MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS OF GENERAL ORDER

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The object of this paper is the extension to linear partial differential equations of order m in N independent variables, of the existence theorems for mixed initial and boundary value problems which have been established for systems of first order equations in **(3)**. In such mixed problems an initial surface S and a boundary surface T are the carriers of the two types of data, and the number of datum functions to be assigned on T depends on the configuration of the characteristic surfaces relative to S and T.

For the first part of the paper (§§ 1-5) the coefficients in the differential equation, the initial and boundary surfaces, and the data prescribed are all taken to be real analytic in the variables $x^1 \ldots x^N$. In this "analytic" case an existence theorem is established for boundary conditions of considerable generality. We assume that the differential equation is regularly hyperbolic with respect to S and T, a notion which is stated precisely in § 1, and is weaker than the usual regular hyperbolic condition. Then the single equation of higher order is reduced to a system of equations of first order, of the type treated in **(3)**, and the existence theorem there established is taken over to obtain the result, which is stated as Theorem 1 in § 5 below. For this purpose we require a certain algebraic lemma relating to the characteristic roots.

The non-analytic problem for regularly hyperbolic equations is treated in §§ 6-10, by adaptation of the energy integral method. A general sufficient condition for the existence of a solution is given in § 6. As it appears that this condition is not always fulfilled, it is necessary to discuss particular cases. In § 8 and § 9 are treated two such special problems, each of which is a generalization of the known results for second order equations. The first of these concerns the problem wherein the number of boundary conditions is one less than the number of initial conditions. The second requires an assumption of symmetry relative to the boundary surface, and the number of boundary conditions is half the number of initial conditions.

1. The differential equation. We consider an analytic linear partial differential equation of order m in the N independent variables x^i:

$$(1.1) \qquad Lu = \sum_{h=0}^{m} a_{(h)\,i_1 \ldots i_h} \frac{\partial^h u}{\partial x^{i_1} \ldots \partial x^{i_h}} = 0.$$

The dependent variable is $u = u(x^i)$. The coefficients

$$a_{(h)\,i_1 \ldots i_h}$$

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of order h are assumed in this first part to be convergent real power series of the real variables x^i.

Let $S : \phi(x^i) = 0$ be an "initial" surface not characteristic for the linear operator L; and let $T : \psi(x^i) = 0$ be a "boundary" surface likewise not characteristic. We assume that both S and T are analytic and that they have an $(N - 2)$-dimensional intersection C.

The characteristic surfaces $G : \chi(x^i) = 0$ of the operator L satisfy the equation

$$C[\chi] = a_{(m)\,i_1\ldots i_m} \frac{\partial\chi}{\partial x^{i_1}} \cdots \frac{\partial\chi}{\partial x^{i_m}} = 0, \tag{1.2}$$

and in general there will be m (or fewer) characteristic surfaces which pass through the edge C. As seen below we assume that there are actually m. We suppose that at least $k_0 (k_0 < m)$ of these lie in a fixed "quadrant" R defined by S and T: and we select k_0 of these surfaces G_i, $(i = 1, \ldots, k_0)$. These shall be referred to as "select" characteristic surfaces, and all others as "non-select."

The mixed problem to be studied below is now formulated as follows. Define $t = \phi(x^i)$, $x = \psi(x^i)$, and assign on S Cauchy data for u with respect to the operator L: that is, values of u and its derivatives with respect to t up to order $m - 1$ inclusive. Assign on T any k_0 of the m quantities:

$$u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{m-1} u}{\partial x^{m-1}},$$

subject to compatibility conditions of order $m - 1$ on the edge C. We seek a piecewise analytic solution in R of $Lu = 0$ which takes the given values on S and on T, and is analytic except across the select characteristic surfaces, where it is of class C^{m-1}.

In order to treat this problem we shall need to assume that the operator L is regularly hyperbolic with respect to S and T, in the following sense: *there shall be m distinct characteristic surfaces passing through* C. Another form of this condition is available if we consider the coefficients

$$a_{(m)\,i_1\ldots i_m}$$

of highest order derivatives in Lu, in which the indices i_p take values N and $N - 1$ corresponding to t and x, respectively. To find this condition we note that by the theory of first order partial differential equations, the characteristic surfaces through C are composed of characteristic strips of the reduced characteristic equation

$$C_1[p_\rho, -1] = a_{(m)\,i_1\ldots i_m}\, p_{i_1} \ldots p_{i_m} = 0, \tag{1.3}$$

where $p_N \equiv p_t \equiv -1$ has been substituted so that χ appears in the form $\chi = \chi_0(x_\rho, x) - t$. On the initial "curve" C the initial values for the strip elements are found from (1.3) and the conditions

$$dt = \sum_{\rho=1}^{N-1} p_\rho dx_\rho.$$

Since for $\rho = 1, 2, \ldots, N-2$, the dx_ρ are independent, we have $p_\rho = 0$, $(\rho = 1, 2, \ldots, N-2)$, and only p_x is different from zero. Thus

$$C_1[0, 0, \ldots, 0, p_x, -1] = 0 \tag{1.4}$$

determines the m values of p_x, which we shall suppose are real and distinct on C and in a neighbourhood of C. Setting

$$a_k = a_{(m)N-1, N-1, \ldots, N-1, N, N, \ldots, N},$$

where the index $N-1$ appears k times, we see that (1.4) becomes

$$a_m p_x^m - a_{m-1} p_x^{m-1} + \ldots + (-1)^m a_0 = 0. \tag{1.5}$$

That is, the roots p_x of (1.5) must be distinct and real.

We note that if $Lu = 0$ is regularly hyperbolic in the sense of Leray **(8)** and if S is spacelike, then Lu is regularly hyperbolic with respect to S and to every non-characteristic surface T. If in the regularly hyperbolic case we imagine the edge C to rotate about a fixed point in S, the characteristic surfaces issuing from C remain separated: no two touch. For our purpose it is enough if these surfaces are distinct for the one position of the edge C. Thus our condition is weaker than the customary regular hyperbolic condition. In fact it becomes equivalent for the case of two independent variables, when the edge C reduces to a point.

Analogously, the normal surface of a regularly hyperbolic operator consists essentially of a nest of concentric ovals, such that any line through the origin meets the surface in a maximal number of real points. In our case it is sufficient if a particular line through the origin, namely the normal to C in S, meets the surface in a maximal number of real points. This could be realized, for instance, by a surface with multiple points, or by a surface consisting of ovals external to one another.

If we alter the negative signs in (1.5) and consider the equation

$$a_m \gamma^m + a_{m-1}\gamma^{m-1} + \ldots + a_0 = 0, \tag{1.6}$$

we see that the roots $\gamma_1, \ldots, \gamma_m$ of (1.6) are also distinct. Let us define the first order operator

$$D_a u = \frac{\partial u}{\partial x} + \gamma_a \frac{\partial u}{\partial t}, \tag{1.7}$$

which indicates differentiation along the section of a characteristic surface by a plane $x_\rho = \text{const}$, $(\rho = 1, \ldots, N-2)$.

The operators D_i shall also be termed select or non-select according as the characteristic surface G_i and the characteristic root γ_i are select or not. We note the identity

$$a_m \prod_a D_a\, u = \sum_{k=0}^{m} a_h \frac{\partial^m u}{\partial x^k \partial t^{m-k}} + \ldots \tag{1.8}$$

where the terms omitted are derivatives of order less than m. In consequence, we can write the given differential equation (1.1) in the form

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$$\prod_a D_a u = L_1(u) \tag{1.9}$$

where $L_1(u)$ is a linear operator of order m in which no term has more than $m - 1$ differentiations with respect to x and t combined.

2. Reduction to a system of first order equations. By introducing as new dependent variables v_i suitable combinations of the derivatives of u, we shall perform a formal reduction of (1.1) to a system

$$D_a v_i = \sum c_{ij} \frac{\partial v_i}{\partial x^j} + \sum e_{ij} v_j, \tag{2.1}$$

where each D_a operator will occur several times, in general, and where only derivatives with respect to x_ρ ($\rho = 1, \ldots, n-2$) appear as $\partial/\partial x_\rho$ on the right. This system is of the type studied in (3), as the elementary divisors of A^N relative to A^{N-1} are simple. The dependent variables in (2.1) are also divided into select and non-select classes, a variable v being select if the operator D_a which operates on it in (2.1) is select, and vice versa. The assigned boundary conditions will be transformed into

$$v_i = \sum a_{ij} v_j + f_i \tag{2.2}$$

where on the left shall appear only the select v_i. Thus the existence theorem of **(3)** is applicable and will lead to a solution of the original problem, when that problem is analytic.

The formal reduction and labelling of new variables will follow the pattern of the Cauchy-Kowalewski reduction to normal form, except for those derivatives with respect to x. To handle these we have introduced the D_a operators and will employ them in the fashion of **(1)**. A result of this distinction is the following subdivision of the new variables into groups. We define formally

$$v_{(i)} = v_{ab\ldots h\; i_1\ldots i_q} = D_a D_b \ldots D_h \frac{\partial^q u}{\partial x^{i_1} \ldots \partial x^{i_q}} \tag{2.3}$$

and construct the first order system so as to satisfy these definitions identically.

In the first group we take $q = 0$; and for higher values of q up to $m - 1$ inclusive there is a group of equations corresponding to each distinct array $i_1, \ldots, i_q$, where order is immaterial.

A fixed ordering $a, b, c, \ldots, k, \ldots$ of the operators D_a is used in each of these groups. However, as the exact selection of these indices depends on the boundary conditions, we shall for the present reserve the choice of select and non-select values.

Certain "commutator" expressions appear and we now define them. The symbol

$$C_{a i_q b\ldots hk\; i_1\ldots i_{q-1}}[u] \tag{2.4}$$

shall denote the reduced form of the expression

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$$(2.5)\quad D_a \frac{\partial}{\partial x^{i_q}} D_b \ldots D_h D_k \frac{\partial^{q-1} u}{\partial x^{i_1} \ldots \partial x^{i-q1}} - \frac{\partial}{\partial x^{i_q}} D_k D_h \ldots D_a \frac{\partial^{q-1} u}{\partial x^{i_1} \ldots \partial x^{i_{q-1}}};$$

this reduced form contains derivatives of u of lower orders $a, b, \ldots, h, k, i_1 \ldots i_{q-1}$, with coefficients functions of the γ_a.

By writing

$$C_{a i_q b \ldots h k i_1 \ldots i_{q-1}}[v]$$

we shall indicate that the derivatives of u have been formally replaced by the corresponding variables

$$v_{ab \ldots h i_1 \ldots i_q}$$

as defined in (2.3). This is possible, since we will show that all derivatives of u can be expressed as linear combinations of the variables $v_{(i)}$ in (2.3). In fact we shall prove by induction that all $k+1$ derivatives

$$\frac{\partial^k u}{\partial x^h \partial t^{k-h}} \qquad h = 0, \ldots, k,$$

can be expressed as linear combinations of the $k+1$ variables

$$v_{hg\ldots a}, \quad v_{kg\ldots a}, \quad \ldots, \quad v_{kh\ldots b},$$

in each of which one of the $k+1$ operations is omitted, and of variables with a lesser number of subscripts. To show this, we note that by (1.7) and (2.3)

$$(2.6)\qquad v_{kh\ldots b} = \sum_{h=0}^{k} S_h^a(\gamma) \frac{\partial^k n}{\partial x^h \partial t^{k-h}} + F_{k-1}[u]$$

where $S_h{}^a(\gamma)$ is the symmetric function of degree h of the k quantities γ_k, $\gamma_h, \ldots, \gamma_b$ and with γ_a omitted. Forming similar equations with $b, c, \ldots, h$, k omitted in turn, we see that the system can be solved for the kth order derivatives of u provided that the determinant $|S_h{}^a(\gamma)|$ is not zero. This is proved separately in Lemma 1 below, and thus our assertion is verified.

In (2.6) $F_{k-1}[u]$ denotes an operator in $\partial/\partial x$, $\partial/\partial t$ of order $k-1$, which by the induction hypothesis may be considered to be already expressed as a combination of the v's.

The groups shall be written with definitions and differential equations in parallel columns. For $q = 0$ we have the "triangular" array of equations shown in Table I.

In Table I $a, b, c, \ldots, k, \ldots, n$ denote the distinct numbers from 1 to n in an as yet undefined order. The operator $L_1[v]$ is defined by replacing derivatives of u in $L_1(u)$ (cf. (1.9)) with the appropriate first derivatives or values of the v's. This array or group of equations contains n subgroups, the kth group containing k equations each with a different operator D_a.

For each array $(i)_q = (i_1, \ldots, i_q)$ we have a similar group of equations, in which appear on the right side certain first derivatives with respect to $x_q{}^i$. We let $(i)_{q-1} = (i_1 \ldots i_{q-1})$ and construct Table II, which also contains a triangular array, with subgroups of increasing size.

TABLE I

DEFINITIONS	DIFFERENTIAL EQUATIONS
$v = u$	$D_a v = v_a$
$v_b = D_b u$	$D_a v_b = v_{ba} + C_{ab}[v]$
$v_a = D_a u$	$D_b v_a = v_{ba}$
$v_{cb} = D_c D_b u$	$D_a v_{cb} = v_{cba} + C_{acb}[v]$
$v_{ca} = D_c D_a u$	$D_b v_{ca} = v_{cba} + C_{bca}[v]$
$v_{ba} = D_b D_a u$	$D_c v_{ba} = v_{cba}$
	$\cdots$
$v_{kh\ldots b} = D_k D_h \ldots D_b u$	$D_a v_{kh\ldots b} = v_{khg\ldots a} + C_{akh\ldots b}[v]$
	$\cdot$
$v_{kg\ldots a} = D_k D_g \ldots D_a u$	$D_h v_{kg\ldots a} = v_{khg\ldots a} + C_{hkg\ldots a}[v]$
$v_{hg\ldots a} = D_h D_g \ldots D_a u$	$D_k v_{hg\ldots a} = v_{khg\ldots a}$
	$\cdots$
$v_{nm\ldots b} = D_n D_m \ldots D_b u,$	$D_a v_{nm\ldots b} = L_1[v] + C_{anm\ldots b}[v]$
	$\cdot$
$v_{n1\ldots a} = D_n D_1 \ldots D_a u,$	$D_m v_{n1\ldots a} = L_1[v] + C_{nm1\ldots a}[v]$
$v_{m1\ldots a} = D_m D_1 \ldots D_a u,$	$D_n v_{m1\ldots a} = L_1[v].$

The last of the "subgroups" of Table II contains $m - q$ equations, and the order of $a, b, c \ldots h, k$ is the same in all groups, that is, for all $(i)_q$. It is seen that the number of new variables defined in these groups is equal to the total number of partial derivatives of u with respect to t, x and the x_ρ, up to and including order $m - 1$. We emphasize that not all groups $(i_1 \ldots i_q)$ are here represented, but only one for each set of integers $(i_1 \ldots i_q)$ without regard to order. Thus we may for simplicity assume that $i_1 \leqslant i_2 \leqslant i_3 \leqslant \ldots \leqslant i_q$.

3. Reduction of the boundary conditions. Let the derivatives of u with respect to x up to order $n - 1$ be paired in order with the numbers $a, b, c, \ldots, k$, which label the D_a operators:

$$(3.1) \qquad \begin{array}{ccccc} u & u_x & u_{xx}, & \ldots & u_x^{(m-1)} \\ a & b & c, & \ldots & l. \end{array}$$

We establish an ordered correspondence between the operators of the sequence $D_a, D_b, \ldots, D_k$ in Table I and the derivatives $u, u_x, \ldots, u_x^{m-1}$. The labels $a, b, \ldots, k$ shall be chosen so that to an assigned derivative $u_x^{(j)}$ there corresponds a select operator D_h, and vice versa. This is possible, in general in a number of ways, since the number of select operators has been taken as equal to the number of boundary conditions. It is now assumed that this arrangement has been adopted in advance in Tables I and II. We repeat that the select v's are those which are operated on in (2.1) by a D_a operator which is select according to this scheme.

TABLE II

Definitions	Differential equations
$v_{(i)} = \dfrac{\partial^q u}{\partial x^{i_1} \ldots \partial x^{i_q}} = u_{(i)}$,	$D_a v_{(i)} = \dfrac{\partial}{\partial x_{i_q}} v_{a(i)_{q-1}} + C_{a i_1 \ldots i_q}[v]$
$\ldots$	
$v_{b(i)} = D_b u_{(i)}$,	$D_a v_{b(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{ba(i)_{q-1}} + C_{ab i_1 \ldots i_q}[v]$
$v_{a(i)} = D_a u_{(i)}$,	$D_b v_{a(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{ba(i)_{q-1}} + C_{b i_q a i_1 \ldots i_{q-1}}[v]$
$\ldots$	
$v_{cb(i)} = D_c D_b u_{(i)}$,	$D_a v_{cb(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{cba(i)_{q-1}} + C_{a i_q cb i_1 \ldots i_{q-1}}[v]$
$v_{ca(i)} = D_c D_a u_{(i)}$,	$D_b v_{ca(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{cba(i)_{q-1}} + C_{b i_q ca i_1 \ldots i_{q-1}}[v]$
$v_{ba(i)} = D_b D_a u_{(i)}$,	$D_c v_{ba(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{cba(i)_{q-1}} + C_{c i_q ba i_1 \ldots i_{q-1}}[v]$
$\ldots$	
$\ldots$	
$v_{kh\ldots b(i)} = D_k D_h \ldots D_b u_{(i)}$,	$D_a v_{kh\ldots b(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{khg\ldots a(i)_{q-1}} + C_{a i_q kh \ldots b i_1 \ldots i_{q-1}}[v]$
$\cdot$	
$v_{kg\ldots a(i)} = D_k D_g \ldots D_a u_{(i)}$,	$D_h v_{kg\ldots a(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{khg\ldots a(i)_{q-1}} + C_{h i_q kg \ldots a i_1 \ldots i_{q-1}}[v]$
$v_{hg\ldots a(i)} = D_h D_g \ldots D_a u_{(i)}$,	$D_k v_{hg\ldots a(i)} = \dfrac{\partial}{\partial x^{i_q}} v_{khg\ldots a(i)_{q-1}} + C_{k i_q hg \ldots a i_1 \ldots i_{q-1}}[v]$

These boundary conditions must be reduced to the form (2.2) where the select v's appear only on the left. Let us consider first the group $q = 0$: for the other groups the calculations are similar. If u is given, D_a and v shall be select; and then $v = u$ is a boundary condition of type (2.2). If u is not given then no boundary condition for v will be needed, as D_a is then non-select.

If u_x is given then D_b is select. There are two cases, according as D_a is select or not. If D_a is select, then $v_a = u_x + \gamma_a u_t$ and $v_b = u_x + \gamma_b u_t$ are both assigned and these conditions are of the form (2.2). If D_a is not select, we eliminate u_t between these two relations and find

$$v_b = \frac{\gamma_b}{\gamma_a} v_a + \left(1 - \frac{\gamma_b}{\gamma_a}\right) u_x, \tag{3.2}$$

which again has the form (2.2). However, if u_x is not given, there are two other cases. If D_a is select, and D_b is not ,we can solve (3.2) for v_a, which is then the single necessary boundary condition of form (2.2). If neither D_a nor D_b is select, no boundary condition is needed.

We may now proceed by induction from one subgroup to the next. If the boundary conditions for the kth subgroup have been put in the form (2.2) then all select variables of that group can be expressed in terms of non-select variables. Thus in treating the next sub-group we can allow all variables of the preceding groups to appear on the right side of the boundary conditions, as the select ones can later be eliminated by means of the preceding boundary conditions. If in (2.6) we replace the quantity $F_{k-1}[u]$ by its formal equivalent in terms of the v's, we see that only those v's of the preceding groups will appear. Consequently this term of (2.6) may be considered as non-essential in the remainder of the calculation.

Assuming then that the result holds for the $(k-1)$st subgroup, let us prove it for the kth subgroup. We divide the k equations (2.6) into select and non-select categories according as the v variable on the left is select or not. All derivatives of $u, u_x, \ldots, u_x^{(n)}$ with respect to t are known, or select, on T, according as $u, u_x, \ldots, u_x^{(n)}$ is select or not. Thus if h of the k quantities $u, u_x, \ldots, u_x^{(k)}$ are select, then h of the derivatives written explicitly on the right side of (2.6) are select. Let us pick out the $k-h$ non-select equations of (2.6) and solve them for the $k-h$ non-select derivatives of u in terms of the select derivatives of u and the (non-select) variables v of these equations. The possibility of this depends on the non-vanishing of a determinant of which the elements are symmetric functions of the γ's, with one of the γ's omitted in each row. Supposing, as will be shown in Lemma 1 of § 4, that all such determinants are different from zero, we can carry out this inversion of the non-select equations (2.6), and then replace the $k-h$ non-select derivatives of u in the h select equations of (2.1) by the expressions so found for them. These h select equations will then take the form (2.2) since all earlier groups of select variables can be eliminated from the right sides.

Now consider these groups with $q > 0$. As all differentiations with respect to x_ρ $(\rho = 1, \ldots, N-2)$ are tangential to T, the derivatives

$$\partial^{q+h} u / \partial x^{i_1} \ldots \partial x^{i_q} \, \partial x^h$$

are select according as $\partial^h u / \partial x^h$ is select, or not. Thus equations similar to (2.6) can be written down for any such group, and the coefficients of the terms shown explicitly will be exactly the same, and the structure of the operator $F_{k-1}[u]$ will be unaltered except that instead of u as argument we will have

$$\partial^q u / \partial x^{i_1} \ldots \partial x^{i_q}.$$

It follows that the boundary conditions for all groups with $0 < q \leqslant m-1$ can be put in the same form (2.2).

Remark. Suppose that the k linear boundary conditions are linear and independent relations among the n quantities $u, u_x, u_{xx}, \ldots, u_x^{(n-1)}$, on T.

Such a system of boundary conditions can be reduced to a triangular standard form

$$u_x^{(h_i)} + \sum_{r_i=0}^{h_i-1} c_{r_i} u_x^{(r_i)} = g_i, \qquad (i = 1, \ldots, k_0), \tag{3.3}$$

where the orders h_i of the leading derivatives form an increasing sequence:

$$h_1 < h_2 < \ldots < h_{k_0}.$$

In addition, the indices r_i are all less than h_i, while the coefficients c_{r_i} are analytic functions on T.

To show that this system of boundary conditions can be expressed in the form (2.2), we need modify the previous working only slightly. In the array (3.1) we choose as select operators $D_a, \ldots, D_n$ those corresponding to the

$$u_x^{(h_i)}.$$

We then commence with the quantity u: if and only if it is given in the first array $q = 0$ of Table I, a boundary condition is required. Now, considering u_x, we see that if one of the $h_i = 1$, there is a condition

$$u_x + c_{11}u = g_1,$$

and if u is given it may be replaced by its given values while, if it is not, the corresponding variable v of Table I is non-select and so may appear on the right of the boundary conditions (2.2). Thus the form (2.2) is attained in either case, as in the preceding calculations.

Proceeding by induction on h_i, we see that in the typical condition (3.3) the terms $u_x^{(r_i)}$ are either non-select, in which case they are allowed on the right side of (2.2), or else they can be expressed, by means of the boundary conditions already standardized, in terms of non-select variables and given data. As remarked earlier any variable of a previous group can be allowed on the right side of a boundary condition in the course of such a calculation. This completes the demonstration that (3.3) can be reduced to the standard form of the boundary conditions for the system of first order equations.

4. A lemma on symmetric functions. To justify the reduction of the differential equation as well as the boundary conditions, we establish a lemma which is required in its most general form in the preceding discussion of the boundary conditions.

LEMMA 1. *Let k distinct numbers $\gamma_a \neq 0$ be given, and let $s_r^a(\gamma) = s_r^a$ denote the elementary symmetric function of degree, r of all $k - 1$ γ's with γ_a omitted. Then every subdeterminant formed by deleting an equal number $h(0 \leqslant h \leqslant k - 1)$ of rows and columns from the $k \times k$ determinant*

$$|s_r^a(\gamma)|, \tag{4.1}$$

is different from zero.

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The numbers γ_a corresponding to the deleted rows are the select γ_a; for convenience we shall denote them now by c_a while retaining γ_a for the $l = k-h$ non-select numbers. The deleted columns refer to the assigned derivatives among $u, u_x, u_x, \ldots, u_x^{(m-1)}$.

Let $s_r(c)$ denote the elementary symmetric function of degree r of the c_a: if we now delete the h select rows we observe from (4.1), that all h of the select c_a are present in each of the other rows. Let $\sigma_r^a(\gamma) = \sigma_r^a$ denote the elementary symmetric function of degree r of the non-select γ's, with γ_a omitted. The following property of s_r^a is evident: s_r^a is the convolution of the $s_m(c)$ and $\sigma_{r-m}^a(\gamma)$ of all lower orders:

$$s_r^a(\gamma) = \sum_{m=0}^{r} s_m(c)\sigma_{r-m}^a(\gamma). \tag{4.2}$$

Now let σ_r denote the column vector with $l = k - h$ components σ_r^a. The $k \times (l - h)$ matrix (the select rows have been deleted), may now be written

$$\left(\sum_{i=0}^{k-1} s_i \sigma_{k-i-1}\right).$$

We note that $\sigma_r = 0$ for $r \geqslant l$, so that we may write this array in the form

$$\begin{aligned}(s_0\sigma_0,\ s_0\sigma_1 + s_1\sigma_0,\ s_0\sigma_2 + s_1\sigma_1 + s_2\sigma_0, \ldots, s_{l-1}\sigma_0 + \ldots + s_0\sigma_{l-1},\\ s_l\sigma_0 + \ldots + s_1\sigma_{l-1}, \ldots, s_h\sigma_{l-2} + s_{h-1}\sigma_{l-1}, s_h\sigma_{l-1}).\end{aligned} \tag{4.3}$$

If the select columns are deleted, and the resulting square determinant Δ expanded, we see that it takes the form of a sum of $l \times l$ determinants with columns the σ_r $(r = 0, 1, \ldots, -1)$. Since any one of these with two equal columns is zero, it follows that the only non-vanishing determinant among them is $|\sigma_0\sigma_1 \ldots \sigma_{l-1}|$. Therefore every surviving term has this determinant as a factor. However elementary reasoning shows that

$$|\sigma_0\sigma_1 \ldots \sigma_{l-1}| = \pm \prod_{\gamma_a \neq \gamma_b} (\gamma_a - \gamma_b) \neq 0. \tag{4.4}$$

The array (4.3) is the symbolic product of (4.4) with the array

$$\begin{matrix} s_0 & s_1 & s_2 & . & . & . & s_h & 0 & 0 & . & . & . & 0 \\ 0 & s_0 & s_1 & s_2 & . & . & . & s_h & 0 & . & . & . & 0 \\ 0 & 0 & s_0 & . & . & . & & s_h & 0 & . & . & . & 0 \\ & & & & . & . & . & . & . & & & & \\ 0 & . & . & . & & & 0 & s_0 & s_1 & . & . & . & s_h \end{matrix} \tag{4.5}$$

of l rows and $h + l$ columns, according to the formal rules of determinant multiplication. We have therefore to show that the $l \times l$ determinant which remains when any h columns have been deleted from (4.5) is not zero.

This $l \times l$ determinant has the form of the representation of a Schur function $\{\lambda\}$ corresponding to a certain partition (λ) consisting of h numbers $\lambda_1, \ldots, \lambda_h$ arranged in decreasing order. For the theory of partitions and S-functions we refer to (**11**, chapters 5, 6). The partition (λ) is best defined in this case by means of its conjugate partition (μ), which consists of l positive integers μ_i, not necessarily distinct, arranged in decreasing order. We set

$$\mu_i = h - d(i)$$

where $d(i)$ is the number of select, or assigned, derivatives of the sequence

$$u, u_x, u_{xx}, \ldots, u_x^{(n-1)}$$

which are encountered before the ith non-select derivative. From **(10**, chapter 6, 3.3**)** we see that the Schur function

$$\{\lambda\} = |s_{\mu_i - i+j}| \tag{4.6}$$

has the form of (4.5) after deletion of the select columns and transposition about the secondary diagonal.

On the other hand, from **(10**, chapter 6, 3.1**)** we have

$$|s_{\mu_q - i+j}| = \{\lambda\} = \frac{|c_i^{\lambda_j+h-j}|}{|c_i^{h-j}|}, \qquad (i, j = 1, \ldots, h), \tag{4.7}$$

where h is the number of select c_i's and i and j are indices of position in the determinants. We recall that the c_i are all distinct; the denominator in (4.7) is the Vandermonde's determinant which is equal to

$$\pm \prod_{j \neq i} (c_i - c_j),$$

and so is not zero. The numerator is a slightly more general type of determinant, which has been studied in **(1)** and shown to be different from zero. For this it is necessary that the c_i should be distinct and positive, and the powers $\lambda_j - j$ distinct, and these requirements are satisfied since the λ_j are non-increasing with j, while the c_i, being the select γ_i, are positive and distinct. A direct proof that the Schur function $\{\lambda\}$ is a symmetric polynomial of the c_i with non-negative coefficients has been given recently in **(8**, Theorem 1**)**. Thus (4.7) is different from zero in our case since the c_i are all positive. Combining this with (4.4) we see that the original subdeterminant of (4.1) is not zero, and this concludes the proof of the lemma.

The special case $h = 0$ is needed in connection with (2.6), and a sequence of applications with various values of h and l, one for each subgroup of Tables I and II, is needed in § 3 as stated there.

5. Verification of the solution. By **(3**, Theorem 3**)**, a piecewise analytic solution of (2.1), satisfying (2.2) and appropriate initial conditions, exists. Let the solution of (2.1) with given Cauchy data and boundary conditions (2.2), which is defined by the piecewise analytic expansions of **(3**, Theorem 3**)**, be constructed, and let us show that the solution u of (1.1) which we seek is actually the component v of the first equation of the first group of Table I. To show that v satisfies (1.1) we shall verify that the defining equations in the left columns of Tables I and II hold, in succession, and use the last subgroup of equations of the first table. When the "definitions" are re-established the various boundary conditions will be automatically satisfied, in view of the equivalence (2.6) between the derivatives of u of a given order, and the variables $v_a \ldots$ of the same order. Thus it will be established that

$$u = v \tag{5.1}$$

is a solution of the original problem, since the algebraic verification of the initial conditions will be trivial. We shall use the uniqueness property of solutions of the first order system which are analytic on the closure of the sector domains.

Consider first Table I, and let us verify the relations in the left-hand column subgroup by subgroup. The first such relation, namely $v = u$, is taken as a hypothesis, or rather a definition of u. The second definition of the second subgroup is precisely the first differential equation and so is valid. To show that the first definition of the second subgroup holds, let us define

$$\xi_b = D_b u - v_b. \tag{5.2}$$

Then ξ_b is piecewise analytic on the closure of the sector domains R_i. Also

$$\begin{aligned} D_a\xi_b &= D_a D_b u - D_a v_b \\ &= D_b D_a u + C_{ab}[u] - v_{ab} - C_{ab}[v] \\ &= v_{ba} - v_{ab} + C_{ab}[u] - C_{ab}[v] \\ &= C_{ab}[\xi], \end{aligned} \tag{5.3}$$

where we have used the first three differential equations of Table I. With

$$C_{ab}[u] = D_a D_b u - D_b D_a u = \alpha D_a u + \beta D_b u,$$

where α and β are certain coefficients which we need not calculate explicitly, we have

$$C_{ab}[v] = \alpha v_a + \beta v_b$$

and therefore

$$C_{ab}[\xi] = \alpha\xi_a + \beta\xi_b.$$

Now in this case, $\xi_a = D_a u - v_a \equiv 0$. Thus ξ_b satisfies the homogeneous linear equation

$$D_a\xi_b = \beta\xi_b. \tag{5.4}$$

The initial conditions for ξ_b are also homogeneous, as follows from the definition of initial conditions for the v variables. If D_a is select, there is a homogeneous boundary condition for ξ_b on T. In this case only, discontinuities of the derivatives of ξ_b across G_a are in principle permitted, but the expansions of (3) applied to this equation show that all such jumps are here zero. Since G_a is the only characteristic surface of (5.4), it follows that ξ_b is analytic everywhere and so identically zero.

To verify the third subgroup of definitions, note that the last of these relations is now equivalent to the last differential equation of the preceding subgroup. Define

$$\begin{aligned} \xi_{cb} &= D_c D_b u - v_{cb} = D_c v_b - v_{cb}\ ; \\ \xi_{ca} &= D_c D_a u - v_{ca} = D_c v_a - v_{ca}\ ; \end{aligned} \tag{5.5}$$

then

$$\begin{aligned} D_a \xi_{cb} &= D_a D_c D_b u - D_a v_{cb} \\ &= D_c D_b D_a u + C_{acb}[u] - D_a v_{cb} \\ &= D_c v_{ba} + C_{acb}[u] - v_{cba} - C_{acb}[v] \\ &= C_{acb}[\xi], \end{aligned} \tag{5.6}$$

and likewise

$$D_b \xi_{ca} = C_{bca}[\xi]\ ; \tag{5.7}$$

using the differential equations and previously established definitions. Here the $C_{acb}[\xi]$ expressions are linear homogeneous in the ξ variables with less than three indices: since all one-index ξ's are zero, (5.6) and (5.7) form a linear homogeneous system for ξ_{cb}, ξ_{ca}. Again, these functions satisfy homogeneous initial conditions. As above it follows in either case that ξ_{cb} and ξ_{ca} vanish identically.

The inductive procedure for the kth subgroup is similar: the last definition of the subgroup is true in view of the previously established definitions and the last differential equation of the preceding subgroup. We define $k - 1$ variables $\xi_{kh\ldots b}, \ldots, \xi_{kg\ldots a}$ as follows:

$$\begin{aligned} \xi_{kh\ldots b} &= D_k D_h \ldots D_b u - v_{kh\ldots b}, \\ \xi_{kg\ldots a} &= D_k D_g \ldots D_a u - v_{kg\ldots a}, \end{aligned} \tag{5.7}$$

there being a different D operator missing in each of these sequences of differentiations. Then

$$\begin{aligned} D_a \xi_{kh\ldots b} &= D_a D_k D_h \ldots D_b u - D\ v_{kh\ldots b} \\ &= D_k D_h \ldots D_b D_a u + C_{akh\ldots b}[u] - D_a v_{kh\ldots b} \\ &= D_k v_{hg\ldots ba} + C_{akh\ldots b}[u] - v_{kh\ldots ba} - C_{akh\ldots b}[v] \\ &= C_{akh\ldots b}[\xi]\ ; \end{aligned} \tag{5.8}$$

and there are $k - 2$ similar equations of which the last is

$$D_h \xi_{kg\ldots a} = C_{hkg\ldots a}[\xi]. \tag{5.9}$$

Here the $C_{akh\ldots b}[\xi]$ are linear expressions containing the $\xi_{kh\ldots b}, \ldots, \xi_{kg\ldots a}$, as well as those of lower order (which are now known to be zero). Thus (5.7), ..., (5.8) form a self-contained linear homogeneous system, with homogeneous initial and boundary conditions. Since the $\xi_{kh\ldots b}, \ldots, \xi_{kg\ldots a}$ are analytic on the closure of each sector R_i they are identically zero, in view of the uniqueness theorem in **(3)**.

The proof that the last subgroup of defining relations holds for the solutions of the first order system is similar to the earlier steps of the induction. The only difference is that the operator $L_1[v]$ replaces the variable $v_{khg\ldots a}$ in the general step. Since this quantity does not appear in the final form (5.8) and (5.9) of the equations for the ξ's, this change has no effect on the result. This shows, then, that all defining relations of Table I are valid.

Let us show that the defining equations of Table II hold for each index group $(i)_q = (i_1 \ldots i_q)$ by induction on q for each of these groups. Let $(i)_{q-1} = (i_1 \ldots i_{q-1})$ and let us assume that the result has been proved for the $(i)_{q-1}$ group. First define

$$\xi_{(i)_q} = \frac{\partial^q u}{\partial x^{i_1} \ldots \partial x^{i_q}} - v_{(i)_q} = \frac{\partial}{\partial x^{i_q}} v_{(i)_{q-1}} - v_{(i)_q}. \tag{5.10}$$

We see that

$$\begin{aligned} D_a \xi_{(i)_q} &= D_a \frac{\partial}{\partial x^{i_q}} v_{(i)_{q-1}} - D_a v_{(i)_q} \\ &= D_a \frac{\partial}{\partial x^{i_q}} v_{(i)_{q-1}} - \frac{\partial}{\partial x^{i_q}} v_{a(i)_{q-1}} - C_{a i_1 \ldots i_q}[v], \end{aligned} \tag{5.11}$$

by the first differential equation of the group $(i)_q$. However the right side of (5.11) contains only v variables of q-order $(i)_{q-1}$ or less, and in view of the definition (2.5) of the commutator operator, this side of (5.11) will be zero, since all variables

$$v_{(i)_{q-1}}$$

have been proved equal to the corresponding partial derivatives of u. It follows by differentiation of the initial and boundary conditions that

$$\xi_{(i)_q}$$

vanishes identically.

For the typical kth subgroup of Table II, we have a system of k equations involving the variables

$$\begin{aligned} \xi_{kh\ldots b(i)_q} &= \frac{\partial}{\partial x^{i_q}} v_{kh\ldots b(i)_{q-1}} - v_{kh\ldots b(i)_q}, \\ \xi_{kg\ldots a(i)_q} &= \frac{\partial}{\partial x^{i_q}} v_{kg\ldots a(i)_{q-1}} - v_{kg\ldots a(i)_q}. \end{aligned} \tag{5.12}$$

Then, for instance, from the first differential equation of the subgroup, we have

$$\begin{aligned} D_a \xi_{kh\ldots b(i)_q} &= D_a \frac{\partial}{\partial x^{i_q}} v_{kh\ldots b(i)_{q-1}} - D_a v_{kh\ldots b(i)_q} \\ &= D_a \frac{\partial}{\partial x^{i_q}} v_{kh\ldots b(i)_{q-1}} - \frac{\partial}{\partial x^{i_q}} v_{kh\ldots a(i)_{q-1}} - C_{a i_q kh\ldots b(i)_{q-1}}[v]. \end{aligned} \tag{5.13}$$

Now the right side contains a commutator which includes only v variables of order $q - 1$ or less with respect to the x^{i_p}, or else of order q but from a preceding subgroup of the qth group. As the identification of all these with the corresponding derivatives of u has already been made, we see by the definition (2.5) that the right side of (5.13) reduces to zero. Similarly, all other right sides, obtained by differentiation of the quantities in (5.12) by appropriate $D_b \ldots D_h$ operators, are seen to vanish. Homogeneous auxiliary conditions are applicable as before, and it follows that the variables ξ in (5.12) vanish identically.

Proceeding thus by induction we make all the identifications of the various q groups, and so identify all derivatives of $u = v$ with the appropriate v variables as foreshadowed in (2.3). This proves that u satisfies all the initial and boundary conditions. It remains now to show that u satisfies the original linear partial differential equation of order m. However this follows at once from the very last differential equation of Table I and the definition (1.9) of the operators $L_1(u)$ and $L_1[v]$. The existence proof is therefore complete.

THEOREM 1. *Let $L(u) = 0$ be an analytic linear differential equation of order m which is regularly hyperbolic with respect to analytic initial and boundary surfaces: $S{:}t = 0$ and $T{:}x = 0$. Let k_0 characteristic surfaces G_i issuing from $C = S \cap T$ into the region R be selected, and let k_0 of the quantities*

$$u, u_x, \ldots, u_x^{(m-1)}$$

be assigned on T in addition to Cauchy data on S. Then there exists a piecewise analytic solution u assuming the given initial and boundary values, and analytic except across the G_i where it is of class C^{m-1}.

This analytic solution is piecewise analytic in the strong sense described in (**2**, § 10); that is, it is analytic on the closures of the distinct sector domains D_h which separate the select characteristic surfaces. The solution must still be proved unique even within this class of functions, since there are many ways of setting up the corresponding first order system, and it is necessary to show that these distinct ways all lead to the same discontinuities of derivatives across the select characteristic surfaces.

To prove this, let us suppose that u is piecewise analytic in the above strong sense, that u is C^{m-1}, and analytic except across the select G_i; that $L(u) = 0$ and that homogeneous Cauchy data and boundary conditions have been assigned. Then u and its derivatives satisfy the system of Tables I and II, which is arranged according to any one of these alternative ways. However the data entering this system are all zero. Since a solution of the system is unique in the strongly piecewise analytic function class, (**3**) the solution of the system is identically zero. Hence $u \equiv 0$ as was to be proved.

As a further corollary we add that if every characteristic surface G_i issuing into the domain is select, then our solution is unique in the class of C^n functions. This now follows at once from the uniqueness in the C^1 class of solutions of the first order system, when every characteristic root is real and not zero, and when all positive roots are select **(3**, § 10**)**.

We have seen in § 3 that lower order derivatives with respect to x can be permitted to appear in the boundary conditions. It is also true that lower order derivatives with respect to the other $N - 1$ variables can be accommodated in the same way. This is possible since we could establish the definitions required in working back to the mth order equation in a lexicographic sequence which takes account of first, order of the derivative, second, index (i) of the group, and third, ordering of the D_a operators within each subgroup. However it is not possible to permit oblique derivatives of an order equal to the highest order which occurs in the boundary condition, as can be seen even for hyperbolic equations of first or second order **(2)**. Such conditions will lead to inconsistencies if directional derivatives involved have characteristic directions.

We comment on the fact that non-analytic "kinks" can be chosen to occur on some, but perhaps not all, characteristic surfaces issuing into the region. Each such characteristic surface may be thought of as corresponding to a particular kind of wave, generated at the boundary. For a vibrating beam there will be flexural and shear waves, travelling at different speeds, for example. Our theorem shows that there is a solution, satisfying one boundary condition, in which only one type of wave is generated. With a suitable boundary condition, it is quite possible to have a solution in which only some other type of wave arises. In a physical problem, the appropriate linear combination of these solutions would have to be selected by some further conditions; usually these would be additional boundary conditions.

6. The non-analytic case. An existence theorem for hyperbolic equations of order m has been given by Leray **(9)** under an assumption of finite differentiability, and by means of analytic approximations for which uniform estimates are obtained through the use of energy integrals. More recently Gårding **(5)** has given a direct existence proof using only the energy integrals. These calculations refer to the Cauchy problem, and we shall here investigate their application to mixed problems as in the preceding sections. For second order equations, this aspect of mixed problems has been treated, for example in **(2, 7, 10)**.

The results which we obtain do generalize the known theorems for second order equations in two different ways, which will constitute Cases I and II below. However it has not been possible to attain the generality of the theorem for analytic equations, and a considerable gap remains to be filled. It should be remarked that for the case of two variables a thorough treatment by Picard's method has been given by Campbell and Robinson **(1)**, covering semilinear equations as well. The energy integral method has been applied to the linear problem in two variables by Thomée **(12)**.

In contrast to the analytic case, we must now assume that the differential equation (1.1) is regularly hyperbolic in the sense of Leray: that is, in effect, that there exist timelike directions and that the normal cone is real and has no multiple points except the origin. This criterion will be fulfilled if the initial surface S is so situated that the equation is regularly hyperbolic (in the sense of § 1) with respect to S and to every surface T meeting S in a smooth hypercurve C.

When applied to a mixed problem, the energy integral formulae are modified by the presence of a boundary integral taken over the surface T. To complete the estimates we must show that this boundary integral form is semi-bounded. We therefore begin with an algebraic study of this boundary term, and will use the elegant notation of Hörmander **(6)** for the algebra of energy integrals.

Let the terms of highest order m in (1.1) be written

$$P(D)u, \qquad D_j = \frac{1}{i}\frac{\partial}{\partial x^j}, \tag{6.1}$$

where $P(D)$ is a polynomial of order m; and let $Q(D)$ be a real polynomial operator of order $m - 1$ in D. Then the quadratic form

$$F(D, \bar{D})\, u\, \bar{u} \equiv P(D)\, u\, \bar{Q}(\bar{D})\, \bar{u} - \bar{P}(\bar{D})\, \bar{u}\, .\, Q(D)u \tag{6.2}$$

is a divergence expression

$$- i \sum_j \partial/\partial x^j(G_j(D, \bar{D})\, u\, \bar{u}), \tag{6.3}$$

where the operators $G_i(D, \bar{D})$ are related to $F(D, \bar{D})$ by the equation

$$F(\zeta, \bar{\zeta}) = \sum_j (\zeta_j - \bar{\zeta}_j)\, G_j\, (\zeta, \bar{\zeta}). \tag{6.4}$$

Here $\zeta_j = \xi_j + i\eta_j$ and $\bar{\zeta}_j = \xi_j - i\eta_j$ are complex variables dual to D_i. In forming (6.3) we shall assume that the coefficients of $P(D)$ and $Q(D)$ are constants; this assumption can later be relaxed.

Writing the differential equation (1.1) in the form

$$L_u = P(D)u + B(D)u = f(x^i), \tag{6.5}$$

where $B(D)$ is an operator of order less than m, we integrate the expression $2\, Re\, Lu\, \bar{Q}(\bar{D})\bar{u}$ over a lens-shaped region R such as is described in **(3**, Figure 2**)**. This region is bounded by initial and final surfaces S_0 and S_t (t = const), and by a portion T_t ($x = 0$) of the boundary surface T. We find, on the one hand

$$i \int_R F(D\, \bar{D})\, u\, \bar{u}\, d\, V = \int_R Q(D, \bar{D}, u, \bar{u}, f)\, d\, V, \tag{6.6}$$

where the quadratic form $Q(D, \bar{D}, u, \bar{u}, f)$ is of order $m - 1$ or less in the D_i, and contains factors linear in f. On the other hand, by (6.3) we have

$$\begin{aligned} i \int_R F(D\, \bar{D})\, u\, \bar{u}\, d\, V &= \int_{S_t - S_0} G_t(D\, D)\, u\, \bar{u}\, d\, S_t \\ &\quad - \int_{T_t} G_x(D\, \bar{D})\, u\, \bar{u}\, d\, S_x, \end{aligned} \tag{6.7}$$

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the minus sign in the last integral being due to the convention that x shall be measured as increasing along the interior normal to T_t. Comparing (6.6) and (6.7), we find

$$(6.8)\quad \int_{S_t} G_t(D\,\bar{D})\,u\,\bar{u}\,d\,S_t - \int_{T_t} G_x(D\,\bar{D})\,u\,\bar{u}\,d\,S_x = \int_{S_0} G_t(D\,\bar{D})\,u\,\bar{u}\,d\,S_t + \int_{R_t} Q(D, \bar{D}, u, u, f)\,d\,V.$$

The method of Leray and Gårding is based on the fact that if the auxiliary operator $Q(D)$ is so chosen that the sheets of its normal cone separate the sheets of the normal cone of $P(D)$, then the integral over S_t is positive definite (**4**, Lemma 3.1). For this purpose we may assume that the coefficients of $i^m D_t^m$ in $P(D)$, and of $i^{m-1}D_t^{m-1}$ in $Q(D)$, are both $+ 1$.

Now let us suppose that $P(D)$ and $Q(D)$ have variable but sufficiently often differentiable coefficients. Then (6.3) is modified by the addition of derivatives of order lower than $m - 1$, and a quadratic form in such derivatives of u, $\bar{u}$ will appear in (6.7). These terms may be absorbed in the integral over R_t in (6.8), which is thus not changed in form. Also, in this case, the integral over S_t can be made positive definite in all derivatives of u of all orders $< m - 1$, by the addition of derivatives to the integrand $G_t(D\,D)\,u\,\bar{u}$ of orders not greater than $m - 2$. Again, such terms in the new integrand $G_t(D\,\bar{D})\,u\,\bar{u}$ over S_t can be counterbalanced by terms in the volume integral containing derivatives of order no higher than $m - 1$. Consequently (6.8) holds unchanged in form for the case of variable coefficients as well, with a somewhat different quadratic form $Q(D, \bar{D}, u, \bar{u}, f)$ in the volume integral. Set

$$(6.9)\qquad E_k(t) = \int_{S_t} \sum_{0\leqslant\alpha\leqslant k} |D^\alpha u|^2\,d\,S_t,$$

the summation being taken over all essentially distinct partial derivatives of u of order less than $k + 1$. By the positive definiteness of the integrand in the integral over S_t, we find, (**4**, Theorem 2.1; **5**, Theorem 3.1) that there exists a constant $c > 0$, depending only on the differential equation and the domain, such that

$$(6.10)\qquad \int_{S_t} G_t(D, \bar{D})\,u\,\bar{u}\,d\,S_t \geqslant c^{-1}E_{m-1}(t) - c\,E_{m-2}(t),$$

for every t and all $u \in C^{m-1}$.

We may express $E_{m-2}(t)$ as the integral of the time-derivative of its integrand: this leads to an estimate

$$(6.11)\qquad E_{m-2}(t) \leqslant E_{m-2}(0) + K\int_0^t E_{m-1}(t)dt.$$

It now follows from (6.8) and (6.10) that

$$
\begin{aligned}
c^{-1} E_{m-1}(t) &\leqslant c\, E_{m-2}(t) + \int_{S_t} G_t(D, \bar{D})\, u\, \bar{u}\, d\, S_t \\
&\leqslant c\, E_{m-2}(0) + c\, K \int_0^t E_{m-1}(\tau) d\tau \\
&\quad + \int_{S_0} G_t(D, \bar{D})\, u\, \bar{u}\, d\, S_t + \int_{R_t} Q(D\, \bar{D}\, u\, \bar{u}\, f) dV \\
&\quad + \int_{T_t} G_x(D\, \bar{D})\, u\, \bar{u}\, dS_x.
\end{aligned} \tag{6.12}
$$

By Schwarzian estimations of the third and fourth terms on the right-hand side of this last inequality, we find

$$
\begin{aligned}
E_{m-1}(t) &\leqslant c^2 E_{m-2}(0) + K_1 E_{m-1}(0) + K_2 \int_0^t E_{m-1}(\tau) d\tau + I_B(t) \\
&\leqslant K_0 + K_2 \int_0^t E_{m-1}(\tau)\, d\, D + I_B(t)
\end{aligned} \tag{6.13}
$$

where we have written

$$
I_B(t) = \int_{T_t} G_x(D, \bar{D})\, u\, \bar{u}\, d\, S_x. \tag{6.14}
$$

Here K_0 and K_2 are constants depending on all the data of the problem, but not on u.

Now let us suppose that we can prescribe a similar estimate

$$
I_B(t) \leqslant K_4 + K_5 \int_0^t E_{m-1}(\tau) d\tau, \tag{6.15}
$$

for the boundary integral. By standard methods (**9**; **4**, Lemma 1.2) we can now establish a conventional L^2 estimate

$$
E_{m-1}(t) \leqslant (K_0 + K_4) \exp\, [(K_2 + K_5)t]. \tag{6.16}
$$

Integration with respect to t leads to L^2 estimates over the entire domain R_t, and the process of solution, using analytic approximation together with Sobolev's lemma and Ascoli's selection theorem, then proceeds as in (**9**, p. 162). Further details will not be presented here.

To summarize this discussion, we state

LEMMA 2. *Let initial and boundary surfaces S and T subtend exactly k characteristic surfaces of the regularly hyperbolic equation*

$$
Lu = f, \tag{6.17}
$$

the surfaces and functions present being of class $[\frac{1}{2}N] + h + m$ in the closure of the region R. Let zero Cauchy data be assigned on S, and let k of the derivatives $u, u_x, \ldots, u_x^{m-1}$ be assigned the value zero on T. Then if the boundary integral $I_B(t)$ satisfies an estimate (6.15), *there exists a solution $u \in C^{h+m}$ of* (6.17) *which satisfies these conditions.*

We note in passing that the problem with non-homogeneous boundary conditions can be reduced to the form above by subtraction of a function which satisfies the initial and boundary conditions.

In Cases I and II below we shall need quite different methods to establish the inequality (6.15). The work falls into two parts, namely, an analysis of the case of constant coefficients, and an adaptation of this case to the more general situation with variable coefficients.

7. The boundary form. We consider first the particular case of constant coefficients in the differential operator, and use Fourier transforms to estimate the integral over T_t. Let us denote by $\tilde{u} = \tilde{u}(\xi_i)$ the Fourier transform

$$\tilde{u}(\xi_i) = \int e^{2\pi i(\xi,x)} u(x^\rho, t)\, dx^\rho\, dt$$

of a function of x_ρ and t, defined as equal to $u(x^\rho, t)$ on T_t and zero elsewhere. Here also

$$(\xi, x) = \xi_0 t + \sum_\rho \xi_\rho x_\rho.$$

The inverse transform is easily written down, and we note that ξ_ρ is now dual to $D_\rho = -i\partial/\partial x_\rho$. Thus, we shall need to distinguish differentiation with respect to the transverse variable x (across T), by writing

$$G_x(D, \bar{D}) = G\,(D_i, D_x, \bar{D}_i, \bar{D}_x).$$

Now Parseval's theorem **(6)** shows that

$$(7.1)\qquad I_B(t) = \int_{T_t} G_x(D_i, D_x, \bar{D}_i, \bar{D}_x)\, u\, \bar{u}\, d\, S_t = \int G_x(\xi_i, D_x, \xi_i, \bar{D}_x)\, \tilde{u}\, \bar{\tilde{u}}\, dS_\xi.$$

This last integrand is a quadratic form in the variables $D_x{}^j\, \tilde{u}$, $(j = 0, 1, \ldots, m-1)$, which are independently defined when regarded as functions of the ξ_i. Thus in the case of constant coefficients we are led to study the algebraic properties of this quadratic form. We single out the variable $\zeta_x = \xi_x + i\eta_x$ and set all other variables ζ_i in G_x equal to their real parts ξ_i. From (6.4) it is found that

$$(7.1)\qquad G_x(\xi_i, \zeta_x, \xi_i, \bar{\zeta}_x) = \frac{F(\xi_i, \zeta_x, \xi_i, \bar{\zeta}_x)}{\zeta_x - \bar{\zeta}_x}.$$

Let us now write

$$(7.2)\qquad \begin{aligned} P(\zeta) &= P(\zeta_x) = P(\zeta_x, \xi) = \sum_{r=0}^{m} a_r(\xi_i)\zeta^r, \\ Q(\zeta) &= Q(\zeta_x) = Q(\zeta_x, \xi) = \sum_{s=0}^{m-1} b_s(\xi_i)\zeta^s. \end{aligned}$$

Then, dropping mention of the ξ_i variables $(i \neq x)$, we find

$$G_x(\zeta_x, \bar\zeta_x) = \frac{P(\zeta_x)Q(\bar\zeta_x) - P(\bar\zeta_x)Q(\zeta_x)}{(\zeta_x - \bar\zeta_x)}$$
$$= \frac{\sum_{r=b}^{m} a_r\zeta_x^r \sum_{s=0}^{m-1} b_s\bar\zeta_x^s - \sum_{r=0}^{m} a_r\bar\zeta_x^r \sum_{s=0}^{m-1} b_s\zeta_x^s}{\zeta_x - \bar\zeta_x}$$
$$= \sum_{r=0}^{m}\sum_{s=0}^{m-1} a_r b_s\left(\frac{\bar\zeta_x^s\zeta_x^r - \zeta_x^s\bar\zeta_x^r}{\zeta_x - \bar\zeta_x}\right).$$

Since the expression in parentheses is the sum of the geometric series

$$\pm \sum_{k=0}^{|r-s|-1} \bar\zeta^k\zeta^{|r-s|-1-k},$$

(the $+$ sign being taken if $r > s$, the minus if $r < s$), we find, after some rearrangement,

$$G_x(\zeta_x, \bar\zeta_x) = \sum_{p,q=0}^{m-1} c_{pq}\,\zeta_x^p\,\bar\zeta_x^q, \tag{7.3}$$
$$c_{pq} = \sum_{s=0}^{\min(p,q)} (b_s\, a_{p+q+1-s} - a_s\, b_{p+q+1-s}).$$

Here a_s is taken as zero for $s > m$, while $b_s = 0$ for $s > m - 1$. It may be noted that $c_{pq} = c_{pq}(\xi_i)$ is homogeneous of degree $2m - 2 - p - q$ in the variables ξ_i.

When k homogeneous boundary conditions are assigned, then in effect k rows and columns of the coefficient matrix c_{pq} are deleted, since the corresponding terms fall out. The estimate we require is essentially that the remaining, or residual, quadratic form, be non-positive. We consider two cases here: when it is negative definite, and when it is zero.

Let the residual form be negative definite; then by altering the k deleted rows and columns we can arrange that the new (enlarged) form should also be negative definite. Then an estimate of the type of (**5**, Lemma 3.1) will hold, though in the opposite direction. This property of the constant-coefficients case, which is a local property for variable coefficients, enables us to deduce the analogue of (**5**, Theorem 3.1) which applies to the case of variable coefficients. It now reads

$$I_B(t) = \int_{T_t} G_x(D_i, D_x, \bar D_i, \bar D_x)\, u\,\bar u\, d\,S_t \tag{7.4}$$
$$\leqslant -c^{-1}\tilde E_{m-1}(t) + c\tilde E_{m-2}(t), \qquad c > 0,$$

where

$$\tilde E_j(t) = \int_{T_t} \sum_{|\alpha|\leqslant j} |D^\alpha u|^2\, d\,S_t. \tag{7.5}$$

We must remember that the assigned homogeneous boundary data are inserted on the left in (7.4). The first term on the right in (7.4) can be replaced by zero; we shall now estimate the second one, in much the same way as in (6.11). Let each point of T_t be joined to a point of S_0 by a line $x + t =$ const, $x_\rho =$ const, $\rho = 1, \ldots, N - 2$; and let us express the integrand of $\tilde{E}_{m-2}(t)$ as the integral of its derivative along this line. Thus derivatives of order $m - 1$ or less appear; and integration over T_t leads to integrals over a "triangular" portion of R_t making their appearance. Application of Schwarz' inequality now leads at once to the estimate (6.15) which we require.

This method of negative definite character for the residual quadratic form will be used in Case I below.

For Case II, where the residual quadratic form vanishes identically at every point of T, we must use a different approach to gain the result for variable coefficients. Let us use the fact that the coefficients are continuous, and, given a fixed function u, together with an arbitrary positive number ϵ, subdivide the boundary surface T_t into a finite number of portions T_h, in each of which the oscillation of the coefficients is less than ϵ. Select a point x_0^h in each T_h, and write

$$\int_{T_h} G_x(D_i D_x \bar{D}_i \bar{D}_x)\, u\bar{u}\, d\, S_T$$
$$= \int_{T_h} G_{x_0}(D_i D_x \bar{D}_i \bar{D}_x)\, u\bar{u}\, d\, S_t$$
$$+ \int_{T_h} R(D_i, D_x, \bar{D}_i \bar{D}_x)\, u\bar{u}\, d\, S_T,$$

where

$$G_{x_0}$$

is the boundary form with constant coefficients evaluated at x_0, and the coefficients in the remainder term $R(D, \bar{D})u\bar{u}$ are all less than ϵ in magnitude.

Suppose now that u satisfies the homogeneous boundary conditions; then by hypothesis the first integral on the right vanishes. The second integral can be estimated to be less than

$$\epsilon\, m^2 \int_{T_h} \sum_{|\alpha| \leqslant m-1} |D^\alpha u|^2\, d\, S_T.$$

It follows by summation over the T_h that the boundary integral $I_B(t)$ is less than

$$\epsilon\, m^2 \tilde{E}_{m-1}(t)$$

in magnitude. However, u and therefore $\tilde{E}_{m-1}(t)$, are fixed, and ϵ is arbitrary. Consequently $I_B(t)$ must vanish. For the variable coefficient problem, it is therefore sufficient that the coefficients at each point of T_t should lead to a vanishing residual matrix. We employ this result in Case II below.

8. Case I: $k = m - 1$. The corner element $c_{m-1,m-1}$ of the coefficient matrix of the above quadratic form is a polynomial of degree zero in the

ξ_t; it is in fact $a_m b_{m-1}$. If this element is negative then the conditions of the lemma will be fulfilled when the function and its first $m - 2$ derivatives $u, u_x, \ldots, u_x^{(m-2)}$ are given as zero on T. Now this boundary condition will be appropriate if $k = m - 1$ characteristic surfaces lie between S and T: we assume this. Consequently one characteristic surface lies between T and the portion of S prolonged beyond the edge $C = S \cap T$. Define an auxiliary co-ordinate $z = t \cos \alpha + x \sin \alpha$, and let α range from $\alpha = 0$ to $\alpha = \frac{1}{2}\pi$. The coefficient of D_z^m in $P(D)$ is equal to 1 when $\alpha = 0$ and z coincides with t. Since this coefficient vanishes when the surface $z = \text{const}$ is characteristic, and changes sign at a simple characteristic surface, it vanishes once for $0 < \alpha < \frac{1}{2}\pi$ and is therefore negative. For $\alpha = \frac{1}{2}\pi$ it is a_m which is thus negative.

To make b_{m-1} positive, we should, in view of this discussion and of the fact that the characteristic surfaces of $Q(D)$ must separate those of $P(D)$, arrange that all $m - 1$ of these surfaces should lie between S and T. That is, T must be spacelike with respect to $Q(D)$, or equivalently the normal to T must be timelike. We shall *assume* that it is possible to find an auxiliary operator which has this property. For example, if $m = 2$, the order of $Q(D)$ is 1 and the characteristic surface can be chosen to have any direction between S and T. In the general case, it will be possible to find such an operator whenever T lies sufficiently close to the single characteristic surface G which lies outside the domain between S and T.

Together with Lemma 2 this demonstrates the following.

THEOREM 2. *Let $k = m - 1$ and suppose there exists an operator $Q(D)$ separating the sheets of $P(D)$ such that all $m - 1$ characteristic surfaces of $Q(D)$ lie between S and T in the region R. Then there exists a solution of $Lu = f$, with given Cauchy data on S, and with given values for the $m - 1$ quantities u, u_x, $u_x^{(2)}, \ldots, u_x^{(m-2)}$* on *T*.

When the coefficients of the differential equation are independent of x, it is possible to show that two other sets of $m - 1$ boundary conditions can be reduced to the set just treated. We write $Lu = u_x^{(m)} + \alpha_{m-1} u_x^{(m-1)} + \ldots + \alpha_0 u$, where α_{m-k} is a differential operator of order k in D_t, D_ρ, with coefficients independent of x.

COROLLARY. *Let the coefficients in* (1.1) *be independent of x. Then there exists a solution of the preceding mixed problem when the boundary conditions are*

(a) $$u_x^{(h)} = 0, \qquad h = 0, 1, \ldots, m-3, m-1, \qquad h \neq m-2$$

or

(b) $$u_x^{(h)} = 0, \qquad h = 1, 2, \ldots, m-1, \qquad h \neq 0.$$

To prove (a) let us suppose that v is a solution of this problem in the analytic case, and let us show that $v = u_x + \alpha_{m-1} u$, where u is a solution of a suitably selected problem with $u, u_x, \ldots, u_x^{(m-2)}$ vanishing on T. Since

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all $m - 1$ characteristic surfaces between S and T are select, it follows from the reduction of Theorem 1 and the uniqueness theorem of (3, § 9) that the solution is unique. Now let u satisfy $Lu = g$, where g is a solution, vanishing on T, of the first order linear partial equation

$$\frac{\partial g}{\partial x} + \alpha_{m-1}\, g = f.$$

Since the coefficient of $\partial g/\partial x$ is not zero, and since f is supposed analytic, such an analytic solution g exists and is uniquely determined. Since $g = \int f\, ds$, where the integration is taken along a characteristic curve, we can find L^2 estimates for g and its derivatives if such estimates are given for f.

Formal calculation, using the non-dependence on x of the coefficients, now shows that the combination $w = u_x + \alpha_{m-1}u$ satisfies $w = 0$, $w_x = 0, \ldots,$ $w_x^{(m-3)} = 0$, on T, while $w_x^{(m-1)} = u_x^{(m)} + \alpha_{m-2}u_x = \alpha_{m-2}ux^{(m-2)} - \ldots$ $- \alpha_0 u + g$, which latter expression also vanishes on T by the boundary conditions for u and g.

Now

$$Lw = L(u_x + \alpha_{m-1}\, u) = \left(\frac{\partial}{\partial x} + \alpha_{m-1}\right) Lu = \left(\frac{\partial}{\partial x} + \alpha_{m-1}\right) g = f,$$

so that w is an analytic solution of the case (a). Hence, by the uniqueness property, $v = w = u_x + \alpha_{m-1}u$. Since we have found L^2 bounds for u and its derivatives, it now follows that such bounds can be obtained for v if one degree of differentiability extra is assumed for the non-analytic problem. The remainder of the existence proof now follows the conventional methods and so is omitted.

The demonstration of case (b) is similar in principle, but a different device is used. We note that the "coefficient" α_0, which is a differential operator of order m in the other $N - 1$ derivatives, contains just those terms of Lu not involving d/dx, and so is regularly hyperbolic in the $N - 1$ variables. Also, the edge $C = S \cap T$ is a spacelike surface relative to α_0, so that the Cauchy problem $\alpha_0 z = 0$, with Cauchy data on C, is a correctly set problem within the boundary T. We will show that a solution v of case (b) is equal to a combination

$$w = u_x^{(m-1)} + \alpha_{m-1}u_x^{(m-2)} + \ldots + \alpha u,$$

when certain preliminary reductions have been made. As the non-homogeneous boundary conditions corresponding to Theorem 2 can be set up by a substitution, we shall consider the problem $v_x = f_1$, $v_x^{(2)} = f_2, \ldots v_x^{(m-1)} f_{m-1}$ on T, with $Lv = 0$ in R, and, as usual, zero Cauchy data. Subtracting from this a suitable solution of the problem with $u_x^{(m-1)}$ not given on T, we can suppose without loss of generality that $f_1 = f_2 = \ldots = f_{m-2} = 0$.

Now let u be that solution of $Lu = 0$ with $u = u_x = \ldots = u_x^{(m-3)} = 0$ on T, with $u_x^{(m-2)} = z$ on T, where z is the solution of $\alpha_0 z = -f_{m-1}$ on T.

Straightforward calculation shows that w, defined above, satisfies $Lw = 0$ with $w_x = w_x^{(2)} = \ldots = w_x^{(m-2)} = 0$ on T, while

$$w_x^{(m-1)} = u_x^{(2m-2)} + \alpha_{m-1}u_x^{(2m-3)} + \ldots + \alpha_1 u_x^{(m-1)} = -\alpha_0 u_x^{(m-2)} = f_{m-1}$$

on T. Thus w is an analytic solution of the problem and so is equal to v. Hence estimates for u can be applied now to v, provided that $m - 1$ extra degrees of differentiability are assumed for the original problem. This completes the reduction of case (b) to the conventional energy integral method.

9. Case II: Symmetry with respect to T**.** The second circumstance in which it can be shown that the residual quadratic form can be bounded above is when the hyperplane $T : x = 0$ is a plane of symmetry for the characteristic cone of the hyperbolic differential operator. We shall here restrict consideration to equations of even order, as it is necessary, for the odd order case, to make rather lengthy changes in the "analytic" existence theorems to cover this situation.

Thus let Lu be a regularly hyperbolic operator of even order $m = 2l$, of which the highest order terms contain D_x^2 but not odd powers of D_x, at any rate for $x = 0$. Then T is a plane of symmetry as stated above. If now the terms of order $2l$ are written as in (7.2), it is seen that $a_r(\xi_i) = 0$ for r odd. Let us take $Q(\zeta) = \partial P/\partial\zeta_t$; as shown in (**8**, p. 140) the sheets of the cone of Q will separate those of P as required for the formulation of the estimates. Thus the odd terms $b_s(\xi_i)$ in $Q(\zeta)$ will likewise vanish.

From (7.3) it is seen that in

$$c_{pq} = \sum_{s=0}^{\min(p,q)} (b_s a_{p+q+1-s} - a_s b_{p+q+1-s})$$

the sum of indices of the a and b coefficients is $p + q - 1$, which is odd whenever $p + q$ is even. It follows that each term will contain a vanishing factor when $p + q$ is even, and therefore that $c_{pq} = 0$ for $p + q$ even. Thus about half the terms in the matrix are zero, including all diagonal terms, all terms twice removed from the diagonal, and so on.

From the symmetry of the characteristic surfaces relative to the boundary T, we see that half of the sheets lie in R, and thus l boundary conditions are appropriate.

THEOREM 3. *Let $Lu = f$ be a regularly hyperbolic equation of even order $2l$, such that the boundary surface T is a plane of symmetry relative to the characteristic cone at each point of T. Then there exists a solution of the $C^{2l+[\frac{1}{2}N]+1}$ mixed problem with zero data assigned on T for either*

(a) *l derivatives of even order: $u, u_x^{(2)}, \ldots u_x^{(2l-2)}$ or*
(b) *l derivatives of odd order: $u_x, u_x^{(3)}, \ldots u_x^{(2l-1)}$.*

Proof. In case (a), an element c_{pq} belongs to the residual matrix only if both p and q are odd, and thus $p + q$ is even. Hence the residual part of the quadratic form is identically zero. Similarly, in case (b), the residual part contains elements with both p and q even so that c_{pq} vanishes and the quadratic form is zero. An application of Lemma 2 completes the proof.

For the case $m = 2l = 2$ we can use the Lorentz transformation to show that the symmetry requirement can always be satisfied.

10. Signature of the quadratic form. We conclude with some remarks on the algebraic structure of the quadratic form $G_x(\zeta_x, \bar{\zeta}_x)$. An adaptation of Gårding's analysis (**5**, Lemma 1.1) to the case where complex roots are present shows that the signature of this quadratic form is always compatible with the number of boundary conditions suggested by the arrangement of characteristic surfaces. However, the coefficients of this quadratic form are coefficients of the variables ξ_ρ dual to t and x^ρ, ($\rho = 1, \ldots, N - 2$), and consequently the eigenvectors corresponding to the negative eigenvalues of the coefficient matrix depend on the ξ_ρ. As these eigenvectors are linear combinations of the transforms of the x-derivatives, it follows that in general k linear conditions (with coefficients depending on the ξ_ρ) of the transforms $\tilde{u}_x^{(h)}(\xi_\rho)$ are required as boundary conditions in order that the residual matrix should be bounded above. Upon transformation back to the t, x^ρ variables, these relations would become integral conditions of convolution type on the derivatives of u. It seems probable that Gårding's direct method could be modified to include this rather unconventional type of condition.

11. A remark. I wish to correct a misstatement in the paper (**3**) on first order systems. On page 154, the fourth line from the bottom of the page should read "Let a non-singular analytic family of surfaces S_t fill R in such fashion that through each point of R there passes one and only one surface S_t of the family, and that $S_1 = S_{t=1}$."

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