Can. J. Math., Vol. XXII, No. 3, 1970, pp. 615–625

LOCALIZATION PROBLEM OF THE ABSOLUTE RIESZ AND ABSOLUTE NÖRLUND SUMMABILITIES OF FOURIER SERIES

MASAKO IZUMI AND SHIN-ICHI IZUMI

1. Introduction and theorems.

1.1. Let $\sum a_n$ be an infinite series and s_n its nth partial sum. Let (p_n) be a sequence of positive numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as } n \to \infty.$$

If the sequence

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \qquad (n = 0, 1, 2, \ldots) \tag{1}$$

is of bounded variation, that is, $\sum |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is said to be absolutely $(R, p_n, 1)$ summable or $|R, p_n, 1|$ summable.

Let f be an integrable function with period 2π and let its Fourier series be

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x). \tag{2}$$

Dikshit [4] (cf. Bhatt [1] and Matsumoto [7]) has proved the following theorems.

THEOREM I. *Suppose that* (i) *the sequence* (p_n/P_n) *is monotone decreasing,* (ii) $m_n > 0$, (iii) *the sequence* $(m_n p_n/P_n)$ *decreases monotonically to zero, and* (iv) *the series* $\sum (m_n p_n/P_n)$ *is divergent. If* $0 < a < b < 2\pi$, *there is a function* f *integrable over the interval* (a, b) *and vanishing on the intervals* $(0, a)$ *and* $(b, 2\pi)$ *such that the series* $\sum m_n A_n(x)$ *is not* $|R, p_n, 1|$ *summable at the origin.*

THEOREM II. *Suppose that* (i) *the sequence* (p_n/P_n) *is monotone decreasing and* (ii) *the sequence* $(P_n/n^{1+\theta}p_n)$ *decreases for a* θ, $0 < \theta \leqq 1$. *If*

$$\sum_{n=1}^{\infty} (|A_n(x)| p_n/P_n) < \infty,$$

then the summability $|R, p_n, 1|$ *of the Fourier series* (2) *at the point* x *depends only on the behaviour of the function* f *in the immediate neighbourhood of the point* x.

We shall first prove the following theorem.

Received March 28, 1969.

THEOREM 1. *Suppose that* (p_n) *is a sequence of positive numbers such that* $P_n \to \infty$ *and* $(p_n/P_{n-1}P_n)$ *is decreasing and the sequence* (m_n) *of positive numbers is of bounded variation such that*

$$m_n p_n/P_n \leqq A m_{2n} p_{2n}/P_{2n} \quad \textit{for all } n$$

and $(m_n p_n/P_n)$ *is monotone decreasing. If*

$$\sum_{n=1}^{\infty} (|A_n(x)| m_n p_n/P_n) < \infty, \tag{3}$$

then the summability $|R, p_n, 1|$ *of the series* $\sum m_n A_n(x)$ *at the point* x *depends only on the behaviour of the function* f *in the immediate neighbourhood of the point* x.

1.2. The nth Nörlund mean of the series $\sum a_n$ is defined by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k, \tag{4}$$

s_k being the kth partial sum of the series. If the sequence (t_n) is of bounded variation, then the series $\sum a_n$ is said to be absolutely (N, p_n) summable or $|N, p_n|$ summable.

Daniel [**3**] has proved the following theorems which are a generalization of the theorems of Jurkat and Peyerimhoff [**6**] and Bhatt [**2**].

THEOREM III. *If the positive sequence* (m_n) *satisfies the conditions*

$$\sum (m_n |\cos 2nx| / P_n) < \infty$$

and

$$\sum (m_n/P_n) = \infty,$$

then the summability $|N, p_n|$ *of the series* $\sum m_n A_n(x)$ *at the point* x *is not a local property of* f.

THEOREM IV. *Suppose that the sequences* (p_n) *and* (m_n) *are positive monotone decreasing and that they satisfy the following conditions:*

$$m_{n+1}/m_{n+k} \leqq A, \textit{ uniformly in } k < n, \quad \textit{as } n \to \infty, \tag{5}$$

$$\frac{1}{n m_n} \sum_{k=1}^{n} \frac{m_k}{P_k} \leqq A \quad \textit{as } n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{m_n}{n P_n} < \infty. \tag{7}$$

If

$$\sum_{n=1}^{\infty} (|A_n(x)| m_n/P_n) < \infty, \tag{8}$$

then the summability $|N, p_n|$ *of the series* $\sum m_n A_n(x)$ *depends only on the behaviour of* f *in the immediate neighbourhood of the point* x.

We prove the following result.

THEOREM 2. *Suppose that the sequences* (p_n) *and* (m_n) *are positive, monotone decreasing and*

$$m_n/P_n \leqq Am_{2n}/P_{2n} \quad \text{for all } n.$$

If the condition (8) *is satisfied, then the summability* $|N, p_n|$ *of the series* $\sum m_n A_n(x)$ *depends only on the behaviour of* f *in the immediate neighbourhood of the point* x.

Further, we prove the following.

THEOREM 3. *Suppose that* (m_n) *is a positive, monotone decreasing and convex sequence such that*

$$\Delta m_n \leqq A\,\Delta m_{2n} \quad \text{for all } n,$$

and that the sequence (p_n) *is monotone increasing and satisfies the condition*

$$(9) \qquad \sum_{n=j+1}^{\infty} \frac{p_{n-j} - p_{n-j-1}}{P_{n-1}} \leqq \frac{A}{j+1} \quad \text{for all } j \geqq 0.$$

If

$$(10) \qquad \sum_{n=1}^{\infty} \frac{|A_n(x)|m_n}{n} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |A_n(x)|\Delta m_n \log n < \infty,$$

then the summability $|N, p_n|$ *of the series* $\sum m_n A_n(x)$ *depends only on the behaviour of* f *in the immediate neighbourhood of the point* x.

Theorems 1, 2, and 3 hold also for conjugate series.

2. Proofs of the theorems.

2.1. *Proof of Theorem* 1. We can suppose that f is even and $x = 0$. We shall consider the Riesz means (t_n) of the series $\sum m_n a_n$, then (1) yields

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} P_k m_{k+1} a_{k+1}$$

and then

$$(11) \qquad \sum_{n=1}^{\infty} |t_n - t_{n-1}| \leqq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} P_k m_{k+1} a_{k+1} \right| = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} |T_n|.$$

We can write

$$(12) \quad T_n = \sum_{k=0}^{n-1} P_k m_{k+1} a_{k+1} = \frac{2}{\pi} \int_0^{\pi} f(t) \left(\sum_{k=0}^{n-1} P_k m_{k+1} \cos(k+1)t \right) dt$$
$$= \frac{2}{\pi} \int_0^{\pi} f(t) \left[\sum_{k=1}^{n-1} (P_{k-1}\Delta m_k - p_k m_{k+1}) D_k(t) + P_{n-1} m_n D_n(t) - P_0 m_1 D_0(t) \right] dt,$$

where $D_k(t)$ is the kth Dirichlet kernel [**8**], that is,

$$D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} = \frac{\sin kt}{2\tan\frac{1}{2}t} + \tfrac{1}{2}\cos kt.$$

Hence we put

$$(13)\quad T_n = \frac{2}{\pi}\int_0^\pi f(t)\left[\sum_{k=1}^{n-1}(P_{k-1}\Delta m_k - p_k m_{k+1})\frac{\sin kt}{2\tan\frac{1}{2}t} + P_{n-1}m_n\frac{\sin nt}{2\tan\frac{1}{2}t}\right]dt$$

$$+\frac{1}{\pi}\int_0^\pi f(t)\left[\sum_{k=1}^{n-1}(P_k\Delta m_k - p_k m_{k+1})\cos kt + P_{n-1}m_n\cos nt - P_0 m_1\right]dt$$

$$= T_n' + T_n'',$$

then

$$|T_n''| \leqq A\sum_{k=1}^{n-1}(P_{k-1}|\Delta m_k| + p_k m_{k+1})|a_k| + AP_{n-1}m_n|a_n| + A,$$

and hence, by (3) and since m_k is of bounded variation, we have

$$(14)\quad \sum_{n=1}^{\infty}\frac{p_n|T_n''|}{P_nP_{n-1}} \leqq A\sum_{n=1}^{\infty}\frac{p_n m_n|a_n|}{P_n} + A\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$+ A\sum_{k=1}^{\infty}(P_{k-1}|\Delta m_k| + p_k m_{k+1})|a_k|\sum_{n=k+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$\leqq A + \sum_{k=1}^{\infty}\frac{P_{k-1}|\Delta m_k| + p_k m_{k+1}}{P_k}|a_k| < \infty.$$

By (11), (13), and (14), we obtain

$$(15)\quad \sum_{n=1}^{\infty}|t_n - t_{n-1}| \leqq A + \sum_{n=1}^{\infty}\frac{p_n|T_n'|}{P_nP_{n-1}}.$$

We shall prove that the last sum is finite except for the terms depending on the behaviour of f in the interval $(0, \epsilon)$, ϵ being any positive fixed number.

Now, we define an odd continuous function g, periodic with period 2π, such that

$$g(t) = \tfrac{1}{2}\cot\tfrac{1}{2}t \quad \text{in the interval } (\epsilon, \pi)$$

and that g is differentiable at least four times everywhere. If we write

$$(16)\quad g(t) \sim \sum_{n=1}^{\infty} c_n \sin nt,$$

then $c_n = O(1/n^3)$ as $n \to \infty$. Using this function $g(t)$, we obtain the following formula:

$$(17)\quad \int_0^\pi \frac{f(t)}{2\tan\frac{1}{2}t}\sin nt\,dt = \int_0^\epsilon f(t)\left(\frac{1}{2\tan\frac{1}{2}t} - g(t)\right)\sin nt\,dt$$

$$+ \int_0^\pi f(t)g(t)\sin nt\,dt.$$

https://doi.org/10.4153/CJM-1970-068-6 Published online by Cambridge University Press

Since the first integral on the right side of (17) depends on the values of f in the interval $(0, \epsilon)$, we leave it out of consideration. Hence it is enough to show that

$$(18)\quad \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-1} (P_{k-1}|\Delta m_k| + p_k m_{k+1}) \left| \int_0^{\pi} f(t)g(t) \sin kt\, dt \right| + \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \left| \int_0^{\pi} f(t)g(t) \sin nt\, dt \right| = U + V$$

is finite. By (16), we obtain

$$(19)\quad \int_0^{\pi} f(t)g(t) \sin nt\, dt = \sum_{j=1}^{\infty} c_j \int_0^{\pi} f(t) \sin jt \sin nt\, dt = \frac{\pi}{4} \sum_{j=1}^{\infty} c_j (a_{|j-n|} - a_{j+n})$$

and then

$$(20)\quad V \leqq A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \sum_{j=1}^{\infty} |c_j|(|a_{|j-n|}| + |a_{j+n}|) = V_1 + V_2,$$

where

$$(21)\quad \begin{aligned} V_1 &= A \sum_{j=1}^{\infty} |c_j| \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} |a_{|j-n|}| \\ &= A \sum_{j=1}^{\infty} |c_j| \left(\sum_{n=1}^{j} \frac{m_n p_n}{P_n} |a_{j-n}| + \sum_{n=j+1}^{\infty} \frac{m_n p_n}{P_n} |a_{n-j}| \right) \\ &\leqq A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \sum_{j=n}^{\infty} |c_j| + \sum_{j=1}^{\infty} |c_j| \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} |a_n| \\ &\leqq A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \frac{1}{n^2} + A \sum_{j=1}^{\infty} \frac{1}{j^3} \\ &< \infty, \end{aligned}$$

by using the monotonicity of the sequence $(m_n p_n / P_n)$ and the condition (3), and

$$(22)\quad \begin{aligned} V_2 &= A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \left(\sum_{j=1}^{n} + \sum_{j=n+1}^{\infty} \right) |c_j|\, |a_{j+n}| \\ &\leqq A \sum_{j=1}^{\infty} |c_j| \sum_{n=j}^{\infty} \frac{m_n p_n}{P_n} |a_{2n}| + A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \sum_{j=n+1}^{\infty} \frac{1}{j^3} \\ &\leqq A \sum_{j=1}^{\infty} |c_j| \sum_{n=j}^{\infty} \frac{m_{2n} p_{2n}}{P_{2n}} |a_{2n}| + A \sum_{n=1}^{\infty} \frac{m_n p_n}{n^2 P_n} \\ &< \infty, \end{aligned}$$

by the conditions $m_n p_n / P_n \leqq A m_{2n} p_{2n} / P_{2n}$ and (3).

https://doi.org/10.4153/CJM-1970-068-6 Published online by Cambridge University Press

On the other hand we put

$$(23)\quad U \leqq A \sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}P_n} \sum_{k=1}^{n-1} (P_{k-1}|\Delta m_k| + p_k m_{k+1}) \sum_{j=1}^{\infty} |c_j|(|a_{|j-n|}| + |a_{j+n}|)$$

$$\leqq A \sum_{j=1}^{\infty} |c_j| \sum_{k=1}^{\infty} \frac{P_{k-1}|\Delta m_k|}{P_k}$$

$$+ A \sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}P_n} \sum_{k=1}^{n-1} p_k m_{k+1} \Bigg(\sum_{j=0}^{n-1} |c_{n-j}|\, |a_j| + \sum_{j=1}^{\infty} |c_{n+j}|\, |a_j| + \sum_{j=1}^{\infty} |c_j|\, |a_{j+n}| \Bigg)$$

$$= W + X + Y + Z.$$

W is evidently finite. We write

$$(24)\quad X = A \sum_{j=0}^{\infty} |a_j| \sum_{n=j+1}^{\infty} \frac{p_n|c_{n-j}|}{P_{n-1}P_n} \sum_{k=1}^{n-1} p_k m_{k+1}$$

$$= A \sum_{j=0}^{\infty} |a_j| \sum_{n=j+1}^{\infty} \frac{p_n|c_{n-j}|}{P_{n-1}P_n} \left(\sum_{k=1}^{j-1} + \sum_{k=j}^{n-1} \right) p_k m_{k+1}$$

$$= X_1 + X_2.$$

Since the sequence $(p_n/P_{n-1}P_n)$ decreases as $n \to \infty$, we have

$$(25)\quad X_1 \leqq A \sum_{j=0}^{\infty} |a_j| \sum_{n=j+1}^{\infty} \frac{p_n|c_{n-j}|}{P_{n-1}P_n} \left(P_{j-1}m_j + \sum_{k=1}^{j-2} P_k \Delta m_{k+1} + P_0 m_2 \right)$$

$$\leqq A \sum_{j=0}^{\infty} \frac{p_j m_j |a_j|}{P_j} + A \sum_{j=0}^{\infty} \frac{|a_j| p_j}{P_{j-1}P_j} \sum_{k=1}^{j-2} P_k |\Delta m_{k+1}| + A$$

$$\leqq A + A \sum_{k=1}^{\infty} P_k |\Delta m_{k+1}| \sum_{j=k+2}^{\infty} \frac{|a_j| p_j}{P_{j-1}P_j}$$

$$< \infty,$$

and, by the monotonicity of the sequences $(p_n/P_{n-1}P_n)$ and $(m_n p_n/P_n)$, we can see that

$$(26)\quad X_2 \leqq A \sum_{j=0}^{\infty} |a_j| \sum_{k=j}^{\infty} p_k m_{k+1} \sum_{n=k+1}^{\infty} \frac{p_n |c_{n-j}|}{P_{n-1}P_n}$$

$$\leqq A \sum_{j=0}^{\infty} |a_j| \sum_{k=j}^{\infty} \frac{p_k m_{k+1} p_{k+1}}{P_k P_{k+1} (k - j + 1)^2}$$

$$\leqq A \sum_{j=0}^{\infty} \frac{|a_j| p_j m_j}{P_j}$$

$$< \infty.$$

Further we obtain

$$(27)\qquad Y \leqq A \sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}P_n} \sum_{k=1}^{n-1} p_k m_{k+1} \sum_{j=1}^{\infty} |c_{n+j}| \leqq A \sum_{n=1}^{\infty} \frac{p_n}{n^2 P_n} < \infty$$

and

$$\begin{aligned}(28)\quad Z &\leqq A \sum_{j=1}^{\infty} |c_j| \sum_{n=1}^{\infty} \frac{p_n|a_{n+j}|}{P_{n-1}P_n} \sum_{k=1}^{n-2} p_k m_{k+1} \\ &\leqq A \sum_{j=1}^{\infty} |c_j| \sum_{n=1}^{\infty} \frac{p_n|a_{n+j}|}{P_{n-1}P_n} \left(P_0 m_2 + P_{n-1}m_n + \sum_{k=1}^{n-2} P_k|\Delta m_{k+1}|\right) \\ &\leqq A \sum_{j=1}^{\infty} c_j + A \sum_{n=1}^{\infty} \frac{m_n p_n}{P_n} \sum_{j=1}^{\infty} |c_j|\,|a_{n+j}| + A \sum_{j=1}^{\infty} |c_j| \sum_{k=1}^{\infty} |\Delta m_{k+1}| \\ &< \infty,\end{aligned}$$

by using (22).

Collecting (23)–(28), we can see that U is finite. Combining this with (18), (20), (21), and (22), we obtain the required result.

2.2. *Proof of Theorem* 2. We can suppose that f is even and $x = 0$. Let (t_n) be the nth Nörlund mean of the series $\sum m_n a_n$, then; by (4),

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k',$$

where s_k' is the kth partial sum of the series $\sum m_n a_n$. Hence

$$\begin{aligned}(29)\quad t_n - t_{n-1} &= \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} (p_k P_n - p_n P_k) m_{n-k} a_{n-k} \\ &= \frac{1}{P_n P_{n-1}} \Bigg\{ \sum_{k=1}^{n-1} [P_n(p_k m_{n-k} - p_{k-1} m_{n-k+1}) \\ &\qquad\qquad - p_n(P_k m_{n-k} - P_{k-1} m_{n-k+1})] s_{n-k} \\ &\qquad\qquad - m_1(P_n p_{n-1} - P_{n-1} p_n) s_0 + m_n(p_0 P_n - P_0 p_n) s_n \Bigg\} \\ &= R_n + S_n + T_n,\end{aligned}$$

where s_n is the nth partial sum of the series $\sum a_n$.

Now, the coefficient of s_{n-k} in R_n is

$$\begin{aligned}&\frac{1}{P_n P_{n-1}} \{P_n(p_k m_{n-k} - p_{k-1} m_{n-k+1}) - p_n(P_k m_{n-k} - P_{k-1} m_{n-k+1})\} \\ &\qquad = \frac{1}{P_n P_{n-1}} \{(P_n p_k - p_n P_k)(m_{n-k} - m_{n-k+1}) + (P_{n-1} p_k - P_n p_{k-1}) m_{n-k+1}\},\end{aligned}$$

so that

$$(30)\qquad \sum_{n=1}^{\infty} |R_n| \leqq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{k=1}^{n-1} (P_n p_k - p_n P_k)(m_{n-k} - m_{n-k+1})|s_{n-k}| + \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{k=1}^{n-1} (P_n p_{k-1} - P_{n-1} p_k) m_{n-k+1} |s_{n-k}| = U + V.$$

As in the proof of Theorem 1, we define the function $g(t)$, and we write

$$U = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{j=1}^{n-1} (P_n p_{n-j} - p_n P_{n-j})(m_j - m_{j+1}) \times \left\{\frac{2}{\pi}\left[\int_0^{\epsilon} f(t)\left(\frac{1}{2\tan\frac{1}{2}t} - g(t)\right)\sin jt\,dt + \int_0^{\pi} f(t)g(t)\sin jt\,dt\right] + \frac{1}{2}a_j\right\}$$

$$= U_1 + U_2 + U_3,$$

$$V = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{j=1}^{n-1} (P_n p_{n-j-1} - P_{n-1} p_{n-j}) m_{j+1} \times \left\{\frac{2}{\pi}\left[\int_0^{\epsilon} f(t)\left(\frac{1}{2\tan\frac{1}{2}t} - g(t)\right)\sin jt\,dt + \int_0^{\pi} f(t)g(t)\sin jt\,dt\right] + \frac{1}{2}a_j\right\}$$

$$= V_1 + V_2 + V_3,$$

where U_1 and V_1 depend only on the value of f in the immediate neighbourhood of the origin. Thus it is sufficient to show that U_2, U_3, V_2, and V_3 are finite.

Since the sequence (p_n) decreases monotonically and

$$(31)\qquad \sum_{n=j+1}^{\infty} \frac{p_{n-j-1} - p_{n-j}}{P_{n-1}} \leqq \frac{A}{P_j} \quad \text{for all } j \geqq 0,$$

we have, by (8),

$$(32)\qquad V_3 \leqq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{j=1}^{n-1} (P_n p_{n-j-1} - P_{n-1} p_{n-j}) m_{j+1} |a_j| = \sum_{j=1}^{\infty} m_{j+1} |a_j| \sum_{n=j+1}^{\infty} \left(\frac{p_{n-j-1} - p_{n-j}}{P_n} + p_{n-j-1}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\right) \leqq A \sum_{j=1}^{\infty} \frac{m_j |a_j|}{P_j} < \infty.$$

We see that (cf. [5, formula (17)])

$$(33)\qquad \sum_{n=j+1}^{\infty} \frac{P_n p_{n-j} - p_n P_{n-j}}{P_n P_{n-1}} \leqq A,$$

and thus

$$(34)\qquad U_3 \leqq \sum_{n=1}^{\infty} \frac{1}{P_nP_{n-1}} \sum_{j=1}^{n-1} (P_np_{n-j} - p_nP_{n-j})\Delta m_j|a_j|$$
$$= \sum_{j=1}^{\infty} \Delta m_j|a_j| \sum_{n=j+1}^{\infty} \frac{P_np_{n-j} - p_nP_{n-j}}{P_nP_{n-1}}$$
$$< \infty.$$

By (19) and (33), we obtain

$$(35)\quad U_2 \leqq A \sum_{n=1}^{\infty} \frac{1}{P_nP_{n-1}} \sum_{j=1}^{n-1} (P_np_{n-j} - p_nP_{n-j})\Delta m_j \sum_{k=1}^{\infty} |c_k|\, |a_{|k-j|} + a_{k+j}|$$
$$\leqq A \sum_{j=1}^{\infty} \Delta m_j \sum_{n=j+1}^{\infty} \frac{P_np_{n-j} - p_nP_{n-j}}{P_nP_{n-1}}$$
$$\leqq A \sum_{j=1}^{\infty} \Delta m_j$$
$$< \infty.$$

Finally we shall estimate V_2. We put

$$V_2 \leqq A \sum_{n=1}^{\infty} \frac{1}{P_nP_{n-1}} \sum_{j=1}^{n-1} (P_np_{n-j-1} - P_{n-1}p_{n-j})m_{j+1} \sum_{k=1}^{\infty} |c_k|(|a_{|j-k|}| + |a_{j+k}|)$$
$$= A \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{j=1}^{n-1} (p_{n-j-1} - p_{n-j})m_{j+1} \sum_{k=1}^{\infty} |c_k|(|a_{|j-k|}| + |a_{j+k}|)$$
$$+ A \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \sum_{j=1}^{n-1} p_{n-j}m_{j+1} \sum_{k=1}^{\infty} |c_k|(|a_{|j-k|}| + |a_{j+k}|)$$
$$= X + Y;$$

then, by (31) and the assumption of the theorem, we obtain

$$(36)\quad X \leqq \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} (|a_{|j-k|}| + |a_{j+k}|)m_{j+1} \sum_{n=j+1}^{\infty} \frac{p_{n-j-1} - p_{n-j}}{P_{n-1}}$$
$$\leqq A \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} \frac{m_{j+1}}{P_j} (|a_{|j-k|}| + |a_{j+k}|)$$
$$\leqq A \sum_{k=1}^{\infty} |c_k| \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty} \right) \frac{m_{j+1}}{P_j} (|a_{|j-k|}| + |a_{j+k}|)$$
$$\leqq A \sum_{j=1}^{\infty} \frac{m_{j+1}}{P_j} |a_{k-j}| \sum_{k=j}^{\infty} |c_k| + A \sum_{k=1}^{\infty} |c_k| \sum_{j=k+1}^{\infty} \frac{m_{j-k}}{P_{j-k}} |a_{j-k}|$$
$$+ A \sum_{j=1}^{\infty} \frac{m_{j+1}}{P_j} |a_{j+k}| \sum_{k=j}^{\infty} |c_k| + A \sum_{k=1}^{\infty} |c_k| \sum_{j=k+1}^{\infty} \frac{m_{2j}}{P_{2j}} |a_{j+k}|$$
$$\leqq A \sum_{j=1}^{\infty} \frac{m_{j+1}}{j^2P_j} + A \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} \frac{m_j}{P_j} |a_j| + A \sum_{k=1}^{\infty} |c_k| \sum_{j=k+1}^{\infty} \frac{m_{j+k}}{P_{j+k}} |a_{j+k}|$$
$$< \infty$$

and further, by (31), we similarly have

$$(37)\qquad Y \leqq A \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} m_{j+1}(|a_{|j-k|}| + |a_{j+k}|) \sum_{n=j+1}^{\infty} \frac{p_n p_{n-j}}{P_n P_{n-1}}$$
$$\leqq A \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} \frac{m_{j+1}}{P_j} (|a_{|j-k|}| + |a_{j+k}|)$$
$$< \infty,$$

Combining (36) and (37), we see that V_2 is finite. By (32)–(35), and the finiteness of V_2, we see that Theorem 2 is proved for $\sum |R_n|$.

We shall now consider

$$\sum_{n=1}^{\infty} |T_n| = \sum_{n=1}^{\infty} \frac{m_n |s_n|}{P_n}.$$

By (17), it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{m_n}{P_n} \left| \int_0^{\pi} f(t) g(t) \sin nt\, dt \right| < \infty.$$

This follows from

$$\sum_{n=1}^{\infty} \frac{m_n}{P_n} \sum_{j=1}^{\infty} |c_j| (|a_{|j-n|}| + |a_{j+n}|) < \infty \quad \text{(by (19))},$$

estimated in the same way as (36). Hence we obtain $\sum |T_n| < \infty$. Evidently, $\sum |S_n| < \infty$. Thus the theorem is proved.

2.3. *Proof of Theorem* 3. The proof is similar to that of Theorem 2. Since (p_n) increases, we obtain by condition (9), instead of (31) (see [**8**, formula (15)]),

$$(38)\qquad \sum_{n=j+1}^{\infty} \frac{P_n p_{n-j} - p_n P_{n-j}}{P_n P_{n-1}} \leqq A \log(j+1) \qquad \text{for all } j \geqq 0.$$

We shall only estimate U_2, defined in § 2.2, since the others are quite similar, as in the proof of Theorem 2. By (10), (38), and convexity of the sequence (m_n), we have

$$U_2 \leqq A \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{j=1}^{n-1} (P_n p_{n-j} - p_n P_{n-j}) \Delta m_j \sum_{k=1}^{\infty} |c_k| (|a_{|k-j|}| + |a_{k+j}|)$$
$$\leqq A \sum_{k=1}^{\infty} |c_k| \sum_{j=1}^{\infty} \log(j+1) \Delta m_j (|a_{|k-j|}| + |a_{k+j}|)$$
$$\leqq A \sum_{k=1}^{\infty} |c_k| \left\{ \sum_{j=1}^{2k} \log(j+1) \Delta m_j |a_{|k-j|}| + \sum_{j=2k+1}^{\infty} \log(j+1) \Delta m_j |a_{j-k}| \right.$$
$$\left. + \sum_{j=1}^{k} \log(j+1) \Delta m_j |a_{k+j}| + \sum_{j=k+1}^{\infty} \log(j+1) \Delta m_j |a_{k+j}| \right\}$$
$$\leqq A \sum_{j=1}^{\infty} \log(j+1) \Delta m_j \sum_{k=\frac{1}{2}j}^{\infty} \frac{1}{k^3} + A \sum_{k=1}^{\infty} |c_k| \sum_{j=k+1}^{\infty} \log j \cdot \Delta m_j |a_j|$$
$$+ A \sum_{j=1}^{\infty} \log(j+1) \Delta m_j \sum_{k=j}^{\infty} c_k |a_{k+j}| + A \sum_{k=1}^{\infty} |c_k| \sum_{j=k+1}^{\infty} \log 2j \cdot \Delta m_{2j} |a_{2j}|$$
$$\leqq A \sum_{j=1}^{\infty} \frac{\log(j+1)}{j^2} + A \sum_{k=1}^{\infty} \frac{1}{k^3}$$
$$< \infty.$$

https://doi.org/10.4153/CJM-1970-068-6 Published online by Cambridge University Press

References

1. S. N. Bhatt, *An aspect of local property of* $|R, \log n, 1|$ *summability of Fourier series*, Tôhoku Math. J. (2) *11* (1959), 13–19.
2. ——— *An aspect of local property of absolute Nörlund summability of a Fourier series*, Proc. Nat. Inst. Sci. India Part A *28* (1962), 787–794.
3. E. C. Daniel, *The non-local character of summability* $|N, p_n|$ *of a Fourier series with factors*, Math. Z. *99* (1967), 392–399.
4. G. D. Dikshit, *Localization relating to the summability* $|R, \lambda_n, 1|$ *of Fourier series*, Indian J. Math. *17* (1965), 31–39.
5. M. Izumi and S. Izumi, *The absolute Nörlund summability factors of Fourier series* (to appear in Indian J. Math.).
6. W. Jurkat and A. Peyerimhoff, *Lokalisation bei absoluter Cesàro-Summierbarkeit von Potenzreihen und trigonometrischen Reihen*. II, Math. Z. *64* (1956), 151–158.
7. K. Matsumoto, *Local property of the summability* $|R, \lambda, 1|$, Tôhoku Math. J. (2) *8* (1956), 354–364.
8. A. Zygmund, *Trigonometric series*, 2nd ed., Vol. I (Cambridge Univ. Press, New York, 1959).

The Australian National University,
Canberra (ACT), Australia