MONOTONE HOMOMORPHISMS OF COMPACT SEMIGROUPS

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1. Introduction

The problem of determining the class of homomorphic images of a given class of topological semigroups seems to have received little attention in the literature. In [4] Cohen and Krule determined the homomorphic images of a semigroup with zero on an interval. Anderson and Hunter in [1] proved several theorems in this direction. In general, the problem seems to be rather difficult. However, the difficulty is lessened somewhat if all of the homomorphisms of the semigroups in question must be monotone. Phillips, [7], showed that every homomorphism of a standard thread is monotone and hence every homomorphic image of a standard thread is either a standard thread or a point. In this paper a larger class of topological semigroups which admit only monotone homomorphisms is given. These results are used to determine the topological nature of the homomorphic images of certain classes of topological semigroups. These include products of standard threads with min threads, certain semilattices on a two-cell, and compact connected lattices in the plane.

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2. Preliminaries

In this paper, the term '*semigroup*' will mean topological semigroup. A *homomorphism* from a semigroup S into a semigroup T is a continuous function $f : S \to T$ satisfying $f(ab) = f(a)f(b)$ for all $a, b \in S$. A *right* (*left*) *congruence* on a semigroup S is a subset $\mathscr{C}$ of $S \times S$ which is an equivalence relation and satisfies the property: $(xz, yz) \in \mathscr{C}((zx, zy) \in \mathscr{C})$ if $(x, y) \in \mathscr{C}$ and $z \in S$. If $\mathscr{C}$ is both a left and a right congruence, then $\mathscr{C}$ will be called a *congruence*. A homomorphism f from a semigroup S onto a semigroup T gives rise to the congruence $\mathscr{C} = \{(x, y) \in S \times S | f(x) = f(y)\}$. In this case $\mathscr{C}$ is a closed congruence, that is, is a closed subset of $S \times S$.

A *standard thread* I is a semigroup whose underlying space is homo-

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morphic to a closed interval of real numbers such that one endpoint is a zero for I and the other endpoint is an identity for I. A *min thread* is a thread which is isomorphic to the interval $[0, 1]$ with multiplication defined by $xy = \min\{x, y\}$. A min thread will be denoted by I_1. A *usual thread* (denoted by I_2) is a thread which is isomorphic to $[0, 1]$ with the usual multiplication of real numbers. A *nilpotent thread* (denoted by I_3) is a thread which is isomorphic to $[\frac{1}{2}, 1]$ with multiplication defined by $x \circ y = \max\{xy, \frac{1}{2}\}$ where xy denotes the usual product. For a complete description of the structure of threads, the reader is referred to [3] or [6]

If S is a compact semigroup and $x \in S$, then the set

$$\Gamma(x) = \{x, x^2, x^3, \cdots\}^*$$

is a compact subsemigroup of S whose minimal ideal $K(x)$ is a compact topological group consisting of the cluster points of the sequence $\{x, x^2, x^3, \cdots\}$. For details see Koch [5]. (A^* denotes the topological closure of A.)

Familiarity with the cyclic element theory of Whyburn, [10], will be assumed.

3. Congruences on compact semigroups

If A and B are subsets of a semigroup S, we define:

$$B \mathbin{.\cdot} A = \{t \in S | At \subset B\},$$

and

$$B \mathbin{\cdot.} A = \{t \in S | tA \subset B\}.$$

THEOREM 3.1. *Let $\mathscr{C}$ be a closed right congruence on a compact semigroup S, and let $(a, ax) \in \mathscr{C}$ for some $a, x \in S$. Then $\{a\} \times a[K(x) \mathbin{.\cdot} K(x)] \subset \mathscr{C}$.*

PROOF. Since $\mathscr{C}$ is a right congruence and $(a, ax) \in \mathscr{C}$, we have $(ax, ax^2) \in \mathscr{C}$ and by the transitivity of $\mathscr{C}$, $(a, ax^2) \in \mathscr{C}$. Iterating this procedure, we obtain $(a, ax^n) \in \mathscr{C}$ for each positive integer n. Since $\mathscr{C}$ is closed, it follows that $(a, ak) \in \mathscr{C}$ for each $k \in K(x)$.

Now let $t \in K(x) \mathbin{.\cdot} K(x)$. Then $(at, ax^n t) \in \mathscr{C}$ for each positive integer n. Fix $k \in K(x)$; then $(at, akt) \in \mathscr{C}$. But $kt \in K(x)$, and by the preceding paragraph $(a, akt) \in \mathscr{C}$. We conclude then that $(a, at) \in \mathscr{C}$.

THEOREM 3.1[1]. *Let $\mathscr{C}$ be a closed left congruence on a compact semigroup S and let $(a, xa) \in \mathscr{C}$ for some $a, x \in S$. Then $\{a\} \times [K(X) \mathbin{\cdot.} K(x)]a \subset \mathscr{C}$.*

DEFINITION. Let S be a semigroup. The symbol S^1 will denote S if S has an identity element and will denote S with isolated identity adjoined if S has no identity element.

DEFINITION. Let S be a commutative semigroup, B a closed subset of S, and $a \in S$. Then, $\mathscr{C}(a, B) = \{(c, d) \in aS^1 \times aS^1 | cB = dB\} \cup \Delta$ (Δ denotes the diagonal of $S \times S$.)

It is trivial to show that $\mathscr{C}(a, B)$ is a closed congruence on S.

THEOREM 3.2. *If S is a compact commutative semigroup and $a, x \in S$, then the smallest closed congruence on S containing (a, ax) is $\mathscr{C}(a, K(x))$.*

PROOF. First note that there is a unique minimal closed congruence on S containing (a, ax), namely the intersection of all closed congruences on S containing (a, ax). Let $\mathscr{C}$ denote this congruence. To show that $\mathscr{C} \subset \mathscr{C}(a, K(x))$, it will suffice to show that $(a, ax) \in \mathscr{C}(a, K(x))$. But $xK(x) = K(x)$ and hence $(ax)K(x) = aK(x)$, which gives $(a, ax) \in \mathscr{C}(a, K(x))$.

Now let $(c, d) \in \mathscr{C}(a, K(x))$. If $(c, d) \in \Delta$, then $(c, d) \in \mathscr{C}$.

Suppose then that $c = ay$, $d = aw$, $(y, w \in S^1)$, and $cK(x) = dK(x)$. Then since $(a, ax) \in \mathscr{C}$ we have $(ya, yax) \in \mathscr{C}$, and by Theorem 3.1, $(ya, yat) \in \mathscr{C}$ for each $t \in K(x) \,.\!\cdot K(x)$. Similarly, $(wa, wat) \in \mathscr{C}$ for each $t \in K(x) \,.\!\cdot K(x)$. Let $k_1 \in K(x)$; then there exists $k_2 \in K(x)$ such that $yak_1 = wak_2$ since $(ya)K(x) = (wa)K(x)$. Now $k_1, k_2 \in K(x) \,.\!\cdot K(x)$ and hence

$$(ya, yak_1) = (ya, wak_2) \in \mathscr{C},$$

and $(wa, wak_2) \in \mathscr{C}$. Hence $(ya, wa) = (c, d) \in \mathscr{C}$.

THEOREM 3.3. *Let S be a compact commutative semigroup with an identity element and $a, x \in S$. If $K(x) \,.\!\cdot K(x)$ is connected, then $\mathscr{C}(a, K(x))$ is monotone, (i.e., the congruence classes are connected).*

PROOF. Let C be a nondegenerate congruence class of S with respect to $\mathscr{C}(a, K(x))$ and $c, d \in C$. Then there exist $y, w \in S^1$ such that $c = ay$, $d = aw$, and $cK(x) = dK(x)$. It was noted in the proof of Theorem 3.2 that (c, ct) and (d, dt) are in $\mathscr{C}(a, K(x))$ for each $t \in K(x) \,.\!\cdot K(x)$. Therefore

$$c[K(x) \,.\!\cdot K(x)] \cup d[K(x) \,.\!\cdot K(x)] \subset C.$$

But

$$K(x) \subset K(x) \,.\!\cdot K(x) \text{ and } cK(x) = dK(x).$$

Hence

$$c[K(x) \,.\!\cdot K(x)] \cap d[K(x) \,.\!\cdot K(x)] \neq \square,$$

and thus

$$c[K(x) \,.\!\cdot K(x)] \cup d[K(x) \,.\!\cdot K(x)]$$

is connected. It follows that C is connected.

THEOREM 3.4. *Let S be a compact semigroup with an identity element* 1 *and suppose that*

i) *for each pair* $a, b \in S$, *there exists an element* $e \in S$ *such that either* $ae = a$ *and* $be \in aS$; *or* $be = b$ *and* $ae \in bS$; *and*

ii) $K(x) \,.\cdot\, K(x)$ *is connected for each* $x \in S$.
Then every closed right congruence on S *is monotone.*

PROOF. Let $\mathscr{C}$ be a closed right congruence on S and let C be a congruence class of S with respect to $\mathscr{C}$. Let $a, b \in C$. Then there exists an element $e \in S$ satisfying one of the two conditions of i). Suppose $ae = a$ and $be = ax$ for some $x \in S$. Then $(ae, be) = (a, ax) \in \mathscr{C}$ and by Theorem 3.1 we have that $a[K(x) \,.\cdot\, K(x)] \subset C$. But also $(ae, be) = (a, be) \in \mathscr{C}$ and hence $(b, be) \in \mathscr{C}$. Again by Theorem 3.1, we have $b[K(e) \,.\cdot\, K(e)] \subset C$. Now the sets

$$a[K(x) \,.\cdot\, K(x)] \text{ and } b[K(e) \,.\cdot\, K(e)]$$

are connected and

$$ax = be \in a[K(x) \,.\cdot\, K(x)] \cap b[K(e) \,.\cdot\, K(e)].$$

Hence

$$a[K(x) \,.\cdot\, K(x)] \cup b[K(e) \,.\cdot\, K(e)]$$

is connected, is a subset of C, and contains a and b since

$$1 \in [K(x) \,.\cdot\, K(x)] \cap [K(e) \,.\cdot\, K(e)].$$

It follows that C is connected and hence $\mathscr{C}$ is monotone.

COROLLARY 3.5. *Every closed congruence on a standard thread is monotone.* (This theorem is originally due to Phillips, [7].)

PROOF. For $a, b \in T$, take $e = 1$. This will give condition i) of Theorem 3.4. To see that condition ii) is satisfied, we note that, for $x \in T$, $K(x) \,.\cdot\, K(x)$ is the interval $[h, 1]$ where $h = \sup \{f \in T | fx = f\}$ and therefore is connected.

COROLLARY 3.6. *If* S *is a compact semilattice* (*commutative idempotent semigroup*) *and if* $K(x) \,.\cdot\, K(x)$ *is connected for each* $x \in S$, *then every closed congruence on* S *is monotone.*

PROOF. For condition i) of Theorem 3.4, take $a = e$. Condition ii) is given in the hypothesis. (We remark that the set $K(x) \,.\cdot\, K(x)$ in this case is just the set $\{t \in S | xt = x\}$ and is usually denoted by $M(x)$.)

If S and T are compact semigroups each of which admits only monotone left (right) closed congruences, then the product $S \times T$ (with coordinate-wise multiplication) need not have this property. As an example, let I denote a usual thread and let $S = I \times I$. Let $a = (\frac{1}{2}, 1)$ and $b = (1, \frac{1}{2})$. Let V be a neighborhood of $(1, 1)$ such that $S \setminus V$ is an ideal of S, $Va \cap Sb = \square$, $Vb \cap Sa = \square$, and let $A = (Sa \cup Sb)(S \setminus V)$.

Now let $\mathscr{C} = \{(xa, xb) | x \in V\} \cup \{(xb, xa)\} | x \in V\} \cup (A \times A) \cup \Delta$. It is not difficult to see that $\mathscr{C}$ is a closed congruence on S and that $\{a, b\}$ is a congruence class. However, the following theorem is a result in this direction.

THEOREM 3.7. *Let S be a compact semigroup with an identity element 1 which admits only monotone closed right (left) congruences, and let T be a compact idempotent semigroup such that $x \,.\!\cdot\, x$ is connected for each $x \in T$. Then $S \times T$ admits only monotone closed right (left) congruences. Conversely if S and T are compact semigroups and if $S \times T$ admits only monotone closed right (left) congruences, then so also do each of S and T.*

PROOF. The proof will be given for closed right congruences. Let $\mathscr{C}$ be a closed right congruence on $S \times T$ and let C be a congruence class of $S \times T$ with respect to $\mathscr{C}$. Let $a = (s_1, t_1)$ and $b = (s_2, t_2)$ be elements of C. Let $e_1 = (1, t_1)$ and $e_2 = (1, t_2)$. Then $ae_1 = a$ and $be_2 = b$. Hence

$$(ae_1, be_1) = (a, be_1) \in \mathscr{C} \text{ and } (ae_2, be_2) = (ae_2, b) \in \mathscr{C},$$

and by the transitivity and symmetry of $\mathscr{C}$, we have that $(a, ae_2) \in \mathscr{C}$ and $(b, be_1) \in \mathscr{C}$. By Theorem 3.1,

$$a[K(e_2) \,.\!\cdot\, K(e_2)] \cup b[K(e_1) \,.\!\cdot\, K(e_1)] \subset C.$$

It is clear that $K(e_i) \,.\!\cdot\, K(e_i)$ is isomorphic to $K(t_i) \,.\!\cdot\, K(t_i)$, $i = 1, 2$, and hence is connected. The set $C \cap [S \times \{t_1 t_2\}]$ is a congruence class of the subsemigroup $S \times \{t_1 t_2\}$ (which is isomorphic to S) with respect to the closed right congruence $\mathscr{C} \cap [S \times \{t_1 t_2\}) \times (S \times \{t_1 t_2\})]$ on $S \times \{t_1 t_2\}$, and hence is connected by the hypothesis. We now have

$$\begin{aligned} &a \in a[K(e_2) \,.\!\cdot\, K(e_2)], \\ &b \in b[K(e_1) \,.\!\cdot\, K(e_1)], \\ &ae_2 \in a[K(e_2) \,.\!\cdot\, K(e_2)] \cap [C \cap (S \times \{t_1 t_2\})], \text{ and} \\ &be_1 \in b[K(e_1) \,.\!\cdot\, K(e_1)] \cap [C \cap (S \times \{t_1 t_2\})]. \end{aligned}$$

Hence the set

$$a[K(e_2) \,.\!\cdot\, K(e_2)] \cup [C \cap (S \times \{t_1 t_2\})] \cup b[K(e_1) \,.\!\cdot\, K(e_1)]$$

is connected, is contained in C, and contains a and b. It follows that C is connected and that $\mathscr{C}$ is monotone.

Now suppose that S and T are compact semigroups and that $S \times T$ admits only monotone closed right congruences. Let $\mathscr{C}$ be a closed right congruence on S. Then the subset $\mathscr{C}'$ of $(S \times T) \times (S \times T)$ defined by

$$\mathscr{C}' = \{((s_1, t_1), (s_2, t_2)) \mid (s_1, s_2) \in \mathscr{C}, (t_1, t_2) \in T \times T\}$$

is a closed right congruence on $S \times T$ and therefore is monotone. A congruence class C of S with respect to $\mathscr{C}$ is of the form $C = \pi_1(C')$ where C'

is a congruence class of $S \times T$ with respect to $\mathscr{C}'$ and π_1 is the natural projection of $S \times T$ onto S. Hence C is connected.

COROLLARY 3.8. *The product of an arbitrary thread with a min thread admits only monotone closed congruences.*

COROLLARY 3.9. *If S is a compact idempotent semigroup with identity and with $x \,.\!\cdot\, x$ connected for each $x \in S$, then S admits only monotone closed right congruences.*

PROOF. Let I be a usual thread. Then by the first part of Theorem 3.7, $S \times I$ admits only monotone closed right congruences. The corollary follows from the second part of Theorem 3.7.

4. Homomorphisms of $I_1 \times I$

In this section I_1 will denote a min thread and I an arbitrary standard thread. For convenience of reference, we introduce the following notation:

$$T_{11} = I_1 \times \{0\},\ T_{12} = \{1\} \times I,\ T_1 = T_{11} \cup T_{12},\ T_{21} = \{0\} \times I,$$
$$T_{22} = I_1 \times \{1\},\ T_2 = T_{21} \cup T_{22},\ B = T_1 \cup T_2.$$

We will denote $I_1 \times I$ by S, $(0, 0)$ by 0, and $(1, 1)$ by 1. Of course, B is the boundary of S if S be considered as a subset of the plane. For $a, b \in S$, we will say that $a \leqq b$ if $a \in bS$. In terms of coordinates, $(a_1, a_2) \leqq (b_1, b_2)$ if and only if $a_1 \leqq b_1$ and $a_2 \leqq b_2$.

We will prove the following theorem:

THEOREM 4.1. *If f is a continuous homomorphism of S onto T, then T is a cyclic chain from $f(0)$ to $f(1)$, (in the sense of Whyburn, [10]) where each true cyclic element is a two-cell.*

PROOF. Let $t = f(x) \in T$. There exists a standard thread E in S from 0 to 1 containing x. Then $f(E)$ is a standard thread from $f(0)$ to $f(1)$ in T which contains t. Since $f(E)$, and hence t, is contained in any A-set in T containing $f(0)$ and $f(1)$, [10, p. 69], it follows that T is the cyclic chain from $f(0)$ to $f(1)$.

Now let D be a true cyclic element of T. Then $(T \setminus D)^* \cap D$ consists of at most two points which are cutpoints of T. If T has no cutpoints, (in which case $T = D$), then by Theorem 2.4 of [10], T is either a point (if $f(0) = f(1)$) or a two-cell (if $f(0) \neq f(1)$), and the theorem is proved. We will assume that $(T \setminus D)^* \cap D$ contains two points. The other case is similar and simpler. Denote these points by t_1 and t_2. We proceed by several steps.

1) For $i = 1, 2$, $f^{-1}(t_i) \cap T_1 \neq \square \neq f^{-1}(t_i) \cap T_2$. For if $f^{-1}(t_i) \cap T_1 = \square$,

then since $f^{-1}(t_i)$ separates S, $f^{-1}(t_i)$ must separate T_1 from some point $c = (c_1, c_2)$ of S.

The set E formed by taking the union of the set $\{c_1\} \times I$ with the closed subintervals $[0, (c_1, 0)]$ and $[(c_1, 1), 1]$ of T_{11} and T_{22}, respectively, is a standard thread with endpoints 0 and 1. Then $f^{-1}(t_i) \cap E$ must separate $\{0, 1\}$ from c, but this is a contradiction since $f|E$ is monotone and hence $f^{-1}(t_i) \cap E$ is a closed subinterval of E. A similar argument shows that $f^{-1}(t_i) \cap T_2 \neq \square$.

2) If $\qquad f^{-1}(t_i) \cap T_{21} \neq \square$, then $f^{-1}(t_i) \cap T_{11} = \square$.

For suppose $x \in f^{-1}(t_i) \cap T_{21}$ and $y \in f^{-1}(t_i) \cap T_{11}$. Then $f(x) = f(y)$ implies that $f(y) = f(y^2) = f(y)f(y) = f(x)f(y) = f(xy) = f(0)$, and $f(0)$ cannot be a cutpoint of T.

3) At this point there are several cases to consider, but the proofs of all of the cases are similar. We will consider only the following representative case:

$$f^{-1}(t_1) \cap T_{11} \neq \square \neq f^{-1}(t_1) \cap T_{22},$$

and

$$f^{-1}(t_2) \cap T_{12} \neq \square \neq f^{-1}(t_2) \cap T_{22}.$$

Let

$$x_1 = (r_1, 0) \in f^{-1}(t_1) \cap T_{11},\ x_2 = (r_2, 1) \in f^{-1}(t_1) \cap T_{22},$$
$$y_1 = (s_1, 1) \in f^{-1}(t_2) \cap T_{22},\ y_2 = (1, s_2) \in f^{-1}(t_2) \cap T_{12},$$

and let $e = (1, 0)$. We will assume that $x_2 < y_1$ on T_{22}. Then noting that $x_1 e = x_1$, we have $f(x_1) = f(x_1 e) = f(x_1)f(e) = f(x_2)f(e) = f(x_2 e) = f(x_2)$. Then by Theorem 3.1, $x_2[K(e) \,.\!\cdot K(e)] \subset f^{-1}(t_1)$. But $K(e) = \{e\}$, $K(e) \,.\!\cdot K(e) = T_{12}$, and $x_2(K(e) \,.\!\cdot K(e)) = \{r_2\} \times I \subset f^{-1}(t_1)$.

We also have $f(y_1) = f(y_1)f(y_1) = f(y_1)f(y_2) = f(y_1 y_2)$, and again by Theorem 3.1, we have that $y_1[K(y_2) \,.\!\cdot K(y_2)] \subset f^{-1}(t_2)$. Now $K(y_2) = h$ where $h = \sup \{g \in T_{12} | g y_1 = g\}$, and $K(y_2) \,.\!\cdot K(y_2)$ is the closed subinterval $[h, 1]$ of T_{12}. Hence the set $y_1[K(y_2) \,.\!\cdot K(y_2)]$ contains the set

$$E_1 = \{(s_1, s) | s_2 \leqq s \leqq 1\},$$

and therefore $E_1 \subset f^{-1}(t_2)$. A similar argument shows that the set

$$E_2 = \{(s, s_2) | s_1 \leqq s \leqq 1\} \subset f^{-1}(t_2).$$

We now have an arc $(\{r_2\} \times I)$ with endpoints $(r_2, 0)$ and x_2 which is contained in $f^{-1}(t_1)$, and an arc $(E_1 \cup E_2)$ with endpoints y_1 and y_2 which is contained in $f^{-1}(t_2)$.

4) Let D' be the closed two-cell subset of S bounded by $\{r_2\} \times I$, $E_1 \cup E_2$, and the closed subintervals $[x_2, y_1]$, $[(r_2, 0), y_1]$ of T_2, T_1, respectively. We will show that $f(D') \subset D$. Let $x \in D'$. Then x lies on a standard thread E

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from 0 to 1 in S, and hence $f(E)$ is a standard thread in T. Now $f(E \cap D')$ is a closed subinterval of $f(E)$ with endpoints t_1 and t_2, and hence $f(E \cap D') \subset D$, [10, p. 72], and in particular $f(x) \in D$. Therefore $f(D') \subset D$.

5) If $f(D')$ has a cutpoint, say q, then $f^{-1}(q)$ would separate D' and it follows easily that q is a cutpoint of D. This is contradictory to the definition of a true cyclic element and hence $f(D')$ has no cutpoints. It now follows from Theorem 2.4 of [10] that $f(D')$ is a two-cell.

6) It remains to be shown only that $f(D') = D$. Suppose there is an element $z \in S \setminus D'$ such that $f(z) \in D$. Then either $z < x_2$ or $z > (s_1, s_2)$. Assume that $z < x_2$. Let z_0 be a minimal element of $f^{-1}(D)$. Then since $0 \notin f^{-1}(D)$, each neighborhood of z_0 contains elements of the set $\{x | x < z_0\}$, and it follows that $f(z_0)$ is a boundary point of $T \setminus D$. But by [10], the only boundary points of $T \setminus D$ are t_1 and t_2. Hence $f(z_0) = t_1$. Now either z is minimal in $f^{-1}(D)$, in which case $f(z) = t_1$, or $z_0 < z < x_2$. In the latter case we would have $t_1 = f(z_0) \leqq f(z) < f(x_2) = t_1$ and again it would follow that $f(z) = t_1$. A similar argument will show that if the case $z > (s_1, s_2)$ holds, then $f(z) = t_2$. Hence $f(S \setminus D') = T \setminus D \cup \{t_1, t_2\}$, and it follows that $f(D') = D$.

More can be said about the homomorphic images of $I_1 \times I$ if the standard thread I is known to be one of I_1, I_2 or I_3. The following theorems will be stated for completeness but proofs will not be given here.

THEOREM 4.2. *If I is either I_2 or I_3 in the hypothesis of Theorem 4.1, then T is the union of an arc A (possibly degenerate) and a two-cell B (possibly degenerate). The arc A is an ideal of T and $A \cap B$ consists of a single point which is contained in the boundary of B.*

THEOREM 4.3. *Suppose $I = I_1$, and let T be any cyclic chain in the plane from a to b such that each true cyclic element is a two-cell. Then there is a topological semigroup structure on T and a homomorphism $f : I_1 \times I_1 \to T$ onto T such that $f(0) = a$ and $f(1) = b$.*

In [2], D. R. Brown proved that if S is a topological semilattice on a two-cell satisfying:

i) $0 \in Bd(S)$, and

ii) $M(x) = \{y | xy = x\}$ is connected for each $x \in M$, then S is a homomorphic image of $I_1 \times I_1$. As a consequence of this result and Theorem 4.1, we have the

COROLLARY 4.4. *If S is a topological semilattice on a two-cell satisfying* i) *and* ii) *above, then every homomorphic image of S is a cyclic chain from $f(0)$ to $f(1)$ and each true cyclic element is a two-cell.*

A. D. Wallace has shown in [8] that a topological lattice on a two-cell satisfies properties i) and ii) above. Hence the conclusion of Theorem 4.1 holds for homomorphic images of topological lattices on a two-cell. Wallace has also shown, [9], that every compact connected lattice which can be imbedded in the plane is a cyclic chain from 0 to 1 and each true cyclic element is a two-cell and a convex sublattice. Such an object may be thought of as a sequence of disjoint topological lattices on two-cells connected by min threads. Using the fact that a homomorphic image of a min thread is again a min thread or a point and applying Corollary 4.4 to the two-cell sublattices we obtain

COROLLARY 4.5. *If L is a compact connected topological lattice which is imbeddable in the plane then every homomorphic image of L is a cyclic chain from $f(0)$ to $f(1)$ where each true cyclic element is a two-cell and a sublattice of L.*

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