The Intrinsic Properties of Curves

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Any property of a curve which depends solely upon its form may be styled intrinsic. Thus the circle of curvature at a point, the relation between s and ψ, the envelope of the normals, the envelope of straight lines making a constant angle with the curve, etc., are all intrinsic properties. It is proposed to investigate a few of these properties in this paper.*

I.

(1) Let a straight line PQ make a constant angle λ with a given curve at any point P; then the envelope of PQ may be geometrically found as follows.

If PC, P′C (Fig. 17) be consecutive normals to the given curve, and PQ, P′Q consecutive positions of the straight line making a constant angle with the curve, then evidently TPCQP′ is a circle. Therefore $\angle$ CQT is right and ultimately $\angle$ CQP is a right angle. Hence the envelope touches the straight line at the foot of the perpendicular from C, the centre of curvature. Also $PQ = \rho\sin\lambda$, where ρ is the radius of curvature at P.

(2) Again, if Q′ be a neighbouring point on the envelope,

$$QQ' = P'Q' - PQ + PP'\cos\lambda,$$

i.e., $\delta s' = (\rho + \delta\rho)\sin\lambda - \rho\sin\lambda + \delta s\cos\lambda = \delta\rho\sin\lambda + \delta s\cos\lambda$

the accented letters referring to the locus of Q.

Also $\delta\psi' = \delta\psi$, since $\psi' - \psi = \lambda$

$$\therefore \frac{\delta s'}{\delta\psi'} = \frac{\delta\rho}{\delta\psi}\sin\lambda + \frac{\delta s}{\delta\psi}\cos\lambda$$

$$\therefore \rho' \text{ or } \frac{ds'}{d\psi'} = \rho_1\sin\lambda + \rho\cos\lambda, \text{ where } \rho_1 = \frac{d\rho}{d\psi}$$

which is equal to the radius of curvature of the evolute.

* [Full references to the literature of the problem dealt with in I. will be found in Loria, *Spezielle algebraische und transscendente ebene Kurven.* Leipzig, 1902, p. 626. Ed. *E.M.S.P.*]

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(3) Let E be the centre of curvature of the evolute; then, if EN be perpendicular to QC, $QN = \rho\cos\lambda + \rho_1\sin\lambda$, so that N is the centre of curvature of the locus of Q. In other words, the centre of curvature at Q is the projection of the centre of curvature of the evolute on the normal at Q; and it is related to the evolute just as Q is related to the original curve.

(4) Further, since $\delta s' = \delta\rho\sin\lambda + \delta s\cos\lambda$, by integrating we have $s' = \rho\sin\lambda + s\cos\lambda +$ a constant. From this the intrinsic equation to the envelope of PQ may be expressed.

(5) From the relation $\rho' = \rho_1\sin\lambda + \rho\cos\lambda_1$ we can deduce that

(i) $$\frac{d\rho'}{d\psi'} = \rho_2\sin\lambda + \rho_1\cos\lambda$$

(ii) $$\frac{d^2\rho'}{d\psi'^2} = \rho_3\sin\lambda + \rho_2\cos\lambda$$

and so on; which means that the centre of curvature of the evolute of Q is the projection of the centre of curvature of the second evolute of the original curve, and so on.

(6) The locus of Q may be regarded as the involute of a curve similarly related to the evolute of the original. Also a system of parallel curves (P) will have a system of parallel curves, (Q), whose evolutes, viz., (C) and (N) are similarly related.

These considerations show that the locus of N may be styled an *oblique* (λ) evolute of the original curve P.

II.

(1) Let PQ be the diameter of curvature at P and CD that of the evolute, then the locus of Q will have QD for normal.

For if PQ, P′Q′ (Fig. 18) be consecutive positions of the diameter, then $PQ = 2\rho$ and $P'Q' = 2(\rho + \delta\rho)$. By projection we have in the limit

$$\tan\phi = \mathrm{L}\,\frac{QR}{Q'R} = \mathrm{L}\,\frac{\rho\delta\psi}{2\delta\rho} = \frac{\rho}{2\rho_1}$$

Hence $\phi = \angle QDC$, that is DQ is the normal at Q.

(2) If QQ′ be denoted by $\delta s'$,

$$(\delta s')^2 = QR^2 + Q'R^2$$

$$\therefore \left(\frac{\delta s'}{\delta\psi}\right)^2 = \left(\frac{\delta s}{\delta\psi}\right)^2 + 4\left(\frac{\delta\rho}{\delta\psi}\right)^2 \text{ and } \therefore \left(\frac{ds'}{d\psi}\right)^2 = \rho^2 + 4\rho_1^2 = QD^2,$$

whence $$\frac{ds'}{d\psi} = QD.$$

Also if the tangents at Q and P make angles ψ' and ψ with a fixed straight line,

$$\psi = \psi' + \frac{\pi}{2} - \phi$$

and therefore

$$\frac{d\psi'}{d\psi} = 1 + \frac{d\phi}{d\psi}. \quad \text{But } \sec^2\phi\frac{d\phi}{d\psi} = \frac{\rho_1^2 - \rho\rho_2}{2\rho_1^2},$$

therefore $$\frac{d\psi'}{d\psi} = 1 + \frac{2(\rho_1^2 - \rho\rho_2)}{4\rho_1^2 + \rho^2}.$$

Hence $$\rho' = \frac{ds'}{d\psi'} = \frac{ds'}{d\psi}\Big/\frac{d\psi'}{d\psi} = \frac{(\rho^2 + 4\rho_1^2)^{\frac{3}{2}}}{\rho^2 + 6\rho_1^2 - \rho\rho_2}.$$

(3) The centre of curvature at Q may be geometrically determined as follows.

Differentiating $$2\rho_1\sin\phi - \rho\cos\phi = 0$$

we get $$\frac{d\phi}{d\psi} = \frac{\rho_1\cos\phi - 2\rho_2\sin\phi}{2\rho_1\cos\phi + \rho\sin\phi}.$$

Now, if EF be the diameter of curvature at E of the second evolute and H the projection of F on QD.

$$\frac{d\phi}{d\psi} = -\frac{HD}{QD} \text{ and therefore } \frac{d\psi'}{d\psi} = 1 - \frac{DH}{TD} = \frac{TH}{TD}$$

Hence $\rho' = QD^2/TH$, since $QD = TD$.

III.

More generally, let PQ be the chord of curvature subtending a constant angle 2λ at the centre, then the properties of the locus of Q may be similarly determined.

1. PQ, P′Q′ (Fig. 19) being consecutive positions, it is obvious that they intersect at N so that C, N, P, P′ are cyclic, and therefore CN is ultimately perpendicular to PN.

If PQ be projected on P′Q′, we see that, in the limit,

$$\tan\phi = \rho\sin\lambda/(2\rho_1\sin\lambda - \rho\cos\lambda)$$

(2) Let CD be the chord of curvature of the evolute subtending an angle 2λ at its centre E; then

$$\tan\phi = \frac{\mathrm{QN}}{\mathrm{CD} - \mathrm{CN}} = \frac{\mathrm{QN}}{\mathrm{ND}} = \tan\mathrm{QDN}.$$

Hence $\phi = \mathrm{QDN}$; that is, the normal at Q passes through D as in case II.

(3) Also $\qquad (\mathrm{QQ}')^2 = (\mathrm{Q}q)^2 + (\mathrm{Q}'q)^2$

$$\therefore \left(\frac{ds'}{d\psi}\right)^2 = \rho^2\sin^2\lambda + (2\rho_1\sin\lambda - \rho\cos\lambda)^2$$

$$= \rho^2 - 4\rho\rho_1\sin\lambda\cos\lambda + 4\rho_1{}^2\sin^2\lambda$$

But $\qquad \psi + \phi - \psi' = \lambda$ and therefore $\dfrac{d\psi'}{d\psi} = 1 + \dfrac{d\phi}{d\psi}$

$$\therefore \rho' \equiv \frac{ds'}{d\psi'} = \frac{(\rho^2 + 4\rho_1{}^2\sin^2\lambda - 4\rho\rho_1\sin\lambda\cos\lambda)^{\frac{3}{2}}}{\rho^2 - 4\rho\rho_1\sin\lambda\cos\lambda + 6\rho_1{}^2\sin^2\lambda - 2\rho\rho_2\sin^2\lambda}$$

after reduction.

(4) Again $\qquad \tan\phi = \rho\sin\lambda/(2\rho_1\sin\lambda - \rho\cos\lambda)$

may be written in the form

$$(2\rho_1\sin\lambda - \rho\cos\lambda)\sin\phi - \rho\sin\lambda\cos\phi = 0$$

Differentiating with respect to ψ, we have

$$\{(2\rho_2\sin\lambda - \rho\cos\lambda)\sin\phi - \rho_1\sin\lambda\cos\phi\} +$$
$$\{(2\rho_1\sin\lambda - \rho\cos\lambda)\cos\phi + \rho\sin\lambda\sin\phi\}\frac{d\phi}{d\psi} = 0$$

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this leads to the geometrical construction in Figure 20 where EF is the chord of curvature of the second evolute subtending an angle 2λ at its centre. Hence as in II. we find

(i) $$\frac{d\phi}{d\psi} = -\frac{\text{DH}}{\text{QD}},$$

(ii) $$\frac{d\psi'}{d\psi} = \frac{\text{TH}}{\text{TD}},$$

and (iii) $$\rho' = \text{QD}^2/\text{TH}.$$

IV.

If in II. or III. the tangent at Q be drawn to the circle of curvature, the envelope of this tangent is closely connected with the evolute of P.

(1) If R (Fig. 21) is the intersection of consecutive tangents RQ′QN is a circle, and therefore $\angle$QRN is ultimately equal to $\phi = \angle$QDN. Thus QNDR is cyclic and $\angle$QRD is a right angle. Hence the envelope touches the tangent at the foot of the perpendicular from D, where CD is the chord of curvature of the evolute subtending the angle 2λ at the centre. Also

$$\text{QR} = \text{CD}\sin\lambda = 2\rho_1\sin^2\lambda.$$

(2) If R′ be a neighbouring point on this envelope

$$\begin{aligned} \text{RR}' &= -\text{QQ}'\cos(\phi+\lambda) + \text{Q}'\text{R}' - \text{QR} \\ &= -\text{QQ}'(\cos\phi\cos\lambda - \sin\phi\sin\lambda) + 2(\rho_1 + \delta\rho_1)\sin^2\lambda - 2\rho_1\sin^2\lambda \\ &= -\cos\lambda(2\rho_1\sin\lambda - \rho\cos\lambda)\delta\psi + \rho\sin^2\lambda\delta\psi + 2\delta\rho_1\sin^2\lambda \end{aligned}$$

$$\therefore\ \rho' = \frac{ds'}{d\psi'} = 2\rho_2\sin^2\lambda + \rho - 2\rho_1\sin\lambda\cos\lambda,$$

since $\psi' - \psi = \pi - 2\lambda$ and therefore $\delta\psi' = \delta\psi$.

(3) If EF be the chord of the second evolute as in III., its projection is $\text{DG} = 2\rho_2\sin^2\lambda$, and RD = sum of projections of $\text{CQ}_1\text{CD} = \rho - 2\rho_1\sin\lambda\cos\lambda$. Hence $\text{RG} = \rho'$, that is the centre of curvature of the envelope is the projection of F on the normal at R.

V.

If on the tangent at P a length PQ equal to the radius of curvature be measured, the locus of Q has the following properties.

By projection we have (Fig. 22)

$$\tan\phi = \mathrm{L}\ \rho d\psi/(\mathrm{PQ} + \mathrm{PP}' - \mathrm{P}'\mathrm{Q}') = \mathrm{L}\ \rho\delta\psi/(\delta s - \delta\rho)$$
$$= \rho/(\rho - \rho_1) = \tan \mathrm{EPQ},$$

if E be the centre of curvature of the evolute. Hence the tangent at Q is the reflection of EQ in the tangent at P.

Again $$(\mathrm{QQ}')^2 = \rho^2\delta\psi^2 + (\delta s - \delta\rho)^2$$

and therefore $$\frac{ds'}{d\psi'} = \mathrm{EQ}. \quad \text{But } \psi' = \psi - \phi,$$

whence $$\rho' = \frac{ds'}{d\psi'} = \frac{ds'}{d\psi} \Big/ \frac{d\psi'}{d\psi} = \frac{\{\rho^2 + (\rho - \rho)^2\}^{\frac{3}{2}}}{2\rho^2 - 2\rho\rho_1 + 2\rho_1^2 - \rho\rho_2}$$

If a length be measured on the opposite side of the tangent, we find $\tan\phi = \rho/(\rho + \rho_1)$, etc.

For any length $\kappa\rho$ measured along the tangent, similar results may be established with slight modifications.