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TOPOLOGIZING DIFFERENT CLASSES OF REAL FUNCTIONS

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ABSTRACT. The purpose of this paper is to examine which classes $\mathcal{F}$ of functions from $\mathbb{R}^n$ into $\mathbb{R}^m$ can be topologized in a sense that there exist topologies τ_1 and τ_2 on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $\mathcal{F}$ is equal to the class $C(\tau_1, \tau_2)$ of all continuous functions $f: (\mathbb{R}^n, \tau_1) \longrightarrow (\mathbb{R}^m, \tau_2)$. We will show that the Generalized Continuum Hypothesis GCH implies the positive answer for this question for a large number of classes of functions $\mathcal{F}$ for which the sets $\{x : f(x) = g(x)\}$ are small in some sense for all $f, g \in \mathcal{F}, f \neq g$. The topologies will be Hausdorff and connected. It will be also shown that in some model of set theory ZFC with GCH these topologies could be completely regular and Baire. One of the corollaries of this theorem is that GCH implies the existence of a connected Hausdorff topology $\mathcal{T}$ on $\mathbb{R}$ such that the class $\mathcal{L}$ of all linear functions $g(x) = ax + b$ coincides with $C(\mathcal{T}, \mathcal{T})$. This gives an affirmative answer to a question of Sam Nadler. The above corollary remains true for the class $\mathcal{P}$ of all polynomials, the class $\mathcal{A}$ of all analytic functions and the class of all harmonic functions.

We will also prove that several other classes of real functions cannot be topologized. This includes the classes of C^∞ functions, differentiable functions, Darboux functions and derivatives.

1. **Introduction.** There are a number of known classes of real functions that can be represented as families of continuous functions

$$C(\tau_1, \tau_2) = \{f: (\mathbb{R}, \tau_1) \longrightarrow (\mathbb{R}, \tau_2) : f \text{ is continuous}\},$$

where τ_1 and τ_2 are the topologies on $\mathbb{R}$. Evidently, the ordinary continuous functions are $C(\mathcal{T}_O, \mathcal{T}_O)$, where $\mathcal{T}_O$ stands for the ordinary topology. The other obvious examples include the class $C\big(\mathcal{T}_O, \{(a, \infty) : a \in [-\infty, \infty]\}\big)$ of lower semicontinuous functions, the class $C\big(\mathcal{T}_O, \{(-\infty, a) : a \in [-\infty, \infty]\}\big)$ of upper semicontinuous functions and the class $C(\tau_r, \mathcal{T}_O)$ of right continuous functions, where topology τ_r is generated by intervals $[a, b)$. Probably the most interesting non-trivial example of topologized class consists of the class $C_{\mathcal{N}O}$ of approximately continuous functions. This class was introduced by Denjoy in 1915 [6] and since then it was extensively studied, including Zahorski's very deep work on the derivatives [15] from 1950. However, it was not until 1952 when the density topology $\mathcal{T}_{\mathcal{N}}$ on $\mathbb{R}$ and the relation $C_{\mathcal{N}O} = C(\mathcal{T}_{\mathcal{N}}, \mathcal{T}_O)$ was discovered by Haupt and Pauc [10]. Moreover, the paper of Haupt and Pauc was completely unnoticed for years, and the real study of the density topology dates from 1961 when Goffman and Waterman [9] rediscovered the density topology and the relation $C_{\mathcal{N}O} = C(\mathcal{T}_{\mathcal{N}}, \mathcal{T}_O)$.

In the last two decades another approach was used: several classes of real functions were introduced as classes of functions continuous in some topologies on $\mathbb{R}$. This includes

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the classes: $C(\mathcal{T}_{\mathcal{N}}, \mathcal{T}_{\mathcal{N}})$ of the density continuous functions, $C(\mathcal{T}_I, \mathcal{T}_O) = C(\mathcal{T}_{\mathcal{D}}, \mathcal{T}_O)$ of the I-approximately continuous functions, $C(\mathcal{T}_I, \mathcal{T}_I)$ of the I-density continuous functions, and $C(\mathcal{T}_{\mathcal{D}}, \mathcal{T}_{\mathcal{D}})$ of the deep I-density continuous functions, where the I-density topology $\mathcal{T}_I$ is considered to be a category analog of the density topology, and deep I-density topology $\mathcal{T}_{\mathcal{D}}$ is defined as the coarsest topology for the I-approximately continuous functions. (For more information on the subject, see [5]. Compare also [4, 8, 13, 14].)

The purpose of this paper is to examine which of the other known classes of real functions can be topologized. This problem, with the emphasis on the classes of linear and differentiable functions, was first articulated by Sam Nadler in a student's problem session held at West Virginia University. The author discussed this for the class of linear functions with several people. This version of the problem was also restated by Lee Larson on Fifteen Summer Symposium in Real Analysis in Bratislava.

The paper is organized as follows. Section 2 contains a discussion of relations between topologized class $\mathcal{F} = C(\tau_1, \tau_2)$ and topologies τ_1 and τ_2. It also contains the proofs that several classes of real functions cannot be topologized. Section 3 states the main theorem and discuss its corollaries. The two cases of the theorem are proved separately. The first part, which involves only the use of GCH and transfinite induction, is presented in Section 4. The second part of the theorem, which proof involves forcing method, is left for Section 6, the last section of the paper. The reader unfamiliar with the forcing technic can simply skip this section. Section 5 contains a discussion of set theoretical assumptions of the main theorem and points several possible generalizations.

The set theoretical and topological terminology and notation used in this paper is standard and follows [11, 7]. In particular, ordinals are identified with their sets of predecessors and cardinals with the initial ordinals. Symbol ω denotes the first infinite ordinal as well as the first infinite cardinal. $\mathcal{P}(X)$ will denote the power set of X and $|X|$ the cardinality of X. If κ is a cardinal number than κ^+ denotes the cardinal successor of κ and $2^\kappa = |\mathcal{P}(\kappa)|$. GCH will stand for the Generalized Continuum Hypothesis, *i.e.*, the statement that $2^\kappa = \kappa^+$ for every infinite cardinal κ. The functions will be identified with their graphs. The class of all functions $f: X \longrightarrow Y$ from a set X to a set Y is denoted by Y^X. For $f: X \longrightarrow Y$ the restriction of f to a set $A \subset X$ will be denoted by $f|_A$. For a set X and a cardinal number κ we define $[X]^{\leq\kappa} = \{Y \subset X : |Y| \leq \kappa\}$ and $[X]^{<\kappa} = \{Y \subset X : |Y| < \kappa\}$. A family $I \subset \mathcal{P}(X)$ is said to be an *ideal* if $A \cup B \in I$ provided $A, B \in I$, and $B \in I$ provided $B \subset A \in I$. An ideal I is said to be a σ-*ideal* if $\bigcup \mathcal{F} \in I$ for every $\mathcal{F} \in [I]^{\leq\omega}$.

The letters $\mathcal{T}$ and τ, with possible subscripts, will always denote the topologies. In particular, $\mathcal{T}_O^n$, or simply $\mathcal{T}_O$, will denote the ordinary topology on $\mathbb{R}^n$. For topological spaces (X, τ_1) and (Y, τ_2) the class of all continuous functions from (X, τ_1) to (Y, τ_2) is denoted by $C(\tau_1, \tau_2)$. We will also write $C(\tau)$ in place of $C(\tau, \tau)$. Symbol Const will denote the family of all constant functions in currently considered class Y^X. Symbol id_X will stand for the identity function on X and $\mathrm{dom}(f)$ for the domain of a function f. The closure and interior of a set A in topology τ will be denoted by $\mathrm{cl}_\tau(A)$ and $\mathrm{int}_\tau(A)$, respectively.

Symbol $\mathcal{A}$ will be reserved for the class of all real or complex analytic functions. $\mathcal{P} \subset \mathcal{A}$ and $\mathcal{L} \subset \mathcal{A}$ will stand for the class of all polynomials and the class of all linear functions $f(x) = ax + b$, respectively.

The definitions of these classes and other classes of real functions used in this paper can be found in [1] or in [5, 14].

2. **Basic properties of topologized classes and their applications.** In this section we will examine some of the properties of the topologies τ_1 and τ_2 that can be deduced from the properties of $C(\tau_1, \tau_2)$. This will give us a perspective on the difficulties we must face topologizing different classes of real functions. Moreover, the results presented in this section will show the boundaries of the technic presented in the sections to follow.

The following theorem lists some basic properties of classes of functions that can be topologized. Recall also that a family $\mathcal{T}$ of subsets of $\mathbb{R}$ or $\mathbb{C}$ is homothetically closed if $L^{-1}(U) \in \mathcal{T}$ for every $L \in \mathcal{L}$ and $U \in \mathcal{T}$.

THEOREM 1. *Let τ_1 and τ_2 be the topologies on sets X and Y, respectively, and let $\mathcal{F} = C(\tau_1, \tau_2) \neq Y^X$. If τ is a weak topology on X generated by F, i.e., generated by the family $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$, then*

(i) Const $\subset \mathcal{F}$, *$\tau \subset \tau_1$, $\tau_1 \neq \mathcal{P}(X)$, $\tau_2 \neq \{\emptyset, Y\}$ and $\mathcal{F} = C(\tau, \tau_2)$;*

(ii) if $X = Y$ and $\mathrm{id}_X \in \mathcal{F}$ *then $\tau_2 \subset \tau \subset \tau_1$;*

(iii) if $Y = \mathbb{R}$ and $\mathcal{T}_O \subset \tau_2$ then $\mathcal{F}$ is closed under the maximum and minimum operations;

(iv) if $\mathcal{G} \subset Y^Y$ is such that $\mathrm{id}_Y \in \mathcal{G}$ *and $g \circ f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ then $\mathcal{F} = C(\tau, \tau')$, where τ' is a topology generated by $\{g^{-1}(U) : U \in \tau_2, g \in \mathcal{G}\}$; in particular, if $\mathcal{G} = \mathcal{L}$ than we may assume that τ_2 is a homothetically closed T_1 topology;*

(v) if $X = Y$, $\mathrm{id}_X \in \mathcal{F}$ *and $\mathcal{F}$ is closed under the composition, then $\mathcal{F} = C(\tau)$;*

(vi) if $X = Y \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{L} \subset \mathcal{F}$ then τ_1 is a T_1 topology;

(vii) if $X = Y \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{L} \subset \mathcal{F}$ and τ_2 contains two nonempty disjoint sets, then τ_1 is Hausdorff;

(viii) if $X = Y = \mathbb{R}$ and $\mathcal{F} \subset$ Darboux then τ_1 is connected;

(ix) if $X = Y = \mathbb{R}$, $\mathcal{L} \subset \mathcal{F}$ and τ_2 contains a nonempty set which has either upper or lower bound, then $\mathcal{T}_O \subset \tau_1$;

PROOF. (i) and (ii) are obvious.

(iii) Let $f, g \in \mathcal{F}$. We have to prove that $\max\{f, g\} \in \mathcal{F}$.

First notice that the function $\vee : (\mathbb{R} \times \mathbb{R}, \tau_2 \times \tau_2) \longrightarrow (\mathbb{R}, \tau_2)$ defined by $\vee(y, z) = \max\{y, z\}$ is continuous. It follows from that fact for every $U \in \tau_2$ we have

$$\vee^{-1}(U) = (U \times U) \cup \Big(\{(y,z) : y < z\} \cap (\mathbb{R} \times U)\Big) \cup \Big(\{(y,z) : y > z\} \cap (U \times \mathbb{R})\Big)$$

and the set on the right hand side belongs to $\tau_2 \times \tau_2$. It is also easy to see that the following functions are continuous: $\triangle : (X, \tau_1) \longrightarrow (X \times X, \tau_1 \times \tau_1)$, $\triangle(x) = (x, x)$ and $f \times g : (X \times X, \tau_1 \times \tau_1) \longrightarrow (\mathbb{R} \times \mathbb{R}, \tau_2 \times \tau_2)$, $(f \times g)(x_1, x_2) = \big(f(x_1), g(x_2)\big)$. But $\max\{f, g\} = \vee \circ (f \times g) \circ \triangle$, so it is continuous as a composition of continuous functions.

The argument for $\min\{f,g\} \in \mathcal{F}$ is essentially the same.

(iv) Evidently, by (i), $C(\tau,\tau') \subset C(\tau,\tau_2) = C(\tau_1,\tau_2) = \mathcal{F}$, since $\tau_2 \subset \tau'$. So, let $f \in \mathcal{F}$. The topology τ' is generated by the sets of the form $g^{-1}(U)$ where $g \in \mathcal{G}$ and $U \in \tau_2$. But $f^{-1}\big(g^{-1}(U)\big) = (g \circ f)^{-1}(U) \in \tau$, since $g \circ f \in \mathcal{F}$. Hence, $\mathcal{F} \subset C(\tau,\tau')$.

Now, if $\mathcal{G} = \mathcal{L}$ then τ' is clearly homothetically closed and it must be a T_1 topology, since $\tau_2 \neq \{\emptyset, Y\}$. (Compare also (vi) below.)

(v) By (iv) used with $\mathcal{G} = \mathcal{F}$ we have $\mathcal{F} = C(\tau,\tau')$, where $\tau' \subset \tau$.

(vi) Since $\tau_2 \neq \{\emptyset, Y\}$, there are $a, b \in Y$ and $U \in \tau_2$ such that $a \in U$ and $b \notin U$. But the family $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{L}\} \subset \tau \subset \tau_1$ is homothetically closed. So, τ_1 is a T_1 topology.

(vii) is also implied by the fact that $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{L}\} \subset \tau_1$ is homothetically closed.

(viii) is true since the characteristic function of any clopen set is continuous.

(ix) is clear, since $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{L}\} \subset \tau \subset \tau_1$ is homothetically closed.

Theorem 1 is proved.

As an immediate corollary we obtain the following.

COROLLARY 1. *Let $\mathcal{F} \subset X^X$. If $\mathcal{F}$ can be topologized then $\mathcal{F} = C(\tau)$ for some topology τ on X if and only if* $\mathrm{id}_X \in \mathcal{F}$ *and $\mathcal{F}$ if closed under the composition operation.*

PROOF. If $\mathcal{F} = C(\tau)$ then obviously $\mathcal{F}$ is closed under composition and $\mathrm{id}_X \in \mathcal{F}$. The other implication follows from Theorem 1(v).

The main goal of the paper is to prove that a wide range of classes of functions can be topologized. It is clear from Theorem 1(i) that to fulfill this project it is enough to construct only a topology τ_2 on the range of the class of functions. By Corollary 1 this is true even is we like to have the same topology on the domain and the range. Moreover, by Theorem 1(vi) and (viii), any topology on $\mathbb{R}$ topologizing class $\mathcal{L}$ must be T_1 and connected. Condition (vii) of Theorem 1 suggest also that it is wise to construct this topology as Hausdorff. These are the properties that topologies constructed in the next sections will have. Right now, let us list some of the immediate corollaries of Theorem 1.

THEOREM 2. *Let $\mathcal{F}$ be a family of real functions closed under composition and such that $C^\infty \subset \mathcal{F}$. If $\mathcal{F}$ can be topologized then $\mathcal{F}$ is closed under the maximum and minimum operations.*

PROOF. Assume that $\mathcal{F}$ can be topologized. Then, by Corollary 1, $\mathcal{F} = C(\tau)$. Moreover, by Theorem 1(vi), τ is T_1. Let $f(x) = e^{-x^{-2}}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. It is well known that $f \in C^\infty$, so that $f \in \mathcal{F}$. Hence, $(0,\infty) = f^{-1}(\mathbb{R} \setminus \{0\}) \in \tau$, since $\mathbb{R} \setminus \{0\} \in \tau$. But conditions (ix) and (iii) of Theorem 1 imply now that $\mathcal{T}_O \subset \tau$ and that $\mathcal{F}$ is closed under the maximum and minimum operations.

Theorem 2 implies the following corollary. The definitions of most of the classes of the corollary can be found in [1] and [14].

COROLLARY 2. *The classes: C^∞, $\mathcal{D}^n$ of n-times differentiable functions and C^n of functions with continuous n-th derivative cannot be topologized The same is true, when*

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in the above we replace differentiability with symmetric differentiability, approximate differentiability, symmetric approximate differentiability, I-approximate differentiability or symmetric I-approximate differentiability.

PROOF. All these classes contain C^∞ as a subclass and are closed under the composition. Moreover, they are not closed under the maximum operation. It is shown by the function $|x| = \max\{x, -x\}$ for all classes but those with prefix symmetric. To see that the symmetric classes are not closed under supremum, take $a_0 > b_0 > a_1 > b_1 > \cdots$ such that $\lim_n a_n = 0$, 0 is a dispersion and I-dispersion point of $\bigcup_{n\in\mathbb{N}}[a_{n+1}, b_n]$ and 0 is neither right density nor right I-density point of $P_i = \bigcup_{n\in\mathbb{N}}[b_{2n+i}, a_{2n+i}]$ for $i < 2$. Define $f(0) = 0, f(x) = (-1)^i x$ for $x \in P_i$, extend it to a C^∞ function on $(0, \infty)$ staying between the graphs of x and $-x$ and define $f(x) = f(-x)$ for $x < 0$. It is clear that function f is symmetrically infinitely many times differentiable in any of the sense define in the theorem. However, $h(x) = \max\{x, f(x)\}$ is in neither of these classes.

Before we state the next theorem let us define the following class of functions. A function f is in the class $\mathcal{E}_0$ provided there exists a sequence

$$\infty \geq d_1 > c_1 > b_1 > a_1 > d_2 > c_2 > b_2 > a_2 > \cdots$$

such that the right hand side density of the sets $P_f^1 = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ and $P_f^2 = \bigcup_{n\in\mathbb{N}}[c_n, d_n]$ with respect to 0 is equal to $1/2$ and that f is defined as follows: $f(x) = -1$ for $x \in P_f^1$, $f(x) = 1$ for $x \in P_f^2$, f is linear on each of the intervals $[b_n, c_n]$ and $[d_{n+1}, a_n]$, $f(0) = 0$ and $f(x) = f(-x)$ for $x \in (-\infty, 0)$.

THEOREM 3. *If $\mathcal{F}$ is a family of real functions such that $\mathcal{E}_0 \subset \mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$ and $L \circ f \circ M \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $L, M \in \mathcal{L}$, then $\mathcal{F}$ cannot be topologized.*

PROOF. By way of contradiction assume that $\mathcal{F} = C(\tau_1, \tau_2)$. By Theorem 1(iv) we may assume, without loss of the generality, that τ_2 is a T_1 topology and that τ_1 is a weak topology generated by $\mathcal{F}$ and τ_2. Choose $f, g \in \mathcal{E}_0$ such that

$$(0, \infty) = P_f^1 \cup P_f^2 \cup P_g^1 \cup P_g^2 \subset f^{-1}(\{-1, 1\}) \cup g^{-1}(\{-1, 1\}).$$

Then, $\{0\} = f^{-1}(\mathbb{R} \setminus \{-1, 1\}) \cap g^{-1}(\mathbb{R} \setminus \{-1, 1\}) \in \tau_1$. But τ_1 is homothetically closed, since for every $f \in \mathcal{F}$, $U \in \tau_2$ and $M \in \mathcal{L}$ we have $M^{-1}\big(f^{-1}(U)\big) = (f \circ M)^{-1}(U) \in \tau_1$. So, $\tau_1 = \mathcal{P}(\mathbb{R})$ and $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$, contradicting our assumption.

COROLLARY 3. *The following classes cannot be topologized: the class of all derivatives, the Zahorski's classes $\mathcal{M}_i$ for $i = 0, 1, 2, 3, 4$, the class of all symmetrically (symmetrically approximately or symmetrically I-approximately) continuous functions, the class of all Darboux functions, the class of all measurable functions and the class of all functions having the Baire property.*

PROOF. It is easy to see that all the above classes are proper subclasses of $\mathbb{R}^{\mathbb{R}}$ and are closed under interior and exterior compositions with linear functions. The proof that all the functions from the class $\mathcal{E}_0$ are derivatives and in class $\mathcal{M}_4$ can be found in [1, pp. 23 and 87]. They are also evidently symmetrically continuous. The rest follows from Theorem 3.

3. **Main theorem and its corollaries.** In this section we state the main theorem and conclude from it some corollaries. The proof of the theorem will be postponed until the next sections. Recall that family $\mathcal{F} \subset Y^X$ *separates points* if for every distinct points $x_1, x_2 \in X$ there is $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$. The topological space is *Baire* if every first category set in this topology has empty interior.

Let us also stress here that in what follows regular and completely regular topological spaces do not have to be Hausdorff.

THEOREM 4. *Let $|X| = |Y| = 2^\omega$, $\mathcal{R} \in [Y^X]^{\leq 2^\omega}$ and let I be a proper σ-ideal on X containing all singletons.*

(A) If GCH *holds then there is a Hausdorff, connected and locally connected topology τ_2 on Y with the property that for every family $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ such that* Const $\subset \mathcal{F}$ *and*

(1) $\{x \in X : f(x) = g(x)\} \in I$ *for every distinct* $f, g \in \mathcal{F}$

we have

$$\mathcal{F} = C(\tau, \tau_2),$$

where τ is generated by the family $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$. Topology τ is connected and locally connected. It is also Hausdorff, provided $\mathcal{F}$ separates points.

(B) Moreover, it is consistent with the set theory ZFC *plus* GCH *that the topologies τ and τ_2 are completely regular and Baire.*

To see the real meaning of Theorem 4 let us state several of its corollaries. In the first corollary we use Theorem 4 with the σ-ideal I_c of the first category subsets of $\mathbb{R}^n$.

COROLLARY 4. *If* GCH *holds then there is a Hausdorff, connected and locally connected topology τ_C on $\mathbb{R}^m$ such that for any family $\mathcal{F} \subset C(\mathcal{T}_O^m, \mathcal{T}_O^n)$, of ordinary continuous functions, containing all constant functions and such that $\mathrm{int}_{\tau_O}(\{x \in X : f(x) = g(x)\}) = \emptyset$ for every distinct $f, g \in \mathcal{F}$ we have*

$$\mathcal{F} = C(\tau_{\mathcal{F}}, \tau_C),$$

where $\tau_{\mathcal{F}}$ is generated by the family $\{f^{-1}(U) : U \in \tau_C, f \in \mathcal{F}\}$. Moreover, $\tau_{\mathcal{F}}$ is connected and locally connected, and it is Hausdorff provided $\mathcal{F}$ separates points. It is also consistent with ZFC + GCH *that all these topologies are completely regular and Baire.*

In particular, it can be shown that for any different harmonic functions $f, g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ we have $\mathrm{int}_{\tau_O}(\{x \in X : f(x) = g(x)\}) = \emptyset$.[1] Thus, by Corollary 4, the class of all harmonic functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be topologized.

[1] To see it let $f, g: \mathbb{R}^n \longrightarrow \mathbb{R}$ be two harmonic functions that agree on some neighbourhood of $x_0 \in \mathbb{R}^n$. Let $R > 0$ be a supremum of all balls $B[x_0, r] = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$ on which f and g agree. If $R = \infty$ then we are done. So, assume that $R < \infty$. In particular, f and g agree on $B[x_0, R]$. Now, for every point x in a boundary of $B[x_0, R]$ use Cauchy-Kovaleski Theorem [2, Theorem 1.5 p. 330] for Laplacian operator L with initial values given by the derivatives of f on a boundary of $B[x_0, R]$ to find an open neighbourhood U_x of x in which this initial value problem has a unique solution. Since both functions f and g form the solutions

Another σ-ideal that can be used with Theorem 4 is the ideal I_ω of at most countable sets. Since for any two different analytic functions $f, g \in \mathcal{A}$ we have $\{x : f(x) = g(x)\} \in I_\omega$, we can also conclude the following corollary.

COROLLARY 5. *If* GCH *holds then there is a Hausdorff, connected and locally connected topology $\tau_{\mathcal{A}}$ (on $\mathbb{R}$ or $\mathbb{C}$) such that for any family $\mathcal{F} \subset \mathcal{A}$ containing all constant functions we have*

$$\mathcal{F} = C(\tau_{\mathcal{F}}, \tau_{\mathcal{A}}),$$

where $\tau_{\mathcal{F}}$ is generated by the family $\{f^{-1}(U) : U \in \tau_{\mathcal{A}}, f \in \mathcal{F}\}$. Moreover, $\tau_{\mathcal{F}}$ is connected and locally connected, and it is Hausdorff provided $\mathcal{F}$ separates points. It is also consistent with ZFC + GCH *that all these topologies are completely regular and Baire.*

Notice also, that if the family $\mathcal{F}$ in Corollary 5 is closed under the composition and id $\in \mathcal{F}$, then, by Theorem 1(v), $\mathcal{F} = C(\tau_{\mathcal{F}})$. We can write this in the form of next corollary.

COROLLARY 6. *If* GCH *holds and $\mathcal{F}$ is a family of real functions which is closed under the composition and such that* $\{\text{id}\} \cup \text{Const} \subset \mathcal{F} \subset \mathcal{A}$, *then there exists a Hausdorff, connected and locally connected topology $\mathcal{T}_{\mathcal{F}}$ (on $\mathbb{R}$ or $\mathbb{C}$) such that $\mathcal{F} = C(\mathcal{T}_{\mathcal{F}})$. In particular, there exist a* "linear topology" *$\mathcal{T}_{\mathcal{L}}$, a* "polynomial topology" *$\mathcal{T}_{\mathcal{P}}$ and an* "analytic topology" *$\mathcal{T}_{\mathcal{A}}$ which are Hausdorff, connected and locally connected such that $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{\mathcal{P}} \subset \mathcal{T}_{\mathcal{A}}$ and for which $\mathcal{L} = C(\mathcal{T}_{\mathcal{L}})$, $\mathcal{P} = C(\mathcal{T}_{\mathcal{P}})$ and $\mathcal{A} = C(\mathcal{T}_{\mathcal{A}})$. Moreover, it is consistent with* ZFC + GCH *that all these topologies are completely regular and Baire.*

The three corollaries above show that "nice" classes of real functions can be "nicely" topologized. However, Theorem 4 tells us also that variety of "wild" classes of real functions can be topologized as well. This is the case, for example, for the families $\mathcal{F}_1 = \text{Const} \cup \{x^3, e^x\}$, $\mathcal{F}_2 = \text{Const} \cup \{x^5 - 17, \sin x, 1/(x^2+1)\}$ and $\mathcal{F}_3 = \text{Const} \cup \{x, \ln(x^2+2), g\}$, where $g(x) = e^{-x^{-2}}$ for $x \neq 0$, $g(0) = 0$, is well known C^∞ function which is not analytic. Also, the functions in the class $\mathcal{F}$ must be neither measurable nor have the Baire property, since in Corollaries 4 and 5 the families $C(\mathcal{T}_O)$ and $\mathcal{A}$ can be replaced by any family $\mathcal{R}_0$ of real functions as long as $|\mathcal{R}_0| \leq 2^\omega$. In particular,

COROLLARY 7. *If* GCH *holds and $h : \mathbb{R} \longrightarrow \mathbb{R}$ is any one-to-one function then the family* $\text{Const} \cup \{h\}$ *can be topologized.*

The next corollary gives a negative answer for the following question of Lee Larson (private communication): "Let τ_1 and τ_2 be homothetically closed connected topologies on $\mathbb{R}$. Is it true that either $C(\tau_1, \tau_2) = \text{Const}$ or $\mathcal{L} \subset C(\tau_1, \tau_2)$?"

COROLLARY 8. *If* GCH *holds then there exist homothetically closed Hausdorff connected topologies τ_1 and τ_2 on $\mathbb{R}$ such that $C(\tau_1, \tau_2) \neq$ Const while $C(\tau_1, \tau_2) \cap \mathcal{L} =$ Const.*

for this initial value problem (they agree on $B[x_0, R]$, so their derivatives must also agree on the boundary of $B[x_0, R]$) they must be equal on U_x. Now, $\{U_x\}_{|x-x_0|=R}$ is an open cover of a compact set $\{x : |x - x_0| = R\}$ so, we can find finite subcover of it. But this means that we can find $r > R$ such that f and g agree on $B[x_0, r]$, contradicting maximality of R.

Notice also, that this scheme can be used to any class of functions defined by operator for which we can use Cauchy-Kovaleski Theorem.

PROOF. Let $f(x) = x^3$ for all $x \in \mathbb{R}$ and let $\mathcal{F} = \{L \circ f \circ M : L, M \in \mathcal{L}\} \subset \mathcal{A}$. Then, by Corollary 5, $\mathcal{F}$ can be topologized as $C(\tau_1, \tau_2)$. Moreover, by Theorem 1(iv), the topology τ_2 can be taken as homothetically closed. It is also easy to see that if τ_1 is a week topology, as in Theorem 4, then it is homothetically closed, connected and Hausdorff. This finishes the proof.

In fact, the assumption GCH in Corollary 8 is unnecessary. It will follow from the Theorem 6.

4. **Proof of Theorem 4(A).** In what follows we will write $H(A, B)$ for the set of all functions from a finite subset of A into B.

Take $X, Y, I, \mathcal{R}$ as in Theorem 4. In both parts of the theorem the topology τ_2 will be chosen in the following way. We will choose a topological space S, and construct a one-to-one function $e: Y \longrightarrow S^{(2^\omega)^+}$. Topology τ_2 on Y will be defined as a weak topology generated by the function $e: Y \longrightarrow S^{(2^\omega)^+}$. Thus, Y will be identified with the subspace $e[Y]$ of $S^{(2^\omega)^+}$.

Notice that if $\mathcal{B}_0 \subset \mathcal{P}(S)$ is such that $\mathcal{B}_0 \cup \{S\}$ forms a basis for S then the sets

$$[\delta]_Y = \Big\{y \in Y : \big(\forall d \in \mathrm{dom}(\delta)\big)\big(e(y)(d) \in \delta(d)\big)\Big\},$$

with $\delta \in H\big((2^\omega)^+, \mathcal{B}_0\big)$, form a basis for (Y, τ_2). Now, if $\mathcal{F}$ is as in Theorem 4 and we define $e_1: X \longrightarrow S^{\mathcal{F} \times (2^\omega)^+}$ by formula $e_1(x)(f, \xi) = e\big(f(x)\big)(\xi)$ then the sets

$$[\varepsilon]_X = \Big\{x \in X : \big(\forall d \in \mathrm{dom}(\varepsilon)\big)\big(e_1(x)(d) \in \varepsilon(d)\big)\Big\},$$

with $\varepsilon \in H\big(\mathcal{F} \times (2^\omega)^+, \mathcal{B}_0\big)$, will form a basis for the weak topology τ on X generated by $\mathcal{F}$, since for $f \in \mathcal{F}$ and $\{\langle \xi, B\rangle\} \in H\big((2^\omega)^+, \mathcal{B}_0\big)$

$$\begin{aligned} f^{-1}([\langle \xi, B\rangle]_Y) &= \{x : f(x) \in [\langle \xi, B\rangle]_Y\} \\ &= \Big\{x : e_1(x)(f, \xi) = e\big(f(x)\big)(\xi) \in B\Big\} \\ &= \Big\{x : x \in \big[\big\langle \langle f, \xi\rangle, B\big\rangle\big]_X\Big\} \\ &= \big[\big\langle \langle f, \xi\rangle, B\big\rangle\big]_X. \end{aligned}$$

Thus, X can be "identified" with $e_1[X]$. (Notice that the "identifying" function e_1 does not have to be one-to-one.)

It is easy to see that for such topologies we have $\mathcal{F} \subset C(\tau, \tau_2)$. Thus, the problem in our construction will be to show that any function $f \in Y^X \setminus \mathcal{F}$ is not in $C(\tau, \tau_2)$. This will be done by choosing an appropriate space S and an embedding e. Function e will be naturally identified with the mapping from $Y \times (2^\omega)^+$ into S.

In the case of the proof of Theorem 4(A) we will choose S to be the space $P = \{0, 1, 2\}$ with topology $\big\{\emptyset, P, \{0\}, \{1\}, \{0, 1\}\big\}$ and $\mathcal{B}_0 = \big\{\{0\}, \{1\}\big\}$. The construction of e will be done by induction on $\xi < (2^\omega)^+$ by listing all functions from Y^X as a sequence $\langle h_\zeta : \zeta \in (2^\omega)^+\rangle$, constructing an increasing sequence of functions $e|_{Y \times \xi}$ and, in step $\xi \in (2^\omega)^+$, defining $e|_{Y \times \{\xi\}}$ in such a way that $h_\xi^{-1}([\langle \xi, \{0\}\rangle]_Y) \notin \tau$ provided $h_\xi \notin C(\tau, \tau_2)$.

We will need two technical lemmas for our constructions. The first one will be used in both parts of the proof of Theorem 4. More precisely, assumptions (A) and (B) of Lemma 1 correspond to the proofs of parts (A) and (B) of Theorem 4, respectively. (Thus, if the reader likes to skip the proof of Theorem 4(B), he may skip the proof of second part of Lemma 1 as well.)

LEMMA 1. *Let (X,τ_1) and (Y,τ_2) be connected topological spaces of cardinality 2^ω such that Y is Hausdorff and that every $A \in [Y]^{<2^\omega}$ is closed in Y. Moreover, let* $\mathrm{Const} \subset \mathcal{F} \subset C(\tau_1,\tau_2)$ *and let $\mathcal{J}_0$ be a σ-ideal generated by all sets $\{x : f(x) = g(x)\}$ for $f,g \in \mathcal{F}, f \neq g$. If $\mathcal{J}$ is a σ-ideal containing $\mathcal{J}_0$ and $\mathcal{B}$ is a base for X such that*

(2) $\quad U \setminus D$ *is nonempty and connected for every* $U \in \mathcal{B}$ *and* $D \in \mathcal{J}$

then for every $h \in C(\tau_1,\tau_2) \setminus \mathcal{F}$ there is $x \in X$ such that for every $U \in \tau_1$ with $x \in U$, $D \in \mathcal{J}$ and $\mathcal{F}_0 \in [\mathcal{F}]^{\leq\omega}$ the following is true. If either

(A) $\mathcal{F}_0$ is finite; or

(B) $\mathcal{J} = \mathcal{J}_0$, X is a regular space and $\bigcap_{n<\omega} U_n \neq \emptyset$ for every sequence $\{U_n \in \mathcal{B} : n < \omega\}$ such that $\mathrm{cl}(U_{n+1}) \subset U_n$,

then

(3) $$\left|h\big(\{z \in U \setminus D : h(z) \neq f(z) \text{ for all } f \in \mathcal{F}_0\}\big)\right| = 2^\omega.$$

PROOF. Let $h \in C(\tau_1,\tau_2) \setminus \mathcal{F}$.

First notice that there is $x \in X$ such that for every $U \in \tau_1$ with $x \in U$,

(4) $$h_{|U} \neq f_{|U} \quad \text{for all } f \in \mathcal{F}.$$

To see this, assume, by way of contradiction, that for every $z \in X$ there is $f_z \in \mathcal{F}$ and $U_z \in \tau_1$ with $z \in U_z$, such that $h_{|U_z} = (f_z)_{|U_z}$. Let $U \in \tau_1$ be a maximal, nonempty set such that $h_{|U} = f_{|U}$ for some $f \in \mathcal{F}$. If $U \neq X$ then, by connectedness of (X,τ_1), there is $z \in \mathrm{cl}_{\tau_1}(U) \setminus U$. Then $U_z \cap U \neq \emptyset$ and $(f_z)_{|(U_z\cap U)} = h_{|(U_z\cap U)} = f_{|(U_z\cap U)}$, *i.e.*, f and f_z are equal on a nonempty open set. Thus, by (2), $f_z = f$, contradicting maximality of U. Hence, $U = X$. But then, $h = h_{|U} = f_{|U} = f$ contradicting the fact that $h \notin \mathcal{F}$. The condition (4) is proved.

Now, choose x as in (4). We will show that Lemma 1 holds for h and x.

For $f \in \mathcal{F}$ let

$$P_f = \{z \in X : h(z) = f(z)\}.$$

Thus, (4) tells us that

(5) $$U \not\subset P_f \quad \text{for all } f \in \mathcal{F} \text{ and } U \in \tau_1 \text{ with } x \in U.$$

Assume, by way of contradiction, that there are $W \in \tau_1$ with $x \in W$, $D_0 \in \mathcal{J}$ and $\mathcal{F}_0 \in [\mathcal{F}]^{\leq\omega}$ for which condition (3) does not hold, *i.e.*, such that $|h(B \setminus D_0)| < 2^\omega$ where

$$B = \{z \in W : h(z) \neq f(z) \text{ for all } f \in \mathcal{F}_0\} = W \setminus \bigcup_{f\in\mathcal{F}_0} P_f.$$

We will show that this contradicts (A) and (B).

Without loss of generality we may assume that the constant function equal to $h(x)$ is in $\mathcal{F}_0$. Hence, $h(x) \notin h(B)$. Since $h(B \setminus D_0)$ is closed in τ_2, $U = W \setminus h^{-1}\big(h(B \setminus D_0)\big) \in \tau_1$ and $x \in U \subset W \setminus (B \setminus D_0)$. Decreasing U, if necessary, we may assume also that $U \in \mathcal{B}$. Since $U \subset W$ and

$$(U \setminus D_0) \cap \Big(W \setminus \bigcup_{f \in \mathcal{F}_0} P_f\Big) = (U \setminus D_0) \cap B = U \cap (B \setminus D_0) = \emptyset$$

then

(6) $$U \setminus D_0 \subset \bigcup_{f \in \mathcal{F}_0} P_f.$$

Notice also that the sets P_f are closed, since τ_2 is Hausdorff, and that for every $f \in \mathcal{F}$ and $D \in \mathcal{J}$,

(7) $$U \subset P_f \text{ if and only if } U \setminus D \subset P_f.$$

Condition (7) follows from the fact that, by (2), $U \setminus D$ is dense in U and that $U \setminus D \subset P_f$ implies $U \subset \mathrm{cl}(U \setminus D) \subset P_f$.

Notice also that (5) and (7) in particular imply that

(8) $$U \setminus D \not\subset P_f \quad \text{for every } f \in \mathcal{F} \text{ and } D \in \mathcal{J}$$

so that, by (6), $|\mathcal{F}_0| > 1$.

We have two cases to consider.

CASE (A). Decreasing $\mathcal{F}_0$, if necessary, we may assume that $\mathcal{F}_0$ is minimal family satisfying (6), *i.e.*, that we have also

(9) $$P_f \cap (U \setminus D_0) \not\subset \bigcup_{g \in \mathcal{F}_0 \setminus \{f\}} P_g \text{ for every } f \in \mathcal{F}_0.$$

Put $D_1 = \bigcup\{P_f \cap P_g : f, g \in \mathcal{F}_0, f \neq g\} \subset \{z : f(z) = g(z) \text{ for some } f, g \in \mathcal{F}_0, f \neq g\} \in \mathcal{J}$ and $D = D_0 \cup D_1 \in \mathcal{J}$. Then, $U \setminus D$ is connected in τ_1 and the family $\{P_f \cap (U \setminus D)\}_{f \in \mathcal{F}_0}$ forms a partition of $U \setminus D$ by the sets relatively closed in $U \setminus D$. Moreover, $\mathcal{F}_0$ has at least two elements and, by (9), all these sets must be nonempty since $D_1 \subset \bigcup_{g \in \mathcal{F}_1 \setminus \{f\}} P_g$ and $P_f \cap (U \setminus D) = \big(P_f \cap (U \setminus D_0)\big) \setminus D_1$. This contradicts connectedness of $U \setminus D$. Case (A) is completed.

CASE (B). Notice that, by (2), $\mathrm{int}(P_f) \cap \mathrm{int}(P_g) = \emptyset$ for every $f, g \in \mathcal{F}$, $f \neq g$. Replacing sets P_f with sets $P_f \setminus \bigcup\{\mathrm{int}(P_g) : g \in \mathcal{F}_0, g \neq f\}$, if necessary, we can assume that in addition to (6) and (8) we have

(10) $$P_g \cap \mathrm{int}(P_f) = \emptyset \quad \text{for all } f, g \in \mathcal{F}_0, g \neq f.$$

Enumerate $\mathcal{F}_0$ as $\{f_n : 0 < n < \omega\}$. Increasing D_0, if necessary, we can also assume that $D_0 = \bigcup_{0<n<\omega} D_n$ where sets D_n, $0 < n < \omega$, are of the form $\{x : f(x) = g(x)\}$ for

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$f, g \in \mathcal{F}, f \neq g$, *i.e.*, D_n are closed and nowhere dense. We will construct a sequence $\{U_n \in \mathcal{B} : n < \omega\}$ of open subsets of U such that for every $n < \omega$ we will have

$$\mathrm{cl}(U_{n+1}) \subset U_n,\ U_{n+1} \cap (P_{f_{n+1}} \cup D_{n+1}) = \emptyset \text{ and } U_n \not\subset P_f \quad \text{for every } f \in \mathcal{F}.$$

This will finish the proof, since, by (B), we will get $\emptyset \neq \bigcap_{n<\omega} U_n \subset (U \setminus D_0) \setminus \bigcup_{f\in\mathcal{F}_0} P_f$ contradicting (6).

To see that the construction is possible start with putting $U_0 = U$. It satisfies the inductive hypothesis by (8). So, assume that U_n is already constructed for some $n < \omega$. If $P_g \cap (U_n \setminus D_0) \subset P_{f_{n+1}}$ for every $g \in \mathcal{F}_0$ then, by (6), $U_n \setminus D_0 \subset P_{f_{n+1}}$ and, by (7), $U_n \subset P_{f_{n+1}}$ contradicting our inductive assumption. So, choose $g \in \mathcal{F}_0$ such that $\emptyset \neq P_g \cap (U_n \setminus D) \not\subset P_{f_{n+1}}$, put $D'' = P_g \cap P_{f_{n+1}} \subset \{z : g(z) = f_{n+1}(z)\} \in \mathcal{J}$ and $D' = D_0 \cup D'' \in \mathcal{J}$. Then, $P_g \cap (U_n \setminus D') = \big(P_g \cap (U_n \setminus D_0)\big) \setminus P_{f_{n+1}} \neq \emptyset$ and $P_g \cap (U_n \setminus D') \neq U_n \setminus D'$ since the equation $P_g \cap (U_n \setminus D') = U_n \setminus D' = (U_n \setminus D_0) \setminus D''$ would imply $U_n \setminus D_0 \subset P_g \cup D'' \subset P_g$ and, by (7), $U_n \subset P_g$, contradicting our inductive assumption. Thus, $P_g \cap (U_n \setminus D')$ cannot be open in $U_n \setminus D'$ since it is closed in $U_n \setminus D'$ and $U_n \setminus D'$ is connected. So, choose $z \in (U_n \setminus D') \cap \big(P_g \setminus \mathrm{int}(P_g)\big)$. Since $z \notin P_{f_{n+1}} \cup D_{n+1}$ and X is regular, we can choose U_{n+1} such that $z \in U_{n+1} \subset \mathrm{cl}(U_{n+1}) \subset U_n$ and $U_{n+1} \cap (P_{f_{n+1}} \cup D_{n+1}) = \emptyset$. To finish the proof it is enough to show that $U_{n+1} \not\subset P_f$ for every $f \in \mathcal{F}$. So, by way of contradiction assume that there is $f \in \mathcal{F}$ such that $U_{n+1} \subset P_f$. Since $z \notin \mathrm{int}(P_g)$ we have $U_{n+1} \not\subset P_g$ and so, $f \neq g$. But then, $z \in P_g \cap U_{n+1} \subset \mathrm{int}(P_f)$ contradicting (10). Lemma 1 has been proved.

We already noticed that topologies τ and τ_2 should be connected. The next lemma explains how we are going to achieve this goal for the proof of Theorem 4(A). In the next lemma $[\delta]$ will stand for

$$[\delta] = \{g \in P^Z : \delta \subset g\}.$$

LEMMA 2. *Let Z be an arbitrary set and let $M \subset P^Z$ be such that $[\delta] \cap M \neq \emptyset$ for every $\delta \in H(Z,3)$. Then for every $\delta \in H(Z,2)$ the set $[\delta] \cap M$ is connected in P^Z. In particular, M considered as a subspace of P^Z is connected and locally connected.*

PROOF. For the use of this proof let $[\delta]$ denote $[\delta] \cap M$ for $\delta \in H(Z,3)$. Then, for every $\varepsilon, \delta \in H(Z,3)$

$$[\varepsilon] \cap [\delta] \neq \emptyset \text{ if and only if } \varepsilon \cup \delta \in H(Z,3)$$

and

$$[\varepsilon] \subset [\delta] \text{ if and only if } \delta \subset \varepsilon. \tag{11}$$

To argue for (11) it is enough to show that $[\varepsilon] \subset [\delta]$ implies $\delta \subset \varepsilon$, since the other inclusion is obvious. But if $\delta \not\subset \varepsilon$ then there exists $\varepsilon' \in H(Z,3)$ extending ε such that $\varepsilon' \cup \delta \notin H(Z,3)$. Hence, $[\varepsilon'] \cap [\delta] = \emptyset$, while $\emptyset \neq [\varepsilon'] \subset [\varepsilon]$. This contradicts $[\varepsilon] \subset [\delta]$.

Now, let us turn to the proof of connectedness of $[\delta]$, where $\delta \in H(Z,2)$. Let $\varepsilon_0, \varepsilon_1 \in H(Z,2)$ be such that $[\varepsilon_0]$ and $[\varepsilon_1]$ are nonempty disjoint subsets of $[\delta]$. Then, by (11),

$\delta \subset \varepsilon_i$ for $i < 2$, *i.e.*, $\varepsilon_i = \delta \cup \delta_i$ for some $\delta_i \in H(Z, 2)$ such that $\mathrm{dom}(\delta) \cap \mathrm{dom}(\delta_i) = \emptyset$. Let $D = \mathrm{dom}(\delta_0) \cup \mathrm{dom}(\delta_1)$ and define $\eta: D \longrightarrow 3$ by $\eta(d) = 2$ for all $d \in D$. Then, $\emptyset \neq [\eta \cup \delta] \subset [\delta]$. We will see that $[\eta \cup \delta] \subset \mathrm{cl}[\varepsilon_0] \cap \mathrm{cl}[\varepsilon_1]$. This will clearly imply that $[\delta]$ is connected.

So, let $\xi \in H(Z, 2)$ be such that $[\xi] \cap [\eta \cup \delta] \neq \emptyset$. Then, $\xi \cup \eta \cup \delta \in H(Z, 3)$, *i.e.*, for $i < 2$, $\xi \cup \varepsilon_i = \xi \cup \delta_i \cup \delta \in H(Z, 2)$ and so, $[\xi] \cap [\varepsilon_i] \neq \emptyset$. But this means that the set $[\eta \cup \delta] \subset [\delta]$ is in the closure of both $[\varepsilon_0]$ and $[\varepsilon_1]$.

Since sets $[\delta] = [\delta] \cap M$ for $\delta \in H(Z, 2)$ form basis for M we conclude that indeed M is connected and locally connected. Lemma 2 is proved.

Now we are ready for the main part of the proof.

CONSTRUCTION OF EMBEDDING e. Assume GCH and take X, Y, I and $\mathcal{R}$ as in Theorem 4. By the structure of condition (1) it is easy to see that we can assume that $\mathcal{R} \cap \mathrm{Const} = \emptyset$ and

$$f^{-1}(y) \in I \quad \text{for all } f \in \mathcal{R} \text{ and } y \in Y.$$

Let I_0 be a family of all $I \in I$ such that either $|I| = 1$ or $I = \{x \in X : f(x) = g(x)\} \in I$ for some $f, g \in \mathcal{R}$ and let $\mathcal{J} \subset I$ be the σ-ideal generated by I_0. Let $\mathcal{B}_1 = \mathcal{B}_0 \cup \{\{2\}\} = \{\{0\}, \{1\}, \{2\}\}$ and for $T \subset (2^\omega)^+$ define $\mathcal{H}(T)$ as the set of all $\varepsilon \in H(\mathcal{R} \times T, \mathcal{B}_1)$ such that

$$\{x : f(x) = g(x)\} \in I \quad \text{for all } \langle f, \eta\rangle,\ \langle g, \eta\rangle \in \mathrm{dom}(\varepsilon),\ f \neq g.$$

Moreover, let

$$\{\langle h_\zeta, x_\zeta, A_\zeta\rangle : \zeta < (2^\omega)^+\} \tag{12}$$

be an enumeration of $Y^X \times X \times [Y]^{<2^\omega}$. This can be chosen by GCH.

We will construct, by induction on $\zeta < (2^\omega)^+$, an increasing sequence of functions $\{e|_{Y\times\zeta} : \zeta < (2^\omega)^+\}$ that will satisfy the following inductive conditions for all $\eta < \zeta < (2^\omega)^+$, when we adopt the notation introduced on the beginning of this section:

(a) $A_\eta \subset \big[\{\langle \eta, \{1\}\rangle\}\big]_Y$ and $h_\eta(x_\eta) \in \big[\{\langle \eta, \{0\}\rangle\}\big]_Y$ provided $h_\eta(x_\eta) \notin A_\eta$;

(b) $|[\delta]_Y| = 2^\omega$ for all $\delta \in H(\zeta, \mathcal{B}_1)$;

(c) $[\varepsilon]_X \notin \mathcal{J}$ for all $\varepsilon \in \mathcal{H}(\zeta)$;

(d) $[\varepsilon]_X \cap h_\eta^{-1}\Big(\big[\{\langle \eta, \{2\}\rangle\}\big]_Y\Big) \notin \mathcal{J}$ for all $\varepsilon \in \mathcal{H}(\zeta)$ such that for every $I \in \mathcal{J}$

$$\Big|h_\eta\big(\{z \in [\varepsilon]_X \setminus I : h_\eta(z) \neq f(z) \text{ for all } \langle f, \eta\rangle \in \mathrm{dom}(\varepsilon)\}\big)\Big| = 2^\omega. \tag{13}$$

First, let us see how conditions (a)–(c) imply the theorem. So, let $\mathcal{F}$ be as in the theorem and let topology τ_2 be chosen as described in the beginning of the section. Condition (a) clearly implies that every $A \in [Y]^{<2^\omega}$ can be separated from every point $y \in Y \setminus A$. (Simply choose $\eta \in (2^\omega)^+$ such that $A_\eta = A$, $h_\eta \equiv y$.) Thus, Y is Hausdorff and every $A \in [Y]^{<2^\omega}$ is closed in Y. This also implies that e is one-to-one and that (X, τ) is Hausdorff, provided $\mathcal{F}$ separates points.

Condition (b) and Lemma 2 used with $M = e[Y]$ imply that (Y, τ_2) is connected and locally connected, since $M \cap [\{\langle \xi, i\rangle\}] = e\Big[\big[\{\langle \xi, \{i\}\rangle\}\big]_Y\Big]$. Similarly, since $H\big(\mathcal{F} \times (2^\omega)^+, \mathcal{B}_1\big) \subset \mathcal{H}\big((2^\omega)^+\big)$ we can use (c) and Lemma 2 with $M = e_1[X \setminus D]$,

$D \in \mathcal{J}$, to conclude that (X,τ) is connected, locally connected and that $[\varepsilon]_X \setminus D$ is connected for every $D \in \mathcal{J}$ and every $\varepsilon \in H(\mathcal{F} \times (2^\omega)^+, \mathcal{B}_0)$. Therefore, condition (2) from Lemma 1 is satisfied with family $\mathcal{B}$ defined by $\mathcal{B} = \left\{[\varepsilon]_X : \varepsilon \in H\left(\mathcal{F} \times (2^\omega)^+, \mathcal{B}_0\right)\right\}$ and $\mathcal{J}$ defined as above. To finish the argument it is enough to show that $C(\tau,\tau_2) \subset \mathcal{F}$, since the converse inclusion is clear. By way of contradiction assume that there is $h \in C(\tau,\tau_2) \setminus \mathcal{F}$. Let x be as in Lemma 1 and let $\eta \in (2^\omega)^+$ be such that $h = h_\eta$, $x = x_\eta$ and $A_\eta = \emptyset$. Then, by (a), $x \in h^{-1}\left(\left[\left\{\langle \eta, \{0\}\rangle\right\}\right]_Y\right)$. Moreover, by Lemma 1, condition (13) is satisfied for every $[\varepsilon]_X \in \mathcal{B}$ such that $x \in [\varepsilon]_X$. Thus, by (d), $[\varepsilon]_X \cap h^{-1}\left(\left[\left\{\langle \eta, \{2\}\rangle\right\}\right]_Y\right) \neq \emptyset$. Hence, none basic open set $[\varepsilon]_X \in \tau$ with $x \in [\varepsilon]_X$, can be contained in $h^{-1}\left(\left[\left\{\langle \eta, \{0\}\rangle\right\}\right]_Y\right) \neq \emptyset$ and so, h cannot be continuous. This contradiction shows that $\mathcal{F} = C(\tau,\tau_2)$.

To finish the proof it is enough to make the inductive construction. Let $\{I_\xi : \xi < 2^\omega\}$ be an enumeration of I_0 and let us assume that for some $\zeta < (2^\omega)^+$ the construction is indeed done. We will construct $e|_{Y\times\{\zeta\}}$. To do this, let

$$\{\langle \delta_\xi^0, \varepsilon_\xi^0, \alpha_\xi, \delta_\xi, \varepsilon_\xi\rangle : 0 < \xi < 2^\omega\}$$

be an enumeration of $H(\zeta, \mathcal{B}_1) \times \mathcal{H}(\zeta) \times (\zeta+1) \times H(\{\zeta\}, \mathcal{B}_1) \times \mathcal{H}(\{\zeta\})$ with each tuple appearing in the sequence continuum many times. We will construct, by induction on $\xi < 2^\omega$, an increasing sequence of functions $\{e|_{Y_\xi\times\{\zeta\}}\}$, where $Y_\xi \subset Y$, by starting with $Y_0 = A_\zeta \cup \{h_\zeta(x_\zeta)\}$, defining $e(d,\zeta) = 1$ for $d \in A_\zeta$, $e\left(h_\zeta(x_\zeta),\zeta\right) = 0$ if $h_\zeta(x_\zeta) \notin A_\zeta$ and such that the following inductive conditions are hold for $0 < \xi < 2^\omega$:

(i) $Y_\xi \setminus \bigcup_{\eta<\xi} Y_\eta$ is finite;

(ii) there is $y \in Y_\xi \setminus \bigcup_{\eta<\xi} Y_\eta$ such that $y \in [\delta_\xi^0]_Y \cap [\delta_\xi]_Y$;

(iii) there exists $x \in [\varepsilon_\xi^0]_X \setminus \bigcup_{\eta<\xi} I_\eta$ such that $f(x) \in \varepsilon_\xi(f,\zeta) \cap Y_\xi \setminus \bigcup_{\eta<\xi} Y_\eta$ for all $\langle f,\zeta\rangle \in \operatorname{dom}(\varepsilon_\xi)$; moreover, we have $h_{\alpha_\xi}(x) \in \left[\left\{\langle \alpha_\xi, \{2\}\rangle\right\}\right]_Y$ provided for every $I \in \mathcal{J}$

$$\left|h_{\alpha_\xi}\left(\{z \in [\varepsilon_\xi^0]_X \setminus I : h_{\alpha_\xi}(z) \neq f(z) \text{ for all } \langle f,\zeta\rangle \in \operatorname{dom}(\varepsilon_\xi)\}\right)\right| = 2^\omega.$$

Notice that this will imply (a)–(d) when we extend $\bigcup_{\xi<2^\omega} e|_{Y_\xi\times\{\zeta\}}$ to $e|_{Y\times\{\zeta\}}$ arbitrarily. Clearly (a) is implied by the definition of e on $Y_0 \times \{\zeta\}$. Condition (b) is implied by (ii), since functions $\delta_\xi^0 \cup \delta_\xi$ list all $H(\zeta+1, \mathcal{B}_1)$ and each function appears there continuum many times. Finally, (c) and (d) are implied by (iii) in similar way, if we notice that every function from $\mathcal{H}(\zeta+1)$ appears as $\varepsilon_\xi^0 \cup \varepsilon_\xi$ for continuum many ξ and that for every $I \in \mathcal{J}$ there is $\xi < 2^\omega$ such that $I \subset \bigcup_{\eta<\xi} I_\eta$.

So, let us assume that for some $\xi < 2^\omega$ construction is done. Notice that the set $\bigcup_{\eta<\xi} Y_\eta$ has cardinality less that continuum. To get y satisfying (ii) it is enough to choose $y \in [\delta_\xi^0]_Y \setminus \bigcup_{\eta<\xi} Y_\eta$ and define $e(y,\zeta) = i$ where $\delta_\xi = \left\{\langle \zeta, \{i\}\rangle\right\}$. To get (iii) let $\varepsilon_\xi = \left\{\langle f_j, \zeta, \{i_j\}\rangle : j < n\right\}$ and put

$$I = \{x \in X : f_j(x) = f_k(x) \text{ for some } j < k < n\}.$$

Then $I \in I$, since $\varepsilon_\xi \in \mathcal{H}(\{\zeta\})$. The set $T_0 = \{y\} \cup \bigcup_{\eta<\xi} Y_\eta$ has cardinality $< 2^\omega$ so $T = I \cup \bigcup_{j<n} f_j^{-1}(T_0) \in \mathcal{J}$ and, by the inductive hypothesis, we can choose $x \in$

$[\varepsilon^0_\xi]_X \setminus (T \cup \bigcup_{\eta<\xi} I_\eta)$. Since $f_j(x) \neq f_k(x)$ for $j < k < n$ and $f_j(x) \notin T_0$ we can define $e\big(f_j(x), \zeta\big) = i_j$ for $j < n$. This gives the main part of (iii). Moreover, if additional assumption is satisfied than we can choose $x \in \{z \in [\varepsilon^0_\xi]_X \setminus (T \cup \bigcup_{\eta<\xi} I_\eta) : h_{\alpha_\xi}(z) \neq f_j(z)$ for all $j < n\}$ such that $h_{\alpha_\xi}(x) \notin T_0$. So, we can freely define $e(h_{\alpha_\xi}(x), \zeta) = 2$. The construction is finished if we define $Y_\xi = \{f_j(x) : j < n\} \cup \{h_{\alpha_\xi}(x), y\} \cup \bigcup_{\eta<\xi} Y_\eta$. (We define $e\big(h_{\alpha_\xi}(x), \zeta\big)$ arbitrarily, when the additional requirement of (iii) is not satisfied.)

This finishes the proof of Theorem 4(A).

5. **Discussion of the assumptions and generalizations of Theorem 4.** We start here with noticing that all families $\mathcal{F}$ from Theorem 4 can be topologized with the same topology τ_2 on the range. It would be nice to prove Theorem 4 with $\mathcal{R} = Y^X$, *i.e.*, to have the same universal topology τ_2 that could be used for topologizing all families $\mathcal{F} \subset Y^X$ containing constant functions and satisfying condition (1) from Theorem 4. However, this cannot be done at least as long as we assume that $X = Y$ and that the "universal" topology τ_2 contains a set U such that $|U| = |X \setminus U| = 2^\omega$. This is the case, since then for a bijection $f: X \longrightarrow X$ such that $f(X \setminus U) = U$ and $f(x) \neq x$ for all $x \in X$ the family $\mathrm{Const} \cup \{\mathrm{id}_X, f\}$ could be topologized. But the family $\mathrm{Const} \cup \{\mathrm{id}_X, f\}$ cannot be topologized since the domain topology would have to contain both U and $X \setminus U$, and thus, the class of continuous functions would contain also a characteristic function of U.

It is not clear at this point whether the Theorem 4(A) or (B) can be proved without any additional set theoretical assumptions. However, it is easy to see that the real assumptions we have used in the proof is that $2^{2^\omega} = (2^\omega)^+$ and that $\bigcup \mathcal{G} \neq X$ for every $\mathcal{G} \in [I]^{<2^\omega}$. For general ideals, this last assumption does not have to be satisfied if $2^\omega > \omega_1$. However, if we consider only σ-ideal $I = I_\omega = [X]^{\leq\omega}$ then the situation simplifies and essentially the original proof of Theorem 4(A) works with the assumption $2^{2^\omega} = (2^\omega)^+$ in place of GCH. The only change in the proof of Theorem 4(A) that has to be made is to define $\mathcal{J}$ as $[X]^{<2^\omega}$ and remove sets I_η from condition (iii). Thus, we can state this in form of next theorem.

THEOREM 5. *Let $|X| = |Y| = 2^\omega$ and $\mathcal{R} \in [Y^X]^{\leq 2^\omega}$. If $2^{2^\omega} = (2^\omega)^+$ then there is a Hausdorff, connected and locally connected topology τ_2 on Y such that for every family $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ with the property that* $\mathrm{Const} \subset \mathcal{F}$ *and*

$$\{x \in X : f(x) = g(x)\} \in [X]^{\leq\omega} \quad \textit{for every distinct } f, g \in \mathcal{F}$$

we have

$$\mathcal{F} = C(\tau, \tau_2),$$

where τ is generated by the family $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$. Topology τ is connected and locally connected. It is also Hausdorff, provided $\mathcal{F}$ separates points.

In particular, in Corollaries 5, 6 and 7 the assumption of GCH can be replaced by the assumption that $2^{2^\omega} = (2^\omega)^+$.

On the other hand, the assumption $2^{2^\omega} = (2^\omega)^+$ is fundamental for the proofs of Theorems 4 and 5. However, we still are able to get the following version of Theorem 5 without any extra set theoretical assumption.

THEOREM 6. *Let* $|X| = |Y| = 2^\omega$ *and* $\mathcal{R} \in [Y^X]^{\le 2^\omega}$. *If* $\mathcal{K} \in [Y^X]^{\le (2^\omega)^+}$ *then there is a Hausdorff, connected and locally connected topology* τ_2 *on* Y *such that for every family* $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ *with the property that* Const $\subset \mathcal{F}$ *and*

$$\{x \in X : f(x) = g(x)\} \in [X]^{\le \omega} \quad \text{for every distinct } f, g \in \mathcal{F}$$

we have

$$\mathcal{K} \cap \mathcal{F} = \mathcal{K} \cap C(\tau, \tau_2), \tag{14}$$

where τ *is generated by the family* $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$. *Topology* τ *is connected and locally connected. It is also Hausdorff, provided* $\mathcal{F}$ *separates points.*

PROOF. We used the assumption $2^{2^\omega} = (2^\omega)^+$ in Theorem 5 only in (12). However there were two reasons for it's use. The main reason was to enumerate all functions from X^Y in a sequence of length of $(2^\omega)^+$ and make sure that none of the function from the list is in $C(\tau, \tau_2)$, provided it is not in $\mathcal{F}$. If we list that way all the functions from $\mathcal{K}$ than we can deduce (14).

The second use of $2^{2^\omega} = (2^\omega)^+$ in (12) was to enumerate $[Y]^{<2^\omega}$ in sequence of length $(2^\omega)^+$. However, we enumerated the sets from $[Y]^{<2^\omega}$ to make sure that they are closed in (Y, τ_2). But, the only place in the proof we really needed this fact was the proof of Lemma 1. Moreover, we did not need this fact for all sets from $[Y]^{<2^\omega}$ but only for the sets of the particular form of $h\big(\{z \in [\varepsilon]_X \setminus D : h(z) \neq f(z) \text{ for all } f \in \mathcal{F}_0\}\big)$, where $D \in [Y]^{\le\omega}$, $\mathcal{F}_0 \in [\mathcal{F}]^{\le\omega}$ and $h \in X^Y$. In case of this theorem need to consider only h from $\mathcal{K}$. Thus, there is only $(2^\omega)^+$-many sets of this form, and we can list all of them in a sequence of length $(2^\omega)^+$. This finishes the proof of Theorem 6.

Let us notice, that using Theorem 6 we can deduce Corollary 8 without any additional set-theoretical assumptions.

As a last position in this section we like to discuss separation axioms for topologizing topologies.

A disadvantage of the original form of Theorem 4 is that for families $\mathcal{F}$ that do not separate points the topology τ of the domain is not Hausdorff. Can we modify the theorem to make topologizing topologies Hausdorff even if $\mathcal{F}$ does no separates points? The positive answer is given by the next theorem.

THEOREM 7. *Let* $|X| = |Y| = 2^\omega$, $\mathcal{R} \in [Y^X]^{\le 2^\omega}$ *and let* I *be a proper* σ*-ideal on* X *containing all singletons.*

(A) If GCH *holds then there are topologies* τ_1 *and* τ_2 *on* X *and* Y *respectively such that for every family* $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ *with the property that* Const $\subset \mathcal{F}$ *and*

$$\{x \in X : f(x) = g(x)\} \in I \quad \text{for every distinct } f, g \in \mathcal{F}$$

we have

$$\mathcal{F} = C(\tau, \tau_2),$$

where τ *is generated by the family* $\tau_1 \cup \{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$. *Topologies* τ_1, τ *and* τ_2 *are Hausdorff, connected and locally connected.*

(B) Moreover, it is consistent with the usual axioms of set theory ZFC *and* GCH *that the topologies* τ_1, τ *and* τ_2 *are completely regular and Baire.*

Sketch of the proof. Modify the proofs of either of the part of the Theorem 4 as follows. Partition $(2^\omega)^+$ into two sets C and D of the size $(2^\omega)^+$ and choose $k: X \longrightarrow Y$ such that $|\{x \in X : f(x) = k(x)\}| < 2^\omega$ for every $f \in \mathcal{R}$. In case of proof of part (A) make sure that every tuple listed in (12) appears for η in C and in D. Next, consider τ_2 as topology generated by restricted embedding $e': Y \longrightarrow S^C$ of e and τ_1 by restricted embedding $e'_1: X \longrightarrow S^{\{k\}\times D}$ of e_1.

It is not difficult to check that these topologies will have the desired properties.

Topologies τ, τ_1 and τ_2 Theorem 7 are Hausdorff and connected. The similar is true for the topologies of the Theorem 4, if $\mathcal{F}$ separates points. Notice also that from Theorem 1(v) it is clear that the topologies must be connected. But, do they have to be Hausdorff? The next theorem gives the negative answer to this question.

THEOREM 8. *Let* $|X| = |Y| = 2^\omega$, $\mathcal{R} \in [Y^X]^{\leq 2^\omega}$ *and let* I *be a proper* σ*-ideal on* X *containing all singletons. If* GCH *holds then there is a topology* τ_2 *on* Y *such that for every family* $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ *with the property that* Const $\subset \mathcal{F}$ *and*

$$\{x \in X : f(x) = g(x)\} \in I \quad \textit{for every distinct } f, g \in \mathcal{F}$$

we have

$$\mathcal{F} = C(\tau, \tau_2),$$

where τ *is generated by the family* $\{f^{-1}(U) : U \in \tau_2, f \in \mathcal{F}\}$*. Topologies* τ *and* τ_2 *are connected, locally connected and* T_1*. However, they are not Hausdorff.*

PROOF. Change topology on $P = \{0, 1, 2\}$ to $\{\emptyset, P, \{0\}\}$. It is easy to see that it works.

6. **Proof of Theorem 4(B).** In this section we will write $H_\omega(A, B)$ for $\{s \in B^D : D \in [A]^{\leq\omega}\}$.

The idea of the proof of Theorem 4(B) is essentially the same as that of Theorem 4(A), except that we will take as space S the unit interval $[0, 1]$ with the natural topology and we will construct an embedding $e: Y \longrightarrow [0, 1]^{(2^\omega)^+}$ using the forcing method.

So, let V be a model of ZFC in which GCH holds and let

$$P = H_\omega(\omega_2, [0, 1])$$

be a forcing notion in V ordered by the reverse inclusion. Let G be a V-generic filter over P and put $g = \bigcup G: \omega_2 \longrightarrow [0, 1]$. We will show that the statement from the theorem is true in $V[G] = V[g]$.

It is clear that P is ω-closed. In particular, the real numbers in V and in $V[g]$ are the same so, we do not have to be worry about the different sets of real numbers $\mathbb{R}$ in V and in $V[g]$. It is also well known (see *e.g.* [11]) that under CH forcing P satisfies ω_2-chain condition. So, the cardinals are preserved by P and, since $|H_\omega(\omega_2,[0,1])| = \omega_2$, GCH holds in $V[G]$.

Take $X, Y, I, \mathcal{R} \in V[g]$ as in Theorem 4. Clearly it is enough to prove Theorem 4 for any sets X and Y of cardinality 2^ω. In particular, we can assume that $X, Y \in V$ are the sets of ordinal numbers. As in the proof of Theorem 4(A) assume that $\mathcal{R} \cap \mathrm{Const} = \emptyset$ and

$$f^{-1}(y) \in I \quad \text{for all } f \in \mathcal{R} \text{ and } y \in Y.$$

Define I_0 as a family of all sets $I \in I$ such that either $|I| = 1$ or $I = \{x \in X : f(x) = g(x)\}$ for some $f, g \in \mathcal{R}$, let $\mathcal{J}_0$ be a family of all countable unions of sets from I_0 and let $\mathcal{J} \subset I$ be the σ-ideal generated by $\mathcal{J}_0$.

Next, choose $\mathcal{R}' \subset Y^X \setminus (\mathcal{R} \cup \mathrm{Const})$ of cardinality continuum such that $|\{x \in X : f(x) = g(x)\}| < 2^\omega$ for every $f \in \mathcal{R} \cup \mathrm{Const}$ and $g \in \mathcal{R}'$ and such that

$$I_0 = \big\{\{x \in X : f(x) = g(x)\} : f, g \in \mathcal{R}', f \neq g\big\}. \tag{15}$$

Such a family can be easily chosen by transfinite induction. Put $\mathcal{R}_0 = \mathcal{R} \cup \mathcal{R}'$. Notice that if $\mathcal{F} \subset \mathcal{R}'$ satisfies the assumption (1) of Theorem 4, then so does $\mathcal{F} \cup \mathcal{R}'$.

Since P satisfies ω_2-chain condition, $X, Y \in V$ and all sets $X, Y, \mathcal{R}_0, \mathcal{J}_0$ have cardinality $\leq \omega_1$ there is $\xi < \omega_2$ such that $\mathcal{R}_0 \cup \{\mathcal{R}_0\} \cup \mathcal{J}_0 \cup \{\mathcal{J}_0\} \subset V[g|_\xi]$. Thus, we can work in extension $V[g|_\xi]$ of V instead in V. To cut unnecessary notational problems we simple treat $V[g|_\xi]$ as V, *i.e.*, we are assuming that $\mathcal{R}_0 \cup \{\mathcal{R}_0\} \cup \mathcal{J}_0 \cup \{\mathcal{J}_0\} \subset V$.

Since P defined as above is isomorphic to $H_\omega(Y \times \omega_2, [0,1])$ ordered by the reverse inclusion, we also assume that

$$P = H_\omega(Y \times \omega_2, [0,1]).$$

Now, if G is a V-generic filter over P then $g = \bigcup G \colon Y \times \omega_2 \longrightarrow [0,1]$. We define embedding $e \colon Y \longrightarrow [0,1]^{\omega_2}$ by $e(y)(\xi) = g(y,\xi)$.

Let $\mathcal{F} \subset \mathcal{R}_0$, $\mathcal{F} \in V[g]$, be as in Theorem 4. Topologies τ_2 and τ are defined as before. So, it is enough to show that they satisfy the desired properties.

Since for every $A \in [Y]^{\leq\omega}$ and $y \in Y \setminus A$ the set

$$\begin{aligned}\big\{s \in P : (\exists \alpha < \omega_2)(s \Vdash \text{“}A \subset [\{\langle \alpha, \{0\}\rangle\}]_Y \text{ and } y \in [\{\langle \alpha, \{1\}\rangle\}]_Y\text{”})\big\}\\ = \{s \in P : (\exists \alpha < \omega_2)(\forall a \in A)(\langle a, \alpha, 0\rangle, \langle y, \alpha, 1\rangle \in s)\}\end{aligned}$$

is dense in P, we conclude easily that τ_2 is Hausdorff and every $A \in [Y]^{\leq\omega}$ is closed in (Y, τ_2). Then, it is also obvious that the weak topology τ on X generated by $\mathcal{F}$ is Hausdorff if and only if $\mathcal{F}$ separates points. Also, since $e[Y]$ and $e_1[X]$ are subspaces of a product of $[0,1]$ we can easily conclude that (Y, τ_2) and (X, τ) are completely regular.

To prove connectedness of these topologies we need some extra facts and notations. Let $\mathcal{B}_0$ be a countable base for $[0,1]$ and let $\mathcal{B}_Z$ be the standard base for $[0,1]^Z$ associated with $\mathcal{B}_0$, *i.e.*, $\mathcal{B}_Z$ is the family of all sets

$$[\delta] = \big\{g \in [0,1]^Z : \big(\forall z \in \mathrm{dom}(\delta)\big)\big(g(z) \in \delta(z)\big)\big\},$$

where $\delta \in H(Z, \mathcal{B}_0)$. We will need the following analog of Lemma 2 that holds for every $U \in \mathcal{B}_Z$.

(16) If $S \subset U$ is disconnected then $[\varepsilon] \subset U \setminus S$ for some $\varepsilon \in H_\omega(Z, [0, 1])$,

where $[\varepsilon] = \{f \in [0, 1]^Z : \varepsilon \subset f\}$. This is a well known fact and it can be found in [12] or [3].

Thus, to show that (Y, τ_2) is connected and locally connected it is enough to show that $Y \cap [\varepsilon] \neq \emptyset$, for every $\varepsilon \in H_\omega(\omega_2, [0, 1])$. This easily follows from the density, in P, of a set

$$\begin{aligned} E_\varepsilon &= \{s \in P : (\exists y \in Y)(s \Vdash \text{“}y \in [\varepsilon]\text{”})\} \\ &= \Big\{s \in P : (\exists y \in Y)\big(\forall \xi \in \operatorname{dom}(\varepsilon)\big)\big(\langle y, \xi, \varepsilon(\xi)\rangle \in s\big)\Big\} \end{aligned}$$

for every $\varepsilon \in H_\omega(\omega_2, [0, 1])$.

The connectedness and local connectedness of (X, τ) we can be deduced similarly. However we need a stronger fact, *i.e.*, condition (2) of Lemma 1. We will show that for every $I \in \mathcal{J}$ and $\varepsilon \in H(\mathcal{F} \times \omega_2, \mathcal{B}_0)$

(17) $X \cap [\varepsilon] \setminus I$ is nonempty and connected in (X, τ),

i.e., that condition (2) of Lemma 1 is satisfied for topologies τ and τ_2. Notice that it obviously implies that (X, τ) is connected and locally connected.

By (16) in order to prove (17) it is enough to show that

(18) $X \cap [\varepsilon] \setminus I \neq \emptyset$ for every $I \in \mathcal{J}_0$ and $\varepsilon \in H_\omega\big((\mathcal{F} \cup \mathcal{R}') \times \omega_2, \mathcal{B}_0\big)$.

Condition (18) follows in natural way from the density of the sets

$$\begin{aligned} E_I^\varepsilon &= \{s \in P : (\exists x \in X \setminus I)(s \Vdash \text{“}x \in [\varepsilon]\text{”})\} \\ &= \{s \in P : (\exists x \in X \setminus I)\big(\forall \langle f, \xi\rangle \in \operatorname{dom}(\varepsilon)\big)\big(\langle f(x), \xi, \varepsilon(f, \xi)\rangle \in s\big)\} \end{aligned}$$

for all $I \in \mathcal{J}_0$ and $\varepsilon \in H_\omega(\mathcal{F} \times \omega_2, \mathcal{B}_0)$. The density of this set can be, in turn, deduced from the fact that for every $s \in P$

$$\begin{aligned} I' &= \{x \in X : f(x) = g(x) \text{ for } \langle f, \xi\rangle, \langle g, \xi\rangle \in \operatorname{dom}(\varepsilon), f \neq g\} \in \mathcal{J}_0, \\ I'' &= \{x \in X : \langle f, \xi\rangle \in \operatorname{dom}(\varepsilon) \text{ and } \langle f(x), \xi\rangle \in \operatorname{dom}(s)\} \in \mathcal{J}_0 \end{aligned}$$

so that there exists $x \in X \setminus (I \cup I' \cup I'')$.

To finish the proof it is enough to show that τ and τ_2 are Baire and that $C(\tau, \tau_2) \subset \mathcal{F}$, since the inclusion $\mathcal{F} \subset C(\tau, \tau_2)$ is obvious.

By way of contradiction let us assume that we can find $h \in C(\tau, \tau_2) \setminus \mathcal{F}$. Let τ' be a weak topology on X generated by $\mathcal{F} \cup \mathcal{R}'$. Then, in particular, $h \in C(\tau', \tau_2) \setminus \mathcal{F}$. We will use Lemma 1(B) for h, topologies τ' and τ_2 and the σ-ideal $\mathcal{J}$. We already checked that (X, τ') and (Y, τ_2) are regular, connected, locally connected, that Y is Hausdorff and that countable subsets of Y are closed. Condition (2) of Lemma 1 is satisfied by (17), and $\mathcal{J}$ is equal to $\mathcal{J}_0$ from the Lemma 1 by (15). To use Lemma 1(B) we have to show that $\bigcap_{n<\omega}[\varepsilon_n] \neq \emptyset$ for every sequence $\{\varepsilon_n \in H\big((\mathcal{F} \cup \mathcal{R}') \times \omega_2, \mathcal{B}_0\big) : n < \omega\}$ such that $\mathrm{cl}_X([\varepsilon_{n+1}]) \subset [\varepsilon_n]$. But it is easy to see that sets $\mathrm{cl}([\varepsilon_n])$, considered as subsets of the entire $[0,1]^Z$, are compact, so they have nonempty intersection. However, we used only countable many coordinates in definitions of $\bigcap_{n<\omega}\mathrm{cl}([\varepsilon_n])$. So, there exists $\delta \in H_\omega\big((\mathcal{F} \cup \mathcal{R}') \times \omega_2, [0,1]\big)$ such that $[\delta] \subset \bigcap_{n<\omega}\mathrm{cl}([\varepsilon_n]) = \bigcap_{n<\omega}[\varepsilon_n]$. This, and (18), imply (B) of Lemma 1. This also imply easily that (X,τ) is Baire. The proof that (Y,τ_2) is Baire is similar.

So, let x_0 be as in Lemma 1(B) for h. Similarly as in the proof of Theorem 4(A) we will show that there exists an ordinal $\eta < \omega_2$ such that $x_0 \in h^{-1}\big(\big[\{\langle \eta, \{0\}\rangle\}\big]\big)$ and that $[\varepsilon] \cap h^{-1}\big(\big[\{\langle \eta, \{1\}\rangle\}\big]\big) \neq \emptyset$ for every $\varepsilon \in H(\mathcal{F} \times \omega_2, \mathcal{B}_0)$ with $x_0 \in [\varepsilon]$. This will give as a contradiction with the fact that $h \in C(\tau, \tau_2)$, since $h^{-1}\big([\{\langle \eta, [0, 1/2)\rangle\}]\big)$ would be nonempty open set with empty interior.

By ω_2-chain condition of forcing P we can find $\zeta < \omega_2$ such that $h \in V[g|_{Y\times\zeta}]$. Since now we will work in $V[g|_{Y\times\zeta}]$ with forcing notion $P_1 = H_\omega\big(Y \times (\omega_2 \setminus \zeta), [0,1]\big)$. Clearly the set

$$\begin{aligned} E_0 &= \Big\{s \in P_1 : (\exists \eta \in \omega_2 \setminus \zeta)\big(s \Vdash \text{“}h(x_0) \in \big[\{\langle \eta, \{0\}\rangle\}\big]\text{”}\big)\Big\} \\ &= \Big\{s \in P_1 : (\exists \eta \in \omega_2 \setminus \zeta)\big(\langle h(x_0), \eta, 0\rangle \in s\big)\Big\} \end{aligned}$$

is dense in P_1. So, there exists $\eta \in \omega_2 \setminus \zeta$ such that $x_0 \in h^{-1}\big(\big[\{\langle \eta, \{0\}\rangle\}\big]\big)$. Let $s_0 \in P_1$ be such that $s_0 \Vdash$ “$x_0 \in h^{-1}\big(\big[\{\langle \eta, \{0\}\rangle\}\big]\big)$”. To show that

$$[\varepsilon] \cap h^{-1}\big(\big[\{\langle \eta, \{1\}\rangle\}\big]\big) \neq \emptyset$$

for every $\varepsilon \in H(\mathcal{F} \times \omega_2, \mathcal{B}_0)$ with $x_0 \in [\varepsilon]$ fix such an ε. To finish the proof it is enough to show that the set

$$\begin{aligned} E &= \Big\{s \in P_1 : (\exists x \in X)\Big(s \Vdash \text{“}x \in [\varepsilon] \cap h^{-1}\big(\big[\{\langle \eta, \{1\}\rangle\}\big]\big)\text{”}\Big)\Big\} \\ &= \Big\{s \in P_1 : (\exists x \in [\varepsilon|_{\mathcal{F}\times\zeta}])\Big(\big(\langle h(x), \eta, 1\rangle \in s\big) \text{ and } \\ &\qquad (\forall \langle f, \xi\rangle \in \mathrm{dom}(\varepsilon), \xi \geq \zeta)\Big(s\big(f(x), \xi\big) \in \varepsilon\big(f(x), \xi\big)\Big)\Big)\Big\}. \end{aligned}$$

is dense in P_1 below s_0.

To see it, choose $t \in P_1$, $t \leq s_0$. We must find $s \leq t$, $s \in P_1$ and an $x \in [\varepsilon|_{\mathcal{F}\times\zeta}]$ such that $\langle h(x), \eta\rangle \in \mathrm{dom}(s)$, $s\big(h(x), \eta\big) = 1$ and for every $\langle f, \xi\rangle \in \mathrm{dom}(\varepsilon)$, $\xi \geq \zeta$, we have $\langle f(x), \xi\rangle \in \mathrm{dom}(s)$ with $s\big(f(x), \xi\big) \in \varepsilon\big(f(x), \xi\big)$. But let $\mathcal{F}_0$ be the set of all f such that either $\langle f, \alpha\rangle \in \mathrm{dom}(\varepsilon)$ for some α or f is equal to a constant $m \in M$, where $M = \{c : \langle c, \beta\rangle \in \mathrm{dom}(t) \text{ for some } \beta\}$. Then, it is enough to find $x \in [\varepsilon|_{\mathcal{F}\times\zeta}]$ such that x does not belong to

$$I = \big\{z \in X : (\exists f, g \in \mathcal{F}_0)\big(f(z) = g(z) \text{ and } f \neq g\big)\big\} \in \mathcal{J},$$

(*i.e.*, that $f(x) \neq g(x)$ and $f(x) \notin M$ for all $\langle f, \xi\rangle$, $\langle g, \xi\rangle \in \mathrm{dom}(\varepsilon)$, $f \neq g$, $\xi \geq \zeta$), and that

$$h(x) \in h\Big(\{z \in [\varepsilon|_{\mathcal{F}\times\zeta}] \setminus I : h(z) \neq f(z) \text{ for all } f \in \mathcal{F}_0\}\Big) \setminus M.$$

But this can be done by the conclusion of Lemma 1(B).

Theorem 4(B) has been proved.

We will finish this paper with the following two problems.

PROBLEM. Can we prove Theorem 4 or any of the Corollaries 4, 5, 6, 7 without any additional set-theoretical assumptions?

PROBLEM. Can topologies from Theorem 4 or any of the Corollaries 4, 5, 6, 7 be normal? Lindelöf? hereditarily Lindelöf? compact? metrizable?

REFERENCES

1. A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math. **659**, Springer-Verlag, 1978.
2. Jacques Chazarain and Alain Piriou, *Introduction to the Theory of Linear Partial Differential Equations*, North-Holland, 1982.
3. Krzysztof Ciesielski, *Linear subspaces of* $\mathbb{R}^\lambda$ *without dense totally disconnected subset*, Fund. Math. **142**(1993), 85–88.
4. Krzysztof Ciesielski and Lee Larson, *Various continuities with the density, I-density and ordinary topologies on* $\mathbb{R}$, Real Anal. Exchange **17**(1991–92), 183–210.
5. Krzysztof Ciesielski and Lee Larson and Krzysztof Ostaszewski, *I-density continuous functions*, Mem. Amer. Math. Soc. (515), **107**, 1994.
6. A. Denjoy, *Mémoire sur les dérivés des fonctions continues*, J. Math. Pures et Appl. **1**(1915), 105–240.
7. Ryszard Engelking, *General Topology*, Warszawa, 1977.
8. C. Goffman and C. J. Neugebauer and T. Nishiura, *The Density Topology and Approximate Continuity*, Duke Math. J. **28**(1961), 497–506.
9. Casper Goffman and Daniel Waterman, *Approximately Continuous Transformations*, Proc. Amer. Math. Soc. **12**(1961), 116–121.
10. O. Haupt and Ch. Pauc, *La topologie de Denjoy envisagée comme vraie topologie*, C. R. Acad. Sci. Paris **234**(1952), 390–392.
11. K. Kunen, *Set Theory*, North-Holland, 1983.
12. A. W. Miller, *The Baire category theorem and cardinals of countable cofinality*, J. Symbolic Logic **47**(1982), 275–288.
13. J. C. Oxtoby, *Measure and Category*, Springer-Verlag, 1971.
14. W. Wilczyński, *A category analogue of the density topology, approximate continuity, and the approximate derivative*, Real Anal. Exchange **10**(1984–85), 241–265.
15. Z. Zahorski, *Sur la Premiere Dériveé*, Trans. Amer. Math. Soc. **69**(1950), 1–54.

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