$, the fiber inclusion i carries the Schubert variety X_w on P_I/T identically onto the Schubert variety X_w on G/T;

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(ii) for each $w \in W/W_I \subset W$, the projection π restricts to a degree-one map from the Schubert variety X_w on G/T to the corresponding Schubert variety on G/P_I. □

CONVENTION 2.4. In view of Lemma 2.3 and for notational convenience, we shall make no difference in notation between an element in $H^*(G/P_I)$ and its π^* image in $H^*(G/T)$, and between a Schubert class on P_I/T and its i^* pre-image on G/T.

To formulate the ring $H^*(G/T)$ in question from the simpler ones $H^*(P_I/T)$ and $H^*(G/P_I)$, assume that $\{y_1, \dots, y_{n_1}\}$ is a subset of Schubert classes on P_I/T, $\{x_1, \dots, x_{n_2}\}$ is a subset of Schubert classes on G/P_I, and that with respect to them one has the following presentations of the cohomologies:

$$H^*(P_I/T) = \frac{\mathbb{Z}[y_i]_{1\leqslant i\leqslant n_1}}{\langle h_s\rangle_{1\leqslant s\leqslant m_1}}; \quad H^*(G/P_I) = \frac{\mathbb{Z}[x_j]_{1\leqslant j\leqslant n_2}}{\langle r_t\rangle_{1\leqslant t\leqslant m_2}}, \tag{2.3}$$

where $h_s \in \mathbb{Z}[y_i]_{1\leqslant i\leqslant n_1}$, $r_t \in \mathbb{Z}[x_j]_{1\leqslant j\leqslant n_2}$.

LEMMA 2.5. *The inclusions $y_i, x_j \in H^*(G/T)$ induce a surjective map*

$$\varphi : \mathbb{Z}[y_i, x_j]_{1\leqslant i\leqslant n_1, 1\leqslant j\leqslant n_2} \to H^*(G/T).$$

Furthermore, if $\{\rho_s\}_{1\leqslant s\leqslant m_1} \subset \mathbb{Z}[y_i, x_j]$ is a system satisfying

$$\varphi(\rho_s) = 0 \quad \text{and} \quad \rho_s|_{x_j=0} = h_s, \tag{2.4}$$

then φ induces a ring isomorphism

$$H^*(G/T) = \mathbb{Z}[y_i, x_i]_{1\leqslant i\leqslant n_1, 1\leqslant j\leqslant n_2}/\langle \rho_s, r_t\rangle_{1\leqslant s\leqslant m_1, 1\leqslant t\leqslant m_2}. \tag{2.5}$$

Proof. Lemma 2.3, together with Convention 2.4, implies that the map φ is surjective. It remains to show that for a $g \in \mathbb{Z}[y_i, x_j]_{1\leqslant i\leqslant n_1, 1\leqslant j\leqslant n_2}$ the relation $\varphi(g) = 0$ implies $g \in \langle \rho_s, r_t\rangle_{1\leqslant s\leqslant m_1, 1\leqslant t\leqslant m_2}$. By Lemma 2.3 and by the Leray–Hirsch property [**22**, p. 231] of the fibration (2.2), one has the following presentation of $H^*(G/T)$, a module over its subring $H^*(G/P_I)$:

$$H^*(G/T) = H^*(G/P_I)\{1, s_w\}_{w\in W_I}.$$

It follows from the presentation of the ring $H^*(P_I/T)$ in (2.3) and the assumption (2.4) that, for any polynomial $g \in \mathbb{Z}[y_i, x_j]$, one has

$$g \equiv \sum_{w\in W_I} g_w \cdot s_w \mod \langle \rho_s\rangle_{1\leqslant s\leqslant m_1} \quad \text{with } g_w \in \mathbb{Z}[x_j]_{1\leqslant j\leqslant n_2}.$$

From this, we find that $\varphi(g) = 0$ implies $\varphi(g_w) = 0$, $w \in W_I$. That is, $g_w \in \langle r_t\rangle_{1\leqslant t\leqslant m_2}$, $w \in W_I$, by the presentation of the ring $H^*(G/P_I)$ in (2.3). This completes the proof. □

3. *Cohomology of generalized Grassmannians*

If $I = \{k\}$ is a singleton, the flag manifold $G/P_{\{k\}}$ is called the *Grassmannians of G corresponding to the weight ω_k* [**17**]. Using Table 3.1, we associate to each exceptional Lie group G a Grassmannian $G/P_{\{k\}}$ where in the third row the subgroups $P_{\{k\}}$ are presented by their local types determined by Lemma 2.2(i), and where the group $P^s_{\{k\}}$ in the fourth row is the simple part of the group $P_{\{k\}}$. Our approach to the ring $H^*(G/T)$ amounts to applying

Lemma 2.5 to the fibration in flag manifolds

$$P_{\{k\}}/T \overset{i}{\hookrightarrow} G/T \overset{\pi}{\to} G/P_{\{k\}}. \tag{3.1}$$

To this end, an account for the cohomologies of the base spaces $G/P_{\{k\}}$ is required.

In [**17**], a program calculating Schubert presentation of a Grassmannian $G/P_{\{k\}}$ has been compiled, whose function is briefly described below.

Algorithm: *The Chow ring of Grassmannians*

Input: *The Cartan matrix* $C = (c_{ij})_n$ *of* G, and an integer $k \in \{1, \ldots, n\}$;

Output: *A Schubert presentation of the cohomology* $H^*(G/P_{\{k\}})$.

As applications of the algorithm, Schubert presentations for the five Grassmannians as shown in Table 3.1 have been obtained. To state the results, we introduce for each of the Grassmannians $G/P_{\{k\}}$ a set of Schubert classes $\{s_w\}$ on $G/P_{\{k\}}$ in terms of the minimized decomposition $\sigma[i_1, \ldots, i_r]$ of the corresponding w, together with their abbreviations y_i (with $\deg y_i = 2i$), in Table 3.2.

TABLE 3.1. *A Grassmannian associated to each exceptional Lie group.*

G	G_2	F_4	E_6	E_7	E_8
k	1	1	2	2	2
$P_{\{k\}}$	$SU(2) \cdot S^1$	$Sp(3) \cdot S^1$	$SU(6) \cdot S^1$	$SU(7) \cdot S^1$	$SU(8) \cdot S^1$
$P^s_{\{k\}}$	$SU(2)$	$Sp(3)$	$SU(6)$	$SU(7)$	$SU(8)$

TABLE 3.2. *A set of special Schubert classes on* $G/P_{\{k\}}$.

y_i	$G_2/P_{\{1\}}$	$F_4/P_{\{1\}}$	$E_n/P_{\{2\}}, n = 6, 7, 8$
y_3	$\sigma_{[1,2,1]}$	$\sigma_{[3,2,1]}$	$\sigma_{[5,4,2]}, n = 6, 7, 8$
y_4		$\sigma_{[4,3,2,1]}$	$\sigma_{[6,5,4,2]}, n = 6, 7, 8$
y_5			$\sigma_{[7,6,5,4,2]}, n = 7, 8$
y_6		$\sigma_{[3,2,4,3,2,1]}$	$\sigma_{[1,3,6,5,4,2]}, n = 6, 7, 8$
y_7			$\sigma_{[1,3,7,6,5,4,2]}, n = 7, 8$
y_8			$\sigma_{[1,3,8,7,6,5,4,2]}, n = 8$
y_9			$\sigma_{[1,5,4,3,7,6,5,4,2]}, n = 7, 8$
y_{10}			$\sigma_{[1,6,5,4,3,7,6,5,4,2]}, n = 8$
y_{15}			$\sigma_{[5,4,2,3,1,6,5,4,3,8,7,6,5,4,2]}, n = 8$

THEOREM 3.1. *With respect to the special Schubert classes on* $G/P_{\{k\}}$ *given in Table 3.2, the Schubert presentations of the integral cohomology rings of* $G/P_{\{k\}}$ *are*

$$H^*(G_2/SU(2) \cdot S^1) = \mathbb{Z}[\omega_1, y_3]/\langle r_3, r_6\rangle, \text{ where} \tag{3.2}$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = y_3^2.$$

$$H^*(F_4/Sp(3) \cdot S^1) = \mathbb{Z}[\omega_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12}\rangle, \text{ where} \tag{3.3}$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = 2y_6 + y_3^2 - 3\omega_1^2 y_4;$$
$$r_8 = 3y_4^2 - \omega_1^2 y_6;$$
$$r_{12} = y_6^2 - y_4^3.$$

$$H^*(E_6/SU(6)\cdot S^1)=\mathbb{Z}[\omega_2,y_3,y_4,y_6]/\langle r_6,r_8,r_9,r_{12}\rangle,\ \textit{where} \tag{3.4}$$
$$r_6=2y_6+y_3^2-3\omega_2^2y_4+2\omega_2^3y_3-\omega_2^6;$$
$$r_8=3y_4^2-6\omega_2y_3y_4+\omega_2^2y_6+5\omega_2^2y_3^2-2\omega_2^5y_3;$$
$$r_9=2y_3y_6-\omega_2^3y_6;$$
$$r_{12}=y_6^2-y_4^3.$$

$$H^*(E_7/SU(7)\cdot S^1)=\mathbb{Z}[\omega_2,y_3,y_4,y_5,y_6,y_7,y_9]/\langle r_j\rangle_{j\in\mathcal{R}(7)}$$
$$\textit{with } \mathcal{R}(7)=\{6,8,9,10,12,14,18\},\ \textit{where} \tag{3.5}$$
$$r_6=2y_6+y_3^2+2\omega_2y_5-3\omega_2^2y_4+2\omega_2^3y_3-\omega_2^6;$$
$$r_8=3y_4^2-2y_3y_5+2\omega_2y_7-6\omega_2y_3y_4+\omega_2^2y_6+5\omega_2^2y_3^2+2\omega_2^3y_5-2\omega_2^5y_3;$$
$$r_9=2y_9+2y_4y_5-2y_3y_6-4\omega_2y_3y_5-\omega_2^2y_7+\omega_2^3y_6+2\omega_2^4y_5;$$
$$r_{10}=y_5^2-2y_3y_7+\omega_2^3y_7;$$
$$r_{12}=y_6^2+2y_5y_7-y_4^3+2y_3y_9+2y_3y_4y_5+2\omega_2y_5y_6-6\omega_2y_4y_7+\omega_2^2y_5^2;$$
$$r_{14}=y_7^2-2y_5y_9+y_4y_5^2;$$
$$r_{18}=y_9^2+2y_5y_6y_7-y_4y_7^2-2y_4y_5y_9+2y_3y_5^3-\omega_2y_5^2y_7.$$

$$H^*(E_8/SU(8)\cdot S^1)=\mathbb{Z}[\omega_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9,y_{10},y_{15}]/\langle r_j\rangle_{j\in\mathcal{R}(8)}$$
$$\textit{with } \mathcal{R}(8)=\{8,9,10,12,14,15,18,20,24,30\},\ \textit{where} \tag{3.6}$$
$$\begin{aligned}
r_8={}&3y_8-3y_4^2+2y_3y_5-2\omega_2y_7+6\omega_2y_3y_4-\omega_2^2y_6-\omega_2^2z_6-5\omega_2^2y_3^2\\
&-2\omega_2^3y_5+2\omega_2^5y_3;\\
r_9={}&2y_9+2y_4y_5-2y_3y_6+\omega_2y_8-4\omega_2y_3y_5-\omega_2^2y_7+\omega_2^3y_6+2\omega_2^4y_5;\\
r_{10}={}&3y_{10}-2y_5^2+6y_4z_6-2y_3y_7-4\omega_2y_3z_6-\omega_2^2y_8+\omega_2^3y_7+2\omega_2^4z_6;\\
r_{12}={}&y_4^3-y_6^2+4z_6^2-2y_5y_7-6y_4y_8-2y_3y_9-2y_3y_4y_5-2y_3^2z_6-2\omega_2y_5y_6\\
&-4\omega_2y_5z_6+6\omega_2y_4y_7+6\omega_2^2y_4z_6-2\omega_2^2y_3y_7+\omega_2^5y_7;\\
r_{14}={}&y_7^2-6z_6y_8-3y_6y_8-2y_5y_9+3y_4y_{10}+6y_4^2z_6-2y_4y_5^2-4y_3y_5z_6\\
&-4y_3^2y_8+4\omega_2z_6y_7+4\omega_2^2z_6^2+2\omega_2^3y_3y_8;\\
r_{15}={}&2y_{15}-y_7y_8+2y_5y_{10}-2y_5^3+y_4y_5z_6+2y_3z_6^2-2y_3y_4y_8+2\omega_2z_6y_8\\
&-2\omega_2^3y_6^2+\omega_2^3y_4y_8;\\
r_{18}={}&y_9^2+9y_{10}y_8-3y_{10}y_7\omega_2-6y_{10}z_6\omega_2^2-3y_{10}y_5\omega_2^3+4y_9z_6y_3\\
&-2y_9y_5y_4-2y_8^2\omega_2^2+y_8y_7y_3+18y_8z_6y_4+2y_8z_6y_3\omega_2-2y_8y_5^2\\
&-6y_8y_5y_4\omega_2+6y_8y_5y_3\omega_2^2-y_8y_5\omega_2^5-2y_8y_4y_3^2+y_8y_4y_3\omega_2^3-y_7^2y_4\\
&+5y_7z_6y_5-10y_7z_6y_4\omega_2+3y_7y_5y_4\omega_2^2+3y_7y_5y_3^2-2y_7y_5y_3\omega_2^3\\
&-10z_6^3+10z_6^2y_5\omega_2-16z_6^2y_4\omega_2^2+4z_6^2y_3^2-4z_6y_5y_4y_3\\
&+6z_6y_5y_3^2\omega_2-4z_6y_5y_3\omega_2^4+2y_5^3\omega_2^3;\\
r_{20}={}&3(y_{10}+2y_4z_6-y_5^2)^2+y_8(-4y_9y_3+2y_9\omega_2^3-3y_8y_4+2y_8y_3\omega_2\\
&-y_8\omega_2^4+y_7y_5+2z_6y_5\omega_2-8z_6y_3^2+4z_6y_3\omega_2^3+2y_5y_4y_3-y_5y_4\omega_2^3);\\
r_{24}={}&5z_6^4+y_8(6y_{10}z_6-12y_{10}y_6-18y_{10}y_5\omega_2-12y_{10}y_3^2-5y_9y_7\\
&-4y_9z_6\omega_2-2y_9y_6\omega_2+2y_9y_5\omega_2^2-6y_9y_4y_3+3y_9y_4\omega_2^3-2y_9y_3^2\omega_2\\
&+4y_8^2-14y_8y_7\omega_2-6y_8z_6\omega_2^2+6y_8y_6\omega_2^2-14y_8y_5y_3-21y_8y_4^2\\
&+22y_8y_4y_3\omega_2-4y_8y_4\omega_2^4+5y_7^2\omega_2^2+10y_7z_6\omega_2^3-4y_7y_5y_4\\
&+6y_7y_5y_3\omega_2+21y_7y_4^2\omega_2-10y_7y_4y_3\omega_2^2+4y_7y_4\omega_2^5+2y_7y_3^3\\
&+4z_6^2y_3\omega_2+15z_6y_6y_4-12z_6y_6y_3\omega_2-12z_6y_5^2+4z_6y_5y_3\omega_2^2
\end{aligned}$$

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$$-24z_6y_4^2\omega_2^2-12z_6y_4y_3^2+30z_6y_4y_3\omega_2^3-14z_6y_4\omega_2^6-2y_6^2y_3\omega_2$$
$$+4y_6y_5^2-6y_6y_5y_3\omega_2^2+y_6y_4y_3^2+12y_5^3\omega_2+2y_4^4);$$
$$\begin{aligned}r_{30}={}&z_6^5-(y_{10}+2z_6y_4-y_5^2)^3+(y_{15}+y_{10}y_5+z_6^2y_3-z_6^2\omega_2^3+2z_6y_5y_4-y_5^3)^2\\&+y_8(y_{15}y_7-4y_{15}z_6\omega_2+4y_{15}y_4y_3-3y_{15}y_4\omega_2^3+6y_{10}y_9y_3\\&-3y_{10}y_9\omega_2^3-9y_{10}y_8y_4+12y_{10}y_8y_3\omega_2-9y_{10}y_8\omega_2^4-2y_{10}y_7y_5\\&-24y_{10}z_6^2+48y_{10}z_6y_6+50y_{10}z_6y_5\omega_2-48y_{10}z_6y_4\omega_2^2+24y_{10}z_6y_3^2\\&-6y_{10}z_6y_3\omega_2^3-2y_{10}y_5y_4y_3+6y_9y_8y_5-4y_9y_8y_4\omega_2+8y_9y_8y_3\omega_2^2\\&-y_9y_8\omega_2^5+5y_9y_7z_6-2y_9y_7y_6-2y_9y_7y_5\omega_2+3y_9y_7y_4\omega_2^2\\&-5y_9y_7y_3^2+2y_9y_7y_3\omega_2^3-6y_9z_6^2\omega_2-4y_9z_6y_6\omega_2+44y_9z_6y_5\omega_2^2\\&+4y_9z_6y_4y_3-2y_9z_6y_3^2\omega_2+81y_8^2z_6+14y_8^2y_6-11y_8^2y_5\omega_2\\&+16y_8^2y_4\omega_2^2+11y_8^2y_3^2-4y_8^2y_3\omega_2^3-7y_8y_7^2-19y_8y_7z_6\omega_2-6y_8y_7y_6\omega_2\\&+6y_8y_7y_5\omega_2^2-7y_8y_7y_4y_3-8y_8y_7y_3^2\omega_2+2y_8y_7y_3\omega_2^4+96y_8z_6^2\omega_2^2\\&+24y_8z_6y_6\omega_2^2+44y_8z_6y_5y_3-32y_8z_6y_5\omega_2^3-59y_8z_6y_4^2+108y_8z_6y_4y_3\omega_2\\&-27y_8z_6y_4\omega_2^4-16y_8z_6y_3^2\omega_2^2+2y_8y_6^2\omega_2^2+6y_8y_6y_5y_3+y_8y_6y_4^2\\&-2y_8y_6y_4y_3\omega_2+y_8y_6y_4\omega_2^4+6y_8y_5y_4y_3\omega_2^2-3y_8y_4^2y_3^2+y_8y_4^2y_3\omega_2^3\\&-34y_7^2z_6\omega_2^2+y_7z_6^2y_3-109y_7z_6^2\omega_2^3-4y_7z_6y_6y_3+2y_7z_6y_6\omega_2^3\\&+8y_7z_6y_5y_3\omega_2-24y_7z_6y_4^2\omega_2+4y_7z_6y_4y_3\omega_2^2+y_7y_5^3-51z_6^3y_4\\&-92z_6^3\omega_2^4+102z_6^2y_6y_4-6z_6^2y_6y_3\omega_2+8z_6^2y_6\omega_2^4+98z_6^2y_5y_4\omega_2\\&+96z_6^2y_5y_3\omega_2^2-153z_6^2y_4^2\omega_2^2+55z_6^2y_4y_3^2-z_6^2y_4y_3\omega_2^3-4z_6^2y_3^3\omega_2\\&+12z_6y_6^2y_3\omega_2-4z_6y_6^2\omega_2^4-12z_6y_6y_4^2\omega_2^2+8z_6y_6y_4y_3^2+2z_6y_6y_4y_3\omega_2^3\\&-2z_6y_6y_3^3\omega_2+y_5^3y_4\omega_2^3),\end{aligned}$$

and where $z_6=2y_6+y_3^2+2\omega_2y_5-3\omega_2^2y_4+2\omega_2^3y_3-\omega_2^6$.

We note that results in (3.3) and (3.4) have been shown in [**17**, Theorems 1 and 3].

4. *Computing with Weyl invariants*

As mentioned earlier, our approach to the ring $H^*(G/T)$ amounts to applying Lemma 2.5 to the fibration (3.1). It requires in addition to Lemma 3.1 that:

(i) a presentation for the cohomology of the fiber space $P_{\{k\}}/T$;

(ii) a set $\{\rho_s\}_{1\leqslant s\leqslant m_1}$ of relations on $H^*(G/T)$ satisfying (2.4).

These two tasks will be implemented in Lemmas 4.2 and 4.4, respectively.

The Weyl group W of a Lie group G can be regarded as the subgroup of $\mathrm{Aut}(H^2(G/T))$ generated by the elements $\sigma_1,\dots,\sigma_n\in\mathrm{Aut}(H^2(G/T))$ whose action on the set $\{\omega_1,\dots,\omega_n\}$ of weights is (see [**17**, § 2.1])

$$\sigma_i(\omega_k)=\begin{cases}\omega_i & \text{if } k\neq i,\\ \omega_i-\sum\limits_{1\leqslant j\leqslant n}c_{ij}\omega_j & \text{if } k=i,\end{cases}\ 1\leqslant i\leqslant n, \tag{4.1}$$

where c_{ij} is the Cartan number relative to the pair β_i,β_j, $1\leqslant i,j\leqslant n$, of simple roots. Given a subgroup $W'\subseteq W$ and a weight $\omega\in H^2(G/T)$, let $O(\omega,W')\subset H^2(G/T)$ be the W'-orbit through ω, and write $e_r(O(\omega,W'))\in H^*(G/T)$ for the rth elementary symmetric functions on the set $O(\omega,W')$. In Table 4.1, we define for each simple Lie group $G\neq\mathrm{Spin}(n)$ a set $c_r(G)\in H^*(G/T)$ of polynomials in the weights $\omega_1,\dots,\omega_n$, where $W_{\{i\}}$ is the Weyl group of the parabolic subgroup $P_{\{i\}}\subset G$ specified in Table 3.1.

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EXAMPLE 4.1. The expressions of $c_r(G)$ as polynomials in the weights $\omega_1, \ldots, \omega_n$ can be concretely presented. As examples, in the order of $G = SU(n), Sp(n), F_4, E_6$, we get from the formula (4.1), together with the Cartan matrix of G given in [**21**, p. 59], that

$$\begin{aligned} O(\omega_1, W) &= \{\omega_1, \omega_k - \omega_{k-1}, -\omega_{n-1} \mid 2 \leqslant k \leqslant n-1\}, \\ O(\omega_1, W) &= \{\pm\omega_1, \pm(\omega_k - \omega_{k-1}) \mid 2 \leqslant k \leqslant n\}, \\ O(\omega_4, W_{\{1\}}) &= \{\omega_4, \omega_3 - \omega_4, \omega_2 - \omega_3, \omega_1 - \omega_2 + \omega_3, \omega_1 - \omega_3 + \omega_4, \omega_1 - \omega_4\}, \\ O(\omega_6, W_{\{2\}}) &= \{\omega_6, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\}. \end{aligned}$$

In the following results we clarify the roles of the polynomials $c_r(G)$.

LEMMA 4.2. *If $G = SU(n)$ or $Sp(n)$, the inclusion $\omega_i \subset H^2(G/T)$ induces ring isomorphisms*

$$H^*(SU(n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_{n-1}]/\langle c_2, \ldots, c_n\rangle, \quad c_r = c_r(SU(n)), \tag{4.2}$$

$$H^*(Sp(n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n]/\langle c_2, \ldots, c_{2n}\rangle, \quad c_{2r} = c_{2r}(Sp(n)). \tag{4.3}$$

Proof. For $G = SU(n)$ or $Sp(n)$, we have by Borel [**3**] that

$$H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n]/\langle \mathbb{Z}[\omega_1, \ldots, \omega_n]^{+,W}\rangle,$$

where $\mathbb{Z}[\omega_1, \ldots, \omega_n]^{+,W}$ denotes the set of W-invariants in positive degrees. The lemma is verified by the classical results that the sets of polynomials $c_r(G)$ in (4.2) and (4.3) generate the subrings $\mathbb{Z}[\omega_1, \ldots, \omega_n]^{+,W}$. □

For an exceptional Lie group G, let $P_{\{k\}} \subset G$ be the parabolic subgroup given by Table 3.1, and consider the corresponding fibration

$$P^s_{\{k\}}/T' \overset{i}{\hookrightarrow} G/T \overset{\pi}{\to} G/P_{\{k\}}$$

in flag manifolds, where $P^s_{\{k\}}$ is the simple part of the group $P_{\{k\}}$, and where T' is the maximal torus on $P^s_{\{k\}}$ corresponding to T.

LEMMA 4.3. *For each exceptional Lie group G, the polynomials $c_r(G) \in H^*(G/T)$ defined in Table 4.1 satisfy the following relations:*

(i) $c_r(G) \in \operatorname{Im}[\pi^* : H^*(G/P_{\{k\}}) \to H^*(G/T)]$;

(ii) $i^* c_r(G) = c_r(P^s_{\{k\}})$.

Proof. For any parabolic subgroup $P \subset G$ with Weyl group $W(P)$, the induced map π^* is injective by Lemma 2.3, and satisfies

$$\operatorname{Im}\pi^* = H^*(G/T)^{W(P)} \quad \text{(see [2, Proposition 5.1])}, \tag{4.4}$$

where $H^*(G/T)^{W(P)} \subset H^*(G/T)$ is the subring of $W(P)$-invariants. Property (i) follows from $c_r(G) \in H^*(G/T)^{W_{\{k\}}}$ by the definition of the polynomial $c_r(G)$ in Table 4.1, where $k = 1$ for $G = G_2$ or F_4, and $k = 2$ for $G = E_n$ with $n = 6, 7, 8$.

Note that the induced map i^* is $W_{\{k\}}$-equivariant. Relation (ii) comes from $i^*(O(\omega_t, W_{\{k\}})) = O(i^*\omega_t, W_{\{k\}})$, where $t = 2, 4, 6, 7, 8$ in accordance to $G = G_2, F_4, E_6 \cdot E_7, E_8$. □

TABLE 4.1. *The definition of the polynomials $c_r(G)$.*

G	$SU(n), Sp(n)$	G_2	F_4	$E_n, n = 6, 7, 8$
$c_r(G)$	$e_r(O(\omega_1, W))$	$e_r(O(\omega_2, W_{\{1\}}))$	$e_r(O(\omega_4, W_{\{1\}}))$	$e_r(O(\omega_n, W_{\{2\}}))$

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LEMMA 4.4. *For each exceptional Lie group G, a set $\{\rho_s\}$ of relations on the ring $H^*(G/T)$ satisfying the property (2.4) is given in the table below:*

G	$\{\rho_s\}$
G_2	$3\omega_1 - c_1$; $3\omega_1^2 - c_1$
F_4	$3\omega_1 - c_1$; $4\omega_1^2 - c_2$; $6y_3 - c_3$; $3y_4 + 2\omega_1 y_3 - c_4$; $\omega_1 y_4 - c_5$; $y_6 - c_6$
E_6	$3\omega_2 - c_1$; $4\omega_2^2 - c_2$; $2y_3 + 2\omega_2^3 - c_3$; $3y_4 + \omega_2^4 - c_4$;
	$3\omega_2 y_4 - 2\omega_2^2 y_3 + \omega_2^5 - c_5$; $y_6 - c_6$
E_7	$3\omega_2 - c_1$; $4\omega_2^2 - c_2$; $2y_3 + 2\omega_2^3 - c_3$; $3y_4 + \omega_2^4 - c_4$;
	$2y_5 + 3\omega_2 y_4 - 2\omega_2^2 y_3 + \omega_2^5 - c_5$; $y_6 + 2\omega_2 y_5 - c_6$; $y_7 - c_7$
E_8	$3\omega_2 - c_1$; $4\omega_2^2 - c_2$; $2y_3 + 2\omega_2^3 - c_3$; $3y_4 + \omega_2^4 - c_4$;
	$2y_5 + 3\omega_2 y_4 - 2\omega_2^2 y_3 + \omega_2^5 - c_5$;
	$5y_6 + 2y_3^2 + 6\omega_2 y_5 - 6\omega_2^2 y_4 + 4\omega_2^3 y_3 - 2\omega_2^6 - c_6$;
	$y_7 + 4\omega_2 y_6 + 2\omega_2 y_3^2 + 4\omega_2^2 y_5 - 6\omega_2^3 y_4 + 4\omega_2^4 y_3 - 2\omega_2^7 - c_7$;
	$y_8 - c_8$

where the y_i are the π^*-images of the Schubert classes on $G/P_{\{k\}}$ specified in Table 3.2, $c_r = c_r(G)$, and where the sets $\{\rho_s\}$ are presented by the order of the degrees of the enclosed polynomials ρ_s.

Proof. By Lemma 4.3(i) and by the injectivity of the map π^*, we can regard $c_r(G) \in H^*(G/P_{\{k\}})$; see Convention 2.4. Moreover, since $c_r(G)$ is a polynomial in the Schubert classes $\omega_1, \ldots, \omega_n$, the package of 'Giambelli polynomials' [**17**, § 2.6] is functional to expand it as a polynomial in the special Schubert classes on $H^*(G/P_{\{k\}})$ given in Table 3.2. This yields the relations ρ_r on the ring $H^*(G/T)$ presented in the table.

Finally, by Lemma 4.2 and Lemma 4.3(ii), property (2.4) is satisfied by the set $\{\rho_r\}$ of relations on $H^*(G/T)$. $\square$

REMARK 4.5. Results in Lemma 4.4 have geometric interpretations. Taking $G = E_n$ with $n = 6, 7, 8$ as examples, the subgroup $P_{(2)} = SU(n) \cdot S^1$ has a canonical n-dimensional complex representation that gives rise to a complex n-bundle ξ_n on the Grassmannian $E_n/P_{\{2\}}$ [**1**]. It can be shown that if we let $c_r(\xi_n) \in H^r(E_n/P_{\{2\}})$ be the rth Chern class of ξ_n, $1 \leqslant r \leqslant n$, then

$$c_r(\xi_n) = c_r(G), \quad 1 \leqslant r \leqslant n.$$

In this regard, the relations $\rho_s = 0$ indicate formulae that express the Chern classes $c_r(\xi_n)$ by the special Schubert classes on $E_n/P_{\{2\}}$.

Note that if we let $p : \mathbb{C}P(\xi_n) \to E_n/P_{\{2\}}$ be the complex projective bundle associated to ξ_n, then $\mathbb{C}P(\xi_n) = E_n/P_{\{2,n\}}$, and the projection p agrees with the bundle map induced by the inclusion $P_{\{2,n\}} \subset P_{\{2\}} \subset E_n$ of parabolic subgroups.

5. *The ring $H^*(G/T)$ for exceptional Lie groups*

Summarizing the computation of §§ 3 and 4, we have associated each exceptional Lie group G with a fibration $P^s_{\{k\}}/T' \hookrightarrow G/T \to G/P_{\{k\}}$ in which presentations of the cohomologies of the base and fiber spaces by Schubert classes have been obtained in Theorem 3.1 and Lemma 4.2, respectively. In addition, a set of relations on $H^*(G/T)$ satisfying the condition (2.4) has been determined in Lemma 4.4. Therefore, Lemma 2.5 is directly applicable to yield the following result, where the y_i are the Schubert classes on $G/P_{\{k\}}$ given in Table 3.2, and where $c_r = c_r(G)$ as in Lemma 4.4.

THEOREM 5.1. *For each exceptional Lie group G, the cohomology ring $H^*(G/T)$ has the following presentation:*

$$H^*(G_2/T) = \mathbb{Z}[\omega_1, \omega_2, y_3]/\langle \rho_2, r_3, r_6\rangle, \text{ where} \tag{5.1}$$
$$\rho_2 = 3\omega_1^2 - 3\omega_1\omega_2 + \omega_2^2;$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = y_3^2.$$

$$H^*(F_4/T) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4]/\langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12}\rangle \text{ where} \tag{5.2}$$
$$\rho_2 = c_2 - 4\omega_1^2;$$
$$\rho_4 = 3y_4 + 2\omega_1 y_3 - c_4;$$
$$r_3 = 2y_3 - \omega_1^3;$$
$$r_6 = y_3^2 + 2c_6 - 3\omega_1^2 y_4;$$
$$r_8 = 3y_4^2 - \omega_1^2 c_6;$$
$$r_{12} = y_4^3 - c_6^2.$$

$$H^*(E_6/T) = \mathbb{Z}[\omega_1, \ldots, \omega_6, y_3, y_4]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12}\rangle, \text{ where} \tag{5.3}$$
$$\rho_2 = 4\omega_2^2 - c_2;$$
$$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$$
$$\rho_4 = 3y_4 + \omega_2^4 - c_4;$$
$$\rho_5 = 2\omega_2^2 y_3 - \omega_2 c_4 + c_5;$$
$$r_6 = y_3^2 - \omega_2 c_5 + 2c_6;$$
$$r_8 = 3y_4^2 - 2c_5 y_3 - \omega_2^2 c_6 + \omega_2^3 c_5;$$
$$r_9 = 2y_3 c_6 - \omega_2^3 c_6;$$
$$r_{12} = y_4^3 - c_6^2.$$

$$H^*(E_7/T) = \mathbb{Z}[\omega_1, \ldots, \omega_7, y_3, y_4, y_5, y_9]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_i\rangle,$$
$$\text{where } i \in \{6, 8, 9, 10, 12, 14, 18\} \text{ and where} \tag{5.4}$$
$$\rho_2 = 4\omega_2^2 - c_2;$$
$$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$$
$$\rho_4 = 3y_4 + \omega_2^4 - c_4;$$
$$\rho_5 = 2y_5 - 2\omega_2^2 y_3 + \omega_2 c_4 - c_5;$$
$$r_6 = y_3^2 - \omega_2 c_5 + 2c_6;$$
$$r_8 = 3y_4^2 + 2y_3 y_5 - 2y_3 c_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5;$$
$$r_9 = 2y_9 + 2y_4 y_5 - 2y_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6;$$
$$r_{10} = y_5^2 - 2y_3 c_7 + \omega_2^3 c_7;$$
$$r_{12} = y_4^3 - 4y_5 c_7 - c_6^2 - 2y_3 y_9 - 2y_3 y_4 y_5 + 2\omega_2 y_5 c_6 + 3\omega_2 y_4 c_7 + c_5 c_7;$$
$$r_{14} = c_7^2 - 2y_5 y_9 + 2y_3 y_4 c_7 - \omega_2^3 y_4 c_7;$$
$$r_{18} = y_9^2 + 2y_5 c_6 c_7 - y_4 c_7^2 - 2y_4 y_5 y_9 + 2y_3 y_5^3 - 5\omega_2 y_5^2 c_7.$$

$$H^*(E_8/T) = \mathbb{Z}[\omega_1, \ldots, \omega_8, y_3, y_4, y_5, y_6, y_9, y_{10}, y_{15}]/\langle \rho_i, r_j\rangle,$$
$$\text{where } i \in \{2, 3, 4, 5, 6\}, j \in \{8, 9, 10, 12, 14, 15, 18, 20, 24, 30\} \text{ and where} \tag{5.5}$$
$$\rho_2 = 4\omega_2^2 - c_2;$$
$$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$$

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$$\rho_4 = 3y_4 + \omega_2^4 - c_4;$$
$$\rho_5 = 2y_5 - 2\omega_2^2 y_3 + \omega_2 c_4 - c_5;$$
$$\rho_6 = 5y_6 + 2y_3^2 + 10\omega_2 y_5 - 2\omega_2 c_5 - c_6;$$
$$r_8 = 3c_8 - 3y_4^2 - 2y_3y_5 + 2y_3c_5 - 2\omega_2 c_7 + \omega_2^2 c_6 - \omega_2^3 c_5;$$
$$r_9 = 2y_9 + 2y_4y_5 - 2y_3y_6 - 4\omega_2 y_3 y_5 + \omega_2 c_8 - \omega_2^2 c_7 + \omega_2^3 c_6;$$
$$r_{10} = 3y_{10} + 6y_4z_6 - 2y_5^2 - 2y_3c_7 - \omega_2^2 c_8 + \omega_2^3 c_7;$$
$$r_{12} = y_4^3 - 2y_3y_9 - 2y_3^2 z_6 - 2y_3y_4y_5 - c_6^2 + 4c_6z_6 + 2\omega_2 y_5 c_6 - 2\omega_2 c_5 z_6 - 4y_5c_7 + 3\omega_2 y_4 c_7 + c_5c_7 - 6y_4c_8;$$
$$r_{14} \equiv c_7^2 + 3y_4y_{10} + 6y_4^2 z_6 - 2y_3y_5c_6 - \omega_2^2 y_5 c_7 + \omega_2^3 y_5 c_6 \bmod c_8;$$
$$r_{15} \equiv 2y_{15} + 2y_5(y_{10} - y_5^2 + 2y_4z_6) + 2y_3z_6^2 - 2\omega_2^3 z_6^2 \bmod c_8;$$
$$r_{18} \equiv y_9^2 - 10z_6^3 + 5y_5z_6y_7 - y_4y_7^2 - 2y_4y_5y_9 + 4y_3z_6y_9 + 6y_3y_5y_{10} - 4y_3y_5^3 + 8y_3y_4y_5z_6 + 4y_3^2 z_6^2 - y_3^2 y_5 y_7 - 3\omega_2 y_7 y_{10} + 10\omega_2 y_5 z_6^2 - 10\omega_2 y_4 z_6 y_7 - 2\omega_2 y_3^2 y_5 z_6 - 6\omega_2^2 z_6 y_{10} - 16\omega_2^2 y_4 z_6^2 + 3\omega_2^2 y_4 y_5 y_7 - 3\omega_2^3 y_5 y_{10} + 2\omega_2^3 y_5^3 \bmod c_8;$$
$$r_{20} \equiv 3(y_5^2 - y_{10} - 2y_4z_6)^2 - y_5^3 \rho_5 \bmod c_8;$$
$$r_{24} \equiv 5(2y_6 + y_3^2 + 4\omega_2 y_5 - \omega_2 c_5)^4 - 2y_3^7 \rho_3 \bmod c_8;$$
$$r_{30} \equiv (-y_{15} - y_5y_{10} + y_5^3 - 2y_4y_5z_6 - y_3z_6^2 + \omega_2^3 z_6^2)^2 + (y_5^2 - y_{10} - 2y_4z_6)^3 + (2y_6 + y_3^2 + 4\omega_2 y_5 - \omega_2 c_5)^5 - 6y_6^4 \rho_6 \bmod c_8,$$

in which $z_6 = 2y_6 + y_3^2 + 4\omega_2 y_5 - \omega_2 c_5$. □

Concerning the formulation of the presentations (5.1)–(5.5) in Theorem 5.1, we make the following remarks.

(a) Certain Schubert classes y_k on the base space $G/P_{\{k\}}$ can be eliminated against appropriate relations of the type ρ_k. As an example, when $G = E_7$ the generators y_6, y_7 and the relations ρ_6, ρ_7 can be excluded by the formulae ρ_6 $(y_6 = c_6 - 2\omega_2 y_5)$ and ρ_7 $(y_7 = c_7)$ in Lemma 4.4.

(b) For simplicity, the relations r_k on the ring $H^*(E_8/T)$ with $k \geqslant 14$ are presented after module c_8, while their full expressions have been recorded in (3.5).

(c) Without altering the ideal, higher degree relations of the type r_i may be simplified using the lower degree relations. The main idea of performing such simplifications is the following one: for two ordered subsets $\{f_i\}_{1\leqslant i\leqslant n}$ and $\{h_i\}_{1\leqslant i\leqslant n}$ of a graded polynomial ring with

$$\deg f_1 < \ldots < \deg f_n \quad \text{and} \quad \deg h_1 < \ldots < \deg h_n,$$

write $\{h_i\}_{1\leqslant i\leqslant n} \sim \{f_i\}_{1\leqslant i\leqslant n}$ to denote the statements that $\deg h_i = \deg f_i$ and that $(f_i - h_i) \in \langle f_j \rangle_{1\leqslant j<i}$. Then

$$\{f_i\}_{1\leqslant i\leqslant n} \sim \{h_i\}_{1\leqslant i\leqslant n} \quad \text{implies that } \langle h_1, \ldots, h_n\rangle = \langle f_1, \ldots, f_n\rangle. \tag{5.6}$$

6. *Proofs of Theorems 1.2 and 1.3*

Proof of Theorem 1.2. If $G = SU(n)$ or $Sp(n)$, we have $m = 0$, and the presentation (1.2) is shown by Lemma 4.2. If $G = G_2, F_4, E_6, E_7$, the formula (1.2) is verified by the presentations (5.1)–(5.4).

For $G = E_8$, the presentation (5.5) can be summarized as

$$H^*(E_8/T) = \mathbb{Z}[\omega_1, \ldots, \omega_8, y_r]/\langle e_i, f_j, g_t, \phi\rangle_{1\leqslant i\leqslant 3; 1\leqslant j\leqslant 7, t=1,2,3,5}, \tag{6.1}$$

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where $r \in \{3, 4, 5, 6, 9, 10, 15\}$ and where:

(i) $e_i \in \langle \omega_1, \ldots, \omega_8 \rangle$, $1 \leqslant i \leqslant 3$;

(ii) $f_j = p_j y_{d_j} + \alpha_j$, $p_j \in \{2, 3, 5\}$, $\alpha_j \in \langle \omega_1, \ldots, \omega_8 \rangle$, $1 \leqslant j \leqslant 7$;

(iii) $g_t = y_{d_t}^{k_t} + \beta_t$ with $\beta_t \in \langle \omega_1, \ldots, \omega_8 \rangle$, $t = 1, 2, 3, 5$;

(iv) $\phi = 2y_6^5 - y_{10}^3 + y_{15}^2 + \beta$ with $\beta \in \langle \omega_1, \ldots, \omega_8 \rangle$.

Comparing (1.2) with (6.1), we find that:

(a) the polynomial ϕ in (iv) does not belong to any of the three types e_i, f_j, g_j of relations in Theorem 1.2;

(b) the polynomials g_4, g_6, g_7 in (1.2) required to couple f_4, f_6, f_7 (see Theorem 1.2) are absent in (iii).

However, if we set

$$\begin{cases} g_4 = -12\phi + 5y_6^4 f_4 - 4y_{10}^2 f_6 + 6y_{15} f_7, \\ g_6 = -10\phi + 4y_6^4 f_4 - 3y_{10}^2 f_6 + 5y_{15} f_7, \\ g_7 = 15\phi - 6y_6^4 f_4 + 5y_{10}^2 f_6 - 7y_{15} f_7, \end{cases} \tag{6.2}$$

then the obvious properties

$$g_4, g_6, g_7 \in \langle e_i, f_k, g_s, \phi \rangle; \quad \phi = 2g_4 - g_6 + g_7 \in \langle e_i; f_j, g_j \rangle_{1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 7}$$

with $1 \leqslant i \leqslant 3$, $1 \leqslant k \leqslant 7$, $s = 1, 2, 3, 5$, imply the relation

$$\langle e_i; f_j, g_s, \phi \rangle_{1 \leqslant i \leqslant 3; 1 \leqslant j \leqslant 7, s=1,2,3,5} = \langle e_i; f_j, g_j \rangle_{1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 7}.$$

It shows that the formula (1.2) for $G = E_8$ is identical to (6.1).

For the remaining case $G = \mathrm{Spin}(m)$, let y_k be the Schubert class on $\mathrm{Spin}(2n)/T$ associated to the element $w_k = \sigma[n-k, \ldots, n-2, n-1]$ in the Weyl group of $\mathrm{Spin}(2n)$, $2 \leqslant k \leqslant n-1$. According to Marlin [**27**, Proposition 3], one has the presentation

$$H^*(\mathrm{Spin}(2n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_2, \ldots, y_{n-1}]/\langle \delta_i, \xi_j, \mu_k \rangle$$

with

$$\delta_i := 2y_i - c_i(\omega_1, \ldots, \omega_n), \quad 1 \leqslant i \leqslant n-1,$$
$$\xi_j := y_{2j} + (-1)^j y_j^2 + 2 \sum_{1 \leqslant r \leqslant j-1} (-1)^r y_r y_{2j-r}, \quad 1 \leqslant j \leqslant \left[\frac{n-1}{2}\right],$$
$$\mu_k := (-1)^k y_k^2 + 2 \sum_{2k-n+1 \leqslant r \leqslant k-1} (-1)^r y_r y_{2k-r}, \quad \left[\frac{n}{2}\right] \leqslant k \leqslant n-1,$$

where $c_i(\omega_1, \ldots, \omega_n)$ is the ith elementary symmetric function on the orbit set

$$O(\omega_n, W) = \{\omega_n, \omega_i - \omega_{i-1}, \omega_{n-1} + \omega_n - \omega_{n-2}, \omega_{n-1} - \omega_n, 2 \leqslant i \leqslant n-2\}.$$

In view of the relations of the type ξ_j, we note that the generators y_{2j} with $1 \leqslant j \leqslant [(n-1)/2]$ can be eliminated to yield the compact presentation

$$H^*(\mathrm{Spin}(2n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_3, y_5, \ldots, y_{2[(n-1)/2]-1}]/\langle \delta_i', \mu_k' \rangle, \tag{6.3}$$

where δ_i' and μ_k' are the polynomials obtained from δ_i and μ_k by replacing all the classes y_{2r} by the polynomials

$$(-1)^{r-1} y_r^2 + 2 \sum_{1 \leqslant k \leqslant r-1} (-1)^{k-1} y_k y_{2r-k} \quad \text{(by the relation } \xi_r\text{)}.$$

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For $G = \mathrm{Spin}(2n)$, we have formula (1.2) and it is verified by (6.3). Similarly, one obtains formula (1.2) for $G = \mathrm{Spin}(2n+1)$ from [**27**, Proposition 2].

It remains to show that the numbers n and m in (1.2) satisfy $h(G,T) = n+m+1$. This will be done in the proof of Theorem 1.3. □

The sets of integers appearing in the formula (1.2),

$$\{k,m\}, \quad \{\deg e_i\}_{1\leqslant i\leqslant k}, \quad \{d_j;p_j;k_j\}_{1\leqslant j\leqslant m}, \tag{6.4}$$

can be shown to be invariants of the corresponding Lie group G and will be called *the basic data* of G. With the formula (1.2) being made explicit, all the simple Lie groups in Lemma 4.2, in formulae (5.1)–(5.5) and in formula (6.3), one gets the following corollary.

COROLLARY 6.1. *The basic data of the* 1*-connected simple Lie groups are as given in Tables* 6.1 *and* 6.2.

For a Lie group G, we set $\mathcal{A}(G) := H^*(G/T)/\langle\omega_1,\ldots,\omega_n\rangle$. In the associated short exact sequence of graded rings,

$$0 \to \langle\omega_1,\ldots,\omega_n\rangle \to H^*(G/T) \xrightarrow{p} \mathcal{A}(G) \to 0, \tag{6.5}$$

the quotient map p is clearly given by $p(\alpha) = \alpha|_{\omega_1=\ldots=\omega_n=0}$, $\alpha \in H^*(G/T)$. It follows from the formulas (1.2) and (1.3) that

$$\mathcal{A}(G) = \begin{cases} \dfrac{\mathbb{Z}[y_{d_1},\ldots,y_{d_m}]}{\langle p_i\cdot y_{d_i}, y_{d_i}^{k_i}\rangle_{1\leqslant i\leqslant m}} & \text{if } G \neq E_8, \\[2ex] \dfrac{\mathbb{Z}[y_{d_1},\ldots,y_{d_7}]}{\langle p_i y_{d_i}, y_{d_t}^{k_t}, 2y_{d_4}^5 - y_{d_6}^3 + y_{d_7}^2\rangle_{1\leqslant i\leqslant 7, t=1,2,3,5}} & \text{if } G = E_8. \end{cases} \tag{6.6}$$

Inputting the values of the data $\{d_j;p_j;k_j\}_{1\leqslant j\leqslant m}$ given by Corollary 6.1 one gets, in particular, the following corollary.

TABLE 6.1. *Basic data for the classical groups.*

G	$SU(n+1)$	$Sp(n)$	$\mathrm{Spin}(2n)$	$\mathrm{Spin}(2n+1)$
$\{k,m\}$	$\{n,0\}$	$\{n,0\}$	$\left\{\left[\frac{n+3}{2}\right],\left[\frac{n-2}{2}\right]\right\}$	$\left\{\left[\frac{n+2}{2}\right],\left[\frac{n-1}{2}\right]\right\}$
$\{\deg e_i\}$	$\{2i+2\}$	$\{4i\}$	$\{4t,2n,2^{[\log_2(n-1)]+2}\}_{1\leqslant t\leqslant[(n-1)/2]}$	$\{4t,2^{[\log_2 n]+2}\}_{1\leqslant t\leqslant[n/2]}$
$\{d_j\}$			$\{4j+2\}$	$\{4j+2\}$
$\{p_j\}$			$\{2,\ldots,2\}$	$\{2,\ldots,2\}$
$\{k_j\}$			$\{2^{[\log_2((n-1)/(2j+1))]+1}\}$	$\{2^{[\log_2(n/(2j+1))]+1}\}$

TABLE 6.2. *Basic data for exceptional Lie groups.*

G	G_2	F_4	E_6	E_7	E_8
$\{k,m\}$	$\{1,1\}$	$\{2,2\}$	$\{4,2\}$	$\{3,4\}$	$\{3,7\}$
$\{\deg e_i\}$	$\{4\}$	$\{4,16\}$	$\{4,10,16,18\}$	$\{4,16,28\}$	$\{4,16,28\}$
$\{d_j\}$	$\{6\}$	$\{6,8\}$	$\{6,8\}$	$\{6,8,10,18\}$	$\{6,8,10,12,18,20,30\}$
$\{p_j\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3,2,2\}$	$\{2,3,2,5,2,3,2\}$
$\{k_j\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3,2,2\}$	$\{8,3,4,5,2,3,2\}$

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COROLLARY 6.2. *For the five exceptional Lie groups, one has*

$$\begin{aligned}
&\mathcal{A}(G_2) = \mathbb{Z}[y_3]/\langle 2y_3, y_3^2\rangle;\\
&\mathcal{A}(F_4) = \mathbb{Z}[y_3, y_4]/\langle 2y_3, y_3^2, 3y_4, y_4^3\rangle;\\
&\mathcal{A}(E_6) = \mathbb{Z}[y_3, y_4]/\langle 2y_3, y_3^2, 3y_4, y_4^3\rangle;\\
&\mathcal{A}(E_7) = \mathbb{Z}[y_3, y_4, y_5, y_9]/\langle 2y_3, 3y_4, 2y_5, 2y_9, y_3^2, y_4^3, y_5^2, y_9^2\rangle;\\
&\mathcal{A}(E_8) = \mathbb{Z}[y_3, y_4, y_5, y_6, y_9, y_{10}, y_{15}]/\\
&\qquad\langle 2y_3, 3y_4, 2y_5, 5y_6, 2y_9, 3y_{10}, 2y_{15}, y_3^8, y_4^3, y_5^4, y_9^2, 2y_6^5 - y_{10}^3 + y_{15}^2\rangle.
\end{aligned}$$

Proof of Theorem 1.3. By Theorem 1.2, the numbers of generators and relations in the presentation of the ring $H^*(G/T)$ in (1.2) for $G \neq E_8$, and in (1.3) for $G = E_8$, are both $n + m$. We shall show that this number is minimum with respect to any presentation of the ring $H^*(G/T)$ in the form (1.1) *without the constraint that the generating set* $\{x_1, \ldots, x_k\}$ *consists of Schubert classes on* G/T.

Let $\{s_1, \ldots, s_h\} \subset H^*(G/T)$ be a subset that generates the ring $H^*(G/T)$ multiplicatively. Since the set $\{\omega_1, \ldots, \omega_n\}$ of fundamental weights is a basis of the group $H^2(G/T)$, none of which can be expressed as a polynomial in the lower degree ones, we can assume $h \geqslant n$ and $s_i = \omega_i$ for $1 \leqslant i \leqslant n$. Now (6.5) implies that the quotient ring $\mathcal{A}(G)$ is generated by those $p(s_j)$ with $n+1 \leqslant i \leqslant h$. Since m is the minimal number of generators required to present the quotient ring $\mathcal{A}(G)$ by (6.6), we have further that $h - n \geqslant m$. This shows that $m + n$ is the least number of generators of the ring $H^*(G/T)$. In particular, $h(G, T) = n + m + 1$.

To show that $n+m$ is the least number of relations to characterize $H^*(G/T)$, we can assume, by the remark after Definition 1.1, that $\{h_1, \ldots, h_q\}$ is a set of homogeneous polynomials in $\{\omega_i, y_{d_j}\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ which satisfies

$$H^*(G/T) = \mathbb{Z}[\omega_i, y_{d_j}]_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}/\langle h_1, \ldots, h_q\rangle. \tag{6.7}$$

Then one gets in addition to (6.6) another presentation of the quotient ring:

$$\mathcal{A}(G) = \mathbb{Z}[y_{d_1}, \ldots, y_{d_m}]_{1\leqslant j\leqslant m}/\langle \overline{h}_1, \ldots, \overline{h}_q\rangle, \quad \overline{h}_i = h_i|_{\omega_1 = \ldots = \omega_n = 0}.$$

Comparing this with (6.6) and in view of the sets $\{d_j\}_{1\leqslant j\leqslant m}$, $\{k_j\}_{1\leqslant j\leqslant m}$ of integers given by Corollary 6.1, we can assume further $q \geqslant m$ and $\overline{h}_i = p_i y_{d_i}$ for all $1 \leqslant i \leqslant m$, where the latter is equivalent to

$$h_i = p_i y_{d_i} + \gamma_i \quad \text{with } \gamma_i \in \langle \omega_1, \ldots, \omega_n\rangle, \quad 1 \leqslant i \leqslant m. \tag{6.8}$$

Since $H^*(G/T; \mathbb{Q}) = H^*(G/T) \otimes \mathbb{Q}$, one gets by (6.7) and (6.8) that

$$H^*(G/T; \mathbb{Q}) = \mathbb{Q}[\omega_1, \ldots, \omega_n]/\langle \widetilde{h}_{m+1}, \ldots, \widetilde{h}_q\rangle, \tag{6.9}$$

where $\widetilde{h}_t$, $m + 1 \leqslant t \leqslant q$, is the polynomial obtained from h_t by substituting in $y_{d_i} = -(1/p_i)\gamma_i$ by the relations (6.8). Since the variety G/T is finite dimensional, we must have $\dim H^*(G/T; \mathbb{Q}) < \infty$. Consequently, $q - m \geqslant n$ by (6.9). That is, in the presentation (6.7) one must have $q \geqslant n + m$. This completes the proof. □

Let $D(\mathcal{A}(G))$ be the ideal of decomposable elements of the ring $\mathcal{A}(G)$, and let $q : \mathcal{A}(G) \to \overline{\mathcal{A}}(G) := \mathcal{A}(G)/D(\mathcal{A}(G))$ be the quotient map. In view of (6.6), the graded group $\overline{\mathcal{A}}(G)$ is determined by the data $\{d_j; p_j\}_{1\leqslant j\leqslant m}$ as

$$\overline{\mathcal{A}}(G) = \mathbb{Z} \bigoplus_{1\leqslant j\leqslant m} \overline{\mathcal{A}}^{d_j}(G) \quad \text{with } \overline{\mathcal{A}}^{d_j}(G) = \mathbb{Z}_{p_j}. \tag{6.10}$$

The proof of Theorem 1.3 is applicable to show the following theorem.

THEOREM 6.3. *A set $S = \{x_{d_1}, \ldots, x_{d_m}\}$ of Schubert classes on G/T is special if and only if the class $q \circ p(x_{d_j})$ is a generator of the cyclic group $\overline{\mathcal{A}}^{2d_j}(G)$, $1 \leqslant j \leqslant m$.*

We give an application of Theorem 1.2. For a Lie group G with a maximal torus T, consider the corresponding fibration $\pi : G \to G/T$. In [20], Grothendieck introduced the *Chow ring* $A(G^c)$ for the reductive algebraic group G^c corresponding to G, and proved the relation

$$A(G^c) = \mathrm{Im}\,\{\pi^* : H^*(G/T) \to H^*(G)\}. \tag{6.11}$$

On the other hand, resorting to the Leray–Serre spectral sequence of π, one can show that

$$\mathrm{Im}\,\pi^* = H^*(G/T)/\langle \mathrm{Im}\,\tau\rangle \quad \text{(see [11, Lemma 4.3])}, \tag{6.12}$$

where $\tau : H^1(T) \to H^2(G/T)$ is the transgression in the fibration π [28, p. 185]. Granted with the explicit presentation of the rings $H^*(G/T)$, as well as the formula [11, formula (3.4)] for τ, formula (6.12) is ready to apply to yield formulae for the ring $A(G^c)$ by the Schubert classes on G/T.

As examples, if G is 1-connected, then $\langle \mathrm{Im}\,\tau\rangle = \langle \omega_1, \ldots, \omega_n\rangle$ by [11, formula (3.4)]. Formula (6.12) implies that

$$A(G^c) = \mathcal{A}(G) \quad \text{(see Corollary 6.2)}.$$

Similarly, for the adjoint Lie groups PG with $G = SU(n)$, $Sp(n), E_6$ and E_7m, one has (see [11, formula (6.2)])

$$\begin{aligned}
A(PSU(n)^c) &= \frac{\mathbb{Z}[\omega_1]}{\langle b_r\omega_1^r \mid 1 \leqslant r \leqslant n\rangle} \quad \text{with } b_r = \gcd\{C_n^1, \ldots, C_n^r\},\\
A(PSp(n)^c) &= \frac{\mathbb{Z}[\omega_1]}{\langle 2\omega_1, \omega_1^{2^{r+1}}\rangle}, \quad n = 2^r(2s+1),\\
A(PE_6^c) &= \frac{\mathbb{Z}[\omega_1, y_3', y_4]}{\langle 3\omega_1, 2y_3', 3y_4, x_6'^2, \omega_1^9, y_4^3\rangle}, \quad y_3' = y_3 + \omega_1^3,\\
A(PE_7^c) &= \frac{\mathbb{Z}[\omega_2, y_3, y_4, y_5, y_9]}{\langle 2\omega_2, \omega_2^2, 2y_3, 3y_4, 2y_5, 2y_9, y_3^2, y_4^3, y_5^2, y_9^2\rangle}.
\end{aligned} \tag{6.13}$$

REMARK 6.4. For $G = \mathrm{Spin}(n), G_2$ and F_4, Marlin [27] obtained the ring $A(G^c)$ by Schubert classes on G/T. For the simple Lie groups, Kač [23] computed the algebras $A(G^c) \otimes \mathbb{F}_p$ with generators specified by the degrees.

REMARK 6.5. For the earlier works studying the presentation of the ring $H^*(G/T)$, see [3, 5, 27, 29, 30, 33, 34]. A basic requirement of intersection theory [19] is to present the cohomology $H^*(X)$ of a projective variety X by explicit described geometrical cycles, such as the Schubert classes of flag manifolds, so that the intersection multiplicities can be computed by the cup products on the ring $H^*(X)$. In this regard, the approaches due to Bott–Samelson [5] and Marlin [27] are inspiring.

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