INTEGRALS INVOLVING PRODUCTS OF MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

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§ 1. *Introductory.* The formula to be proved is

$$\int_0^\infty e^{-\zeta\lambda}\lambda^{k-1}K_m(\lambda)K_n(z/\lambda)\,d\lambda$$
$$=\sum_{n,-n}\frac{\Gamma(k+m+n)\Gamma(k-m+n)}{\Gamma(k+n+\frac{1}{2})2^{k+1}}\Gamma(\tfrac{1}{2})\Gamma(n)z^{-n}$$
$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}n\,;\,r)(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}n\,;\,r)(\frac{1}{4}z^2)^r}{r!(1-n;r)(1-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n;r)(1-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n;r)(\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n;r)(\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n;r)}$$
$$\times F(\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n-r,\ \tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-r\,;\ k+n+\tfrac{1}{2}-2r\,;\ 1-\zeta^2)$$
$$+\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n)\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n)\Gamma(-m)2^{-m-3}(\tfrac{1}{2}z)^{k+m}$$
$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{4}+\frac{1}{2}m\,;\,r)(\frac{3}{4}+\frac{1}{2}m\,;\,r)(\frac{1}{4}z^2)^r}{r!(1+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n\,;\,r)(1+\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n\,;\,r)(\frac{1}{2}\,;\,r)(\frac{1}{2}+m\,;\,r)(1+m\,;\,r)}$$
$$\times F(-r,\ -m-r\,;\ \tfrac{1}{2}-m-2r\,;\ 1-\zeta^2)$$
$$-\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(-m)2^{-m-3}(\tfrac{1}{2}z)^{k+m+1}$$
$$\times\sum_{r=0}^{\infty}\frac{(\frac{3}{4}+\frac{1}{2}m\,;\,r)(\frac{5}{4}+\frac{1}{2}m\,;\,r)(\frac{1}{4}z^2)^r}{r!(\frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n\,;\,r)(\frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n\,;\,r)(\frac{3}{2}\,;\,r)(1+m\,;\,r)(\frac{3}{2}+m\,;\,r)}$$
$$\times F(-\tfrac{1}{2}-r,\ -\tfrac{1}{2}-m-r\,;\ -\tfrac{1}{2}-m-2r\,;\ 1-\zeta^2). \qquad\text{...........(1)}$$

The integral converges if $R(z)>0$, $R(\zeta)>-1$. The series on the right converge if $|1-\zeta^2|<1$. It will be assumed that ζ is interior to the right-hand loop of the curve $|\zeta^2-1|=1$. When $\zeta=1$ this formula reduces to one given by Ragab (**1**). Formula (1) expresses the integral in series of powers of z.

The formula (**2**)

$$\int_0^\infty \lambda^{l-1}K_m(\lambda)K_n(z/\lambda)d\lambda$$
$$=\sum_{n,-n}\Gamma(\tfrac{1}{2}l+\tfrac{1}{2}m+\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}l-\tfrac{1}{2}m+\tfrac{1}{2}n)\Gamma(n)2^{l+2n-3}z^{-n}$$
$$\times F(\,;\ 1-n,\ 1-\tfrac{1}{2}l-\tfrac{1}{2}m-\tfrac{1}{2}n,\ 1-\tfrac{1}{2}l+\tfrac{1}{2}m-\tfrac{1}{2}n\,;\ z^2/16)$$
$$+\sum_{m,-m}\Gamma(-\tfrac{1}{2}l-\tfrac{1}{2}m-\tfrac{1}{2}n)\Gamma(-\tfrac{1}{2}l-\tfrac{1}{2}m+\tfrac{1}{2}n)\Gamma(-m)2^{-l-2m-3}z^{l+m}$$
$$\times F(\,;\ 1+m,\ 1+\tfrac{1}{2}l+\tfrac{1}{2}m+\tfrac{1}{2}n,\ 1+\tfrac{1}{2}l+\tfrac{1}{2}m-\tfrac{1}{2}n\,;\ z^2/16), \qquad\text{..................(2)}$$

where $R(z)>0$, will be required in the proof.

Other formulae required are

$$F\begin{pmatrix}\alpha,\beta;z\\ \gamma\end{pmatrix}=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F\begin{pmatrix}\alpha,\beta\quad;1-z\\ \alpha+\beta-\gamma+1\end{pmatrix}$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F\begin{pmatrix}\gamma-\alpha,\ \gamma-\beta;\ 1-z\\ \gamma-\alpha-\beta+1\end{pmatrix}, \qquad\text{......(3)}$$

$$F(\alpha,\beta\,;\ \gamma\,;\ z)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta\,;\ \gamma\,;\ z) \qquad\text{.......................(4)}$$

and

$$\Gamma(\tfrac{1}{2})\,\Gamma(2z)=\Gamma(z)\,\Gamma(z+\tfrac{1}{2})2^{2z-1}. \qquad\text{..............................(5)}$$

Formula (1) will be established in section 2. The case when $-1<R(\zeta)<0$ will be considered in section 3.

§ 3. *Proof of the Formula.* Expand the exponential function on the left of (1) in powers of $\zeta\lambda$, and apply (2) to each term, so getting

$$\sum_{n,-n} 2^{k+2n-3}\Gamma(n)z^{-n}\sum_{p=0}^{\infty}\frac{(-2\zeta)^p}{p!}\Gamma\left(\frac{k+p+m+n}{2}\right)\Gamma\left(\frac{k+p-m+n}{2}\right)$$
$$\times F\left(\ ;1-n,\ 1-\frac{k+p+m+n}{2},\ 1-\frac{k+p-m+n}{2};\ \frac{z^2}{16}\right)$$
$$+\sum_{m,-m} 2^{-k-2m-3}\Gamma(-m)z^{k+m}\sum_{p=0}^{\infty}\frac{(-\frac{1}{2}\zeta z)^p}{p!}\Gamma\left(\frac{-k-p-m-n}{2}\right)\Gamma\left(\frac{-k-p-m+n}{2}\right)$$
$$\times F(\ ;1+m,\ 1+\tfrac{1}{2}k+\tfrac{1}{2}p+\tfrac{1}{2}m+\tfrac{1}{2}n,\ 1+\tfrac{1}{2}k+\tfrac{1}{2}p+\tfrac{1}{2}m-\tfrac{1}{2}n;\ z^2/16). \quad\text{.....................(A)}$$

Now in the inner summation in the first two lines of (A) the coefficient of $(z^2/16)^r$ is

$$\frac{1}{r!(1-n;\ r)}\sum_{p=0}\frac{(-2\zeta)^p}{p!}\Gamma\left(\frac{k+p+m+n}{2}-r\right)\Gamma\left(\frac{k+p-m+n}{2}-r\right)$$
$$=\frac{1}{r!(1-n;\ r)}$$
$$\times\left[\begin{array}{l}\Gamma\left(\frac{k+m+n}{2}-r\right)\Gamma\left(\frac{k-m+n}{2}-r\right)F\left(\frac{k+m+n}{2}-r,\ \frac{k-m+n}{2}-r;\ \frac{1}{2};\ \zeta^2\right)\\ -2\zeta\Gamma\left(\frac{k+m+n+1}{2}-r\right)\Gamma\left(\frac{k-m+n+1}{2}-r\right)F\left(\frac{k+m+n+1}{2}-r,\frac{k-m+n+1}{2}-r;\ \frac{3}{2};\ \zeta^2\right)\end{array}\right].$$

Here apply formula (3), and the expression in the bracket becomes

$$\frac{\Gamma(\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n-r)\Gamma(\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n-r)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-k-n+2r)}{\Gamma(\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n+r)\Gamma(\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n+r)}$$
$$\times F\left(\begin{matrix}\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n-r,\ \frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n-r;\ 1-\zeta^2\\ \frac{1}{2}+k+n-2r\end{matrix}\right)$$
$$+\Gamma(\tfrac{1}{2})\Gamma(k+n-\tfrac{1}{2}-2r)(1-\zeta^2)^{\frac{1}{2}-k-n+2r}F\left(\begin{matrix}\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n+r,\ \frac{1}{2}-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n+r:\ 1-\zeta^2\\ \frac{3}{2}-k-n+2r\end{matrix}\right)$$
$$-2\zeta\frac{\Gamma(\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}-r)\Gamma(\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}-r)\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}-k-n+2r)}{\Gamma(1-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n+r)\Gamma(1-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n+r)}$$
$$\times F\left(\begin{matrix}\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}-r,\ \frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}-r;\ 1-\zeta^2\\ \frac{1}{2}+k+n-2r\end{matrix}\right)$$
$$-2\zeta\Gamma(\tfrac{3}{2})\Gamma(k+n-\tfrac{1}{2}-2r)(1-\zeta^2)^{\frac{1}{2}-k-n+2r}F\left(\begin{matrix}1-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n+r,\ 1-\frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}n+r;\ 1-\zeta^2\\ \frac{3}{2}-k-n+2r\end{matrix}\right).$$

On applying (4) to the hypergeometric functions in the last two lines it is seen that the expressions in the second and fourth lines cancel; while the expressions in the first and third lines reduce to

$$\Gamma(\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n-r)\Gamma(\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2}-r)\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-r)\Gamma(\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2}-r)\Gamma(\tfrac{1}{2})$$
$$\times\frac{1}{\pi^2}\left\{\cos\left(\frac{k+m+n}{2}\pi\right)\cos\left(\frac{k-m+n}{2}\pi\right)-\sin\left(\frac{k+m+n}{2}\pi\right)\sin\left(\frac{k-m+n}{2}\pi\right)\right\}$$
$$\times\frac{\pi}{\cos(k+n)\pi\,\Gamma(\frac{1}{2}+k+n-2r)}F\left(\begin{matrix}\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n-r,\ \frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n-r;\ 1-\zeta^2\\ \frac{1}{2}+k+n-2r\end{matrix}\right).$$

From this, making use of formula (5), the first part of the right-hand side of (1) is obtained.

Again, in the inner summation in lines 3 and 4 of (A) the coefficient of $(z^2/16)^r$ is

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$$\frac{\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n)\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n)}{r!(1+m;\ r)(1+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n;\ r)(1+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;\ r)}F\begin{pmatrix}-r,\ -m-r;\ \zeta^2\\ \tfrac{1}{2}\end{pmatrix},$$

and, from (3), the hypergeometric function is equal to

$$\frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}+m+2r)}{\Gamma(\tfrac{1}{2}+r)\Gamma(\tfrac{1}{2}+m+r)}F\begin{pmatrix}-r,\ -m-r;\ 1-\zeta^2\\ \tfrac{1}{2}-m-2r\end{pmatrix},$$

since $1/\Gamma(-r)=0$. This gives the second part of the right-hand side of (1).

Finally, in the inner summation in lines (3) and (4) of (A) the coefficient of $z(z^2/16)^r$ is

$$-\tfrac{1}{2}\zeta\frac{\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2})}{r!(1+m;\ r)(\tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n;\ r)(\tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;\ r)}F\begin{pmatrix}-r,\ -m-r;\ \zeta^2\\ \tfrac{3}{2}\end{pmatrix},$$

the hypergeometric function being equal to

$$\frac{\Gamma(\tfrac{3}{2})\Gamma(\tfrac{3}{2}+m+2r)}{\Gamma(\tfrac{3}{2}+r)\Gamma(\tfrac{3}{2}+m+r)}F\begin{pmatrix}-r,\ -m-r;\ 1-\zeta^2\\ -\tfrac{1}{2}-m-2r\end{pmatrix}.$$

On applying (4) the final part of (1) is obtained.

§ 3. *Evaluation of the Integral for other values of the Parameter.* If $0<R(\zeta)<1$, while ζ lies within the right-hand loop of the curve $|\zeta^2-1|=1$, and assuming that $R(z)>0$, it can be seen, on replacing ζ by $-\zeta$ in the above proof, that

$$\int_0^\infty e^{\zeta\lambda}\lambda^{k-1}K_m(\lambda)K_n(z/\lambda)\,d\lambda$$

$$=\frac{\cos m\pi}{\pi}\sum_{n,-n}\Gamma(k+m+n)\Gamma(k-m+n)\Gamma(\tfrac{1}{2}-k-n)2^{-k-1}\Gamma(\tfrac{1}{2})\Gamma(n)z^{-n}$$

$$\times\sum_{r=0}^{\infty}\frac{(\tfrac{1}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n;\ r)(\tfrac{3}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n;\ r)(\tfrac{1}{4}z^2)^r}{r!(1-n;r)(1-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n;r)(1-\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;r)(\tfrac{1}{2}-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n;r)(\tfrac{1}{2}-\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;r)}$$

$$\times F(\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n-r,\ \tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-r;\ k+n+\tfrac{1}{2}-2r;\ 1-\zeta^2)$$

$$+\sum_{n,-n}2^{k+2n-2}\Gamma(\tfrac{1}{2})\Gamma(n)\Gamma(k+n-\tfrac{1}{2})z^{-n}(1-\zeta^2)^{\frac{1}{2}-k-n}$$

$$\times\sum_{r=0}^{\infty}\frac{(z^2/64)^r(1-\zeta^2)^{2r}}{r!(1-n;\ r)(\tfrac{3}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n;\ r)(\tfrac{5}{4}-\tfrac{1}{2}k-\tfrac{1}{2}n;\ r)}$$

$$\times F(\tfrac{1}{2}-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n+r,\ \tfrac{1}{2}-\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n+r;\ \tfrac{3}{2}-k-n+2r;\ 1-\zeta^2)$$

$$+\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n)\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n)\Gamma(-m)2^{-m-3}(\tfrac{1}{2}z)^{k+m}$$

$$\times\sum_{r=0}^{\infty}\frac{(\tfrac{1}{4}+\tfrac{1}{2}m;\ r)(\tfrac{3}{4}+\tfrac{1}{2}m;\ r)(\tfrac{1}{4}z^2)^r}{r!(1+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n;\ r)(1+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;\ r)(\tfrac{1}{2};\ r)(\tfrac{1}{2}+m;\ r)(1+m;\ r)}$$

$$\times F(-r,\ -m-r;\ \tfrac{1}{2}-m-2r;\ 1-\zeta^2)$$

$$+\sum_{m,-m}\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(-m)2^{-m-3}(\tfrac{1}{2}z)^{k+m+1}$$

$$\times\sum_{r=0}^{\infty}\frac{(\tfrac{3}{4}+\tfrac{1}{2}m;\ r)(\tfrac{5}{4}+\tfrac{1}{2}m;\ r)(\tfrac{1}{4}z^2)^r}{r!(\tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m+\tfrac{1}{2}n;\ r)(\tfrac{3}{2}+\tfrac{1}{2}k+\tfrac{1}{2}m-\tfrac{1}{2}n;\ r)(\tfrac{3}{2};\ r)(1+m;\ r)(\tfrac{3}{2}+m;\ r)}$$

$$\times F(-\tfrac{1}{2}-r,\ -\tfrac{1}{2}-m-r;\ -\tfrac{1}{2}-m-2r;\ 1-\zeta^2). \qquad \text{..................(6)}$$

Note. If in (6) $\zeta=1$ and $R(k\pm n)<\tfrac{1}{2}$, while $R(z)>0$, the integral is convergent, and its value is obtained by putting $\zeta=1$ on the R.H.S. Then the second expression on the right vanishes and the three hypergeometric functions reduce to unity.

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