LATTICE TETRAHEDRA

G. K. WHITE

1. Introduction. A class of problems in the geometry of numbers, for which there are but fragmentary results, may be expressed in general terms, as follows. Let S be a given point-set and let G be a given discrete point-set, both in Euclidean n-space. Suppose that Λ is a lattice which contains G but no point of S not in G. Such lattices, if they exist, will be said to be *admissible* for S with respect to G, and the general problem is to investigate their properties and, if possible, classify them. For our present purpose, and to illustrate the ideas involved, it is best to confine ourselves to two cases:

I. G consists of the vertices $O, P_1, \ldots, P_n$ of an n-dimensional simplex T;

II. G consists of O and the vertices $\pm P_1, \ldots, \pm P_n$ of an n-dimensional octahedron K,

with $S = \mathrm{C}(G)$, the closed convex cover of G, or $S = \mathrm{I}(\mathrm{C}(G))$, its interior. In these cases at least, the particular lattice with basis $\{P_1, \ldots, P_n\}$ is evidently admissible for S with respect to G. Our two cases also serve to distinguish two mutually exclusive properties of our class of lattices. The fact that the origin O is an inner point of S in II implies that $d(\Lambda)$, for any such lattice Λ, is bounded below by some positive constant depending only on G, whereas this is not true for I, as is well known. When $n = 2$, both I and II are straightforward. The problem for II has been solved for $n = 3$ and 4 as well. In **(4)**, Mordell gives simplified proofs, for the case $S = \mathrm{C}(G)$, for solutions obtained by Minkowski **(3)** for $n = 3$, and by Brünngräber **(2)** and Wolff **(5)**, for $n = 4$; while Bantegnie **(1)** supplies a solution for the more difficult problem when $S = \mathrm{I}(\mathrm{C}(G))$. In Theorem 1, I settle the question for $n = 3$ in Case I with $S = \mathrm{C}(G)$ and in the process (without undue elaboration) establish a slightly more general result. In formulating this, we shall say that a plane π for which $\dim(\Lambda \cap \pi) = 2$ is a *lattice plane* of Λ and that two lattice planes π, π' are *adjacent* if π is parallel to π' and no points of Λ lie strictly between π and π'.

THEOREM 1. ($n = 3$). *Let G be the set of vertices of a tetrahedron $T = \mathrm{C}(G)$ and let $E_i = \{\sigma_i, \sigma_i'\}$, $i = 1, 2, 3$, be the set of points belonging to a pair of opposite edges σ_i, σ_i' of T. Then Λ is admissible for*

$$S_i = T - E_i$$

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with respect to G is and only if there is a pair of opposite edges E_j and adjacent lattice planes π_j, π_j' of Λ for which

$$\sigma_j \in \pi_j, \qquad \sigma_j' \in \pi_j'.$$

We remark that since $T \supset S_i$ for each i, any lattice Λ admissible for T with respect to G has the above property $\sigma_j \in \pi_j$, $\sigma_j' \in \pi_j'$ for some j. Also, if E_i contains points of Λ other than the vertices G, then $j = i$. In Theorem 2, the essence of our result is expressed in terms of diophantine approximation.

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2. A result connected with the greatest integer function $[x]$. In the course of our proof of Theorem 1, we shall need a result (Theorem 2) of some intrinsic interest. If x is any real number, $[x]$ denotes the greatest integer $\leqslant x$, and we shall denote the fractional part of x by

$$\{x\} = x - [x].$$

THEOREM 2. *Let a, b, c, d be integers satisfying*

$$d \not\equiv 0, \quad b + c \not\equiv 0, \quad c + a \not\equiv 0, \quad a + b \not\equiv 0 \pmod{d}. \tag{1}$$

Then there is an integer u for which

$$0 < \left\{\frac{au}{d}\right\} + \left\{\frac{bu}{d}\right\} + \left\{\frac{cu}{d}\right\} \leqslant 1, \tag{2}$$

with strict inequality whenever $(a + b + c, d) = 1$.

Our proof depends on the following lemma.

LEMMA. *If d, k, t are integers satisfying*

$$1 \leqslant k < k + t \quad \textit{and} \quad 2(2k + t) < d, \tag{3}$$

then there is an integer u such that

$$\left[\frac{u(k+1)}{d}\right] = 0, \qquad 0 < u < d, \tag{4}$$

$$\left[\frac{u(k+t)}{d}\right] < \left[\frac{u(2k+t)}{d}\right], \tag{5}$$

except when $(d, k, t) = (9, 1, 2)$ or $(15, 2, 3)$.

Proof. By (3), (4), (5), it is sufficient to prove that there is an integer u with $0 < u < d$ in one of the intervals

$$I(n): \quad \frac{d}{2k+t} n \leqslant u < \frac{d}{k+t} n, \tag{6}$$

where n is an integer satisfying

(7) $$0 < n \leqslant \frac{k+t}{k+1}.$$

Note that this follows whenever the length of $I(n)$, $L(n)$ satisfies

(8) $$L(n) = \frac{dkn}{(k+t)(2k+t)} > 1 - \frac{1}{2k+t}.$$

We now express t in the form

(9) $$t = m(k+1) + r, \qquad \text{where } 0 \leqslant r \leqslant k, \quad m \geqslant 0,$$

and consider two cases according as $r > 0$ or $r = 0$.

Case 1. $0 < r \leqslant k$. Observe that

$$(k+t)(k+1)^{-1} = m + (k+r)(k+1)^{-1} \geqslant m+1,$$

and we may select $n = m + 1$ in (7). Thus the length of $I(n)$ satisfies

$$L(m+1) > 2kn(k+t)^{-1} = 2k(m+1)(m(k+1)+k+r)^{-1},$$

by (8) and our hypothesis (3) for d. Moreover,

$$2k(m+1) = (m(k+1)+k+r) + k - r + m(k-1) \geqslant m(k+1)+k+r,$$

since $k \geqslant 1$, $r \leqslant k$, and $m \geqslant 0$. Hence $L(m+1) > 1$, and $I(m+1)$ contains an integer u.

Case 2. $r = 0$. Then by (7), we may select $n = m$. Since $t \geqslant 1$ by (3), we have $m \geqslant 1$. With $n = m$,

$$L(m) > 2kn(k+t)^{-1} = 2km(k+m(k+1))^{-1} \geqslant 1$$

when $(m-1)(k-1) \geqslant 1$. Thus for $m \geqslant 2$, $k \geqslant 2$, we have $L(m) \geqslant 1$. It remains to consider the cases $k = 1$ and $m = 1$.

(i) Suppose $k = 1$. Then $t = 2m$ and by (3), $d = 4m + 4 + \epsilon$ ($\epsilon = 1, 2, \ldots$). Thus

$$\begin{aligned} L(m) - (1-(2k+t)^{-1}) &= dm(2m+1)^{-1}(2m+2)^{-1} - 1 + (2m+2)^{-1} \\ &= (4m^2+4m+\epsilon m - 4m^2 - 4m - 1)(2m+1)^{-1}(2m+2)^{-1} > 0, \end{aligned}$$

unless $\epsilon = m = 1$, when $(d, k, t) = (9, 1, 2)$.

(ii) Suppose $m = 1$. Then we can assume that $k \geqslant 2$, and we have $t = k+1$, $d = 6k + 2 + \epsilon$ ($\epsilon = 1, 2, \ldots$) by (3), so that

$$\begin{aligned} L(m) - (1-(2k+t)^{-1}) &= dk(2k+1)^{-1}(3k+1)^{-1} - 1 + (3k+1)^{-1} \\ &= (6k^2+2k+\epsilon k - 6k^2 - 3k)(2k+1)^{-1}(3k+1)^{-1} > 0, \end{aligned}$$

unless $(\epsilon - 1)k \leqslant 0$, i.e. $\epsilon = 1$, $d = 6k+3$. For this remaining case we choose $u = 5$ in (4) and (5). Then

$$\left[\frac{5(k+1)}{6k+3}\right] = 0, \quad \text{and} \quad 1 = \left[\frac{5(2k+1)}{6k+3}\right] < \left[\frac{5(3k+1)}{6k+3}\right] = 2$$

for $k \geqslant 3$. This leaves only $k = 2$, giving $(d, k, t) = (15, 2, 3)$, and completing the proof of the lemma.

3. Proof of Theorem 2. We remark initially that by our hypothesis (1), d cannot divide all of a, b, c:

$$d \nmid (a, b, c), \tag{10}$$

and there is no loss of generality in assuming that

$$\begin{cases} \text{(i)} \ d \geqslant 2,\ 0 < u < d,\ 0 \leqslant a < d,\ 0 \leqslant b < d,\ 0 \leqslant c < d, \\ \text{(ii)} \ (a, b, c, d) = 1, \end{cases} \tag{11}$$

since if $k \geqslant 2$ and $0 < u < d < kd$, then u, ka, kb, kc, kd satisfy (1) and (2) if and only if (1) and (2) hold for u, a, b, c, d. For convenience, we divide our proof into three parts:

(A) $a + b + c \equiv 1 \pmod{d}$,
(B) $(a + b + c, d) = 1$, and
(C) $(a + b + c, d) > 1$.

(A) Suppose that

$$a + b + c \equiv 1 \pmod{d}. \tag{12}$$

Then by (1) and (11) we have

$$(a - 1)(b - 1)(c - 1) \neq 0. \tag{13}$$

By (11), (12), and (13), $a + b + c = 1 + md$, where $m = 1$ or 2. Since $u(a + b + c) \equiv u \pmod{d}$, the inequality (2) holds with strict inequality for $0 < u < d$ if and only if

$$\left\{\frac{au}{d}\right\} + \left\{\frac{bu}{d}\right\} + \left\{\frac{cu}{d}\right\} = \frac{u}{d}, \tag{14}$$

i.e.

$$\left[\frac{au}{d}\right] + \left[\frac{bu}{d}\right] + \left[\frac{cu}{d}\right] = mu. \tag{15}$$

Now any integers u and v satisfy

$$\left[\frac{v(d - u)}{d}\right] + \left[\frac{vu}{d}\right] = \begin{cases} v & \text{if } d | vu, \\ v - 1 & \text{otherwise.} \end{cases} \tag{16}$$

On defining

$$\psi_m(u) = mu - \left[\frac{au}{d}\right] - \left[\frac{bu}{d}\right] - \left[\frac{cu}{d}\right], \tag{17}$$

we have

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$$(18) \qquad \psi_m(u) \geqslant mu - d^{-1}(a+b+c) = -ud^{-1} \geqslant 0 \qquad (\text{if } 0 < u < d),$$

$$(19) \qquad \psi_m(d-u) + \psi_m(u) \begin{cases} = 2 & \text{if } d \nmid au,\ d \nmid bu,\ d \nmid cu, \\ < 2 & \text{otherwise.} \end{cases}$$

The case $m = 2$ may be settled by taking $u = 1$, for then

$$\psi_2(1) = 2 - [ad^{-1}] - [bd^{-1}] - [cd^{-1}] = 2$$

by (11), and $\psi_2(d-1) = 0$, by (18) and (19), and so $u = d - 1$ is the required integer, by (15). Hence we can suppose that $m = 1, a + b + c = d+1$ and write $\psi(u)$ for $\psi_1(u)$. By (18) and (19), we see that either $\psi(u)$ or $\psi(d-u)$ vanishes for some integer u with $0 < u < d$ (in which case (15) is satisfied) unless

(20) $\quad d \nmid au,\ d \nmid bu,\ d \nmid cu$ for all integers u with $0 < u < d$,

i.e.,

$$(21) \qquad (d, a) = (d, b) = (d, c) = 1.$$

By permuting a, b, c if necessary, we may clearly assume that

$$(22) \qquad a \leqslant b \leqslant c.$$

Also, by (17), we have $\psi(2) = 2$ and $\psi(d-2) = 0$ if c is small, say $c < \frac{1}{2}d$; so we can suppose that $c > \frac{1}{2}d$ ($c = \frac{1}{2}d$ contradicting (20)). Then $b < \frac{1}{2}d$, since $a + b + c = d + 1$ and $a \geqslant 1$ by (21). Thus, assembling our inequalities, we have

$$(23) \qquad a \leqslant b < \tfrac{1}{2}d, \qquad c > \tfrac{1}{2}d.$$

In order to apply our lemma, we introduce the symbols k and t defined by

$$(24) \qquad a = k + 1, \qquad b = k + t,$$

where

$$(25) \qquad 1 \leqslant k < k + t \qquad \text{and} \qquad 2k + t < \tfrac{1}{2}d,$$

since $a \geqslant 2$ by (21) and (13), $t = b - a + 1 \geqslant 1$ by (22), and

$$2k + t = a + b - 1 = d - c < \tfrac{1}{2}d$$

by (23). Now $(d, k, t) \neq (9, 1, 2)$ or $(15, 2, 3)$ by (21) and so, using the lemma, there is an integer u satisfying (4) and (5). Whence, by (16),

$$(26) \qquad \psi(u) = 1 - \left[\frac{au}{d}\right] - \left[\frac{bu}{d}\right] + \left[\frac{(d-c)u}{d}\right] = 1 - \left[\frac{u(k+1)}{d}\right] - \left[\frac{u(k+t)}{d}\right] + \left[\frac{u(2k+t)}{d}\right] \geqslant 2,$$

and so $\psi(d-u) = 0$, by (19). Thus Theorem 2 holds in case (A).

(B) Suppose $a + b + c \equiv s \pmod{d}$, where $(s, d) = 1$. Then there exists an integer s' such that $ss' \equiv 1 \pmod{d}$ and if we set $a' = s'a$, $b' = s'b$, $c' = s'c$, we have $a' + b' + c' \equiv 1 \pmod{d}$. Clearly $(s', d) = 1$, so the conditions in (1) are equivalent to

$$b' + c' \equiv 0,\ c' + a' \equiv 0,\ a' + b' \equiv 0 \pmod{d}.$$

Thus there exists an integer $u' \not\equiv 0 \pmod{d}$ for which

$$0 < \left\{\frac{a'u'}{d}\right\} + \left\{\frac{b'u'}{d}\right\} + \left\{\frac{c'u'}{d}\right\} < 1, \tag{27}$$

by part (A) of our proof. Set $u \equiv s'u' \pmod{d}$, $0 \leqslant u < d$. Then $0 < u < d$ and $a'u' = s'au' = au$, $b'u' = bu$, $c'u' = cu$, so, by (27), u is our required integer.

(C) Finally, suppose $a + b + c \equiv s \pmod{d}$ where $(s, d) = e > 1$. Let $s = s'e$, $d = d'e$, so that $(s', d') = 1$, $0 < d' < d$. Then $d's \equiv 0 \pmod{d}$, but not all of $d'a$, $d'b$, $d'c$ are divisible by d, since otherwise a, b, c, d have a common factor $e > 1$, contrary to (11). Thus if we define

$$\theta(x) = \{xad^{-1}\} + \{xbd^{-1}\} + \{xcd^{-1}\},$$

we have $\theta(d') > 0$. Also, $\theta(d - d') > 0$ and so $\theta(d')$ and $\theta(d - d')$ are positive integers whose sum is at most 3, since $\{x\} + \{-x\} \leqslant 1$. Thus either $\theta(d') = 1$ or $\theta(d - d') = 1$, and one of d' and $d - d'$ must be the desired integer u. This completes our proof of Theorem 2.

4. Proof of Theorem 1.

Case 1. Suppose that Λ is admissible for T with respect to G and choose the co-ordinate system so that the vertices G of T are

$$O = (0, 0, 0), \quad P = (1, 0, 0), \quad Q = (0, 1, 0), \quad R = (0, 0, 1). \tag{28}$$

Then we can choose a point

$$S = (\alpha, \beta, \delta) \qquad \text{with} \qquad \delta > 0 \tag{29}$$

of Λ such that P, Q, S form a basis for Λ. We thus induce a second co-ordinate system $[u, v, w]$ in which

$$(x, y, z) = xP + yQ + zR = uP + vQ + wS = [u, v, w], \tag{30}$$

and for which $\Lambda = \Lambda_0 = \{[u, v, w]; u, v, w = 0, \pm 1, \pm 2, \ldots\}$ and

$$P = [1, 0, 0], \quad Q = [0, 1, 0], \quad R = [a, b, d], \tag{31}$$

where a, b, d are integers satisfying

$$\alpha d + a = 0, \qquad \beta d + b = 0, \qquad \delta d = 1,$$

by (30). Hence

$$d \geqslant 1, \quad \text{and} \quad \delta = d^{-1}, \qquad \beta = -bd^{-1}, \qquad \alpha = -ad^{-1} \tag{32}$$

and any point (x, y, z) of Λ satisfies

(33) $x = u - \dfrac{wa}{d}, \quad y = v - \dfrac{wb}{d}, \quad z = \dfrac{w}{d}, \qquad u, v, w = 0, \pm 1, \pm 2, \ldots .$

Note that the parallel lattice planes $x + y = 0$ and $x + y = 1$ passing through OR and PQ satisfy $u + v - d^{-1}(a + b) = 0, 1$ respectively, and thus are adjacent whenever $a + b \equiv 0 \pmod d$. Similarly, the planes $x + z = 0$ and $x + z = 1$ through OQ and PR are adjacent whenever $a \equiv 1 \pmod d$, and the planes $y + z = 0$ and $y + z = 1$ through the edges OP and QR are adjacent whenever $b \equiv 1 \pmod d$. In the case $d = 1$ (where T is clearly a basis for Λ), all three conditions

$$a \equiv 1, \qquad b \equiv 1, \qquad a + b \equiv 0 \pmod d,$$

are trivially satisfied, and so all three pairs of opposite edges lie on adjacent lattice planes. Now suppose, if possible, that

(34) $$a \not\equiv 1, \qquad b \not\equiv 1, \qquad a + b \not\equiv 0 \pmod d;$$

we shall establish our result by deducing a contradiction. By (33), the set

$$T^* = T - \{O, P, Q, R\} = \{(x, y, z) : \\ 0 \leqslant x < 1,\ 0 \leqslant y < 1,\ 0 \leqslant z < 1,\ 0 < x + y + z \leqslant 1\}$$

will contain a point of Λ if and only if there exists an integer w for which

(35) $$0 < w < d, \qquad 0 < \left\{-\frac{aw}{d}\right\} + \left\{-\frac{bw}{d}\right\} + \frac{w}{d} \leqslant 1$$

since for given w, $x = \{-awd^{-1}\}$, $y = \{-bwd^{-1}\}$ are the only possible choices in T^*, and $w = 0$ implies that $x + y + z = 0$. By Theorem 2, such an integer indeed exists when (34) holds, and consequently Λ is not admissible, contrary to hypothesis. Thus for admissible Λ (with respect to the vertices), either

(36) $$a \equiv 1 \quad \text{or} \quad b \equiv 1 \quad \text{or} \quad a + b \equiv 0 \pmod d,$$

and we have our result. The converse is now trivial.

Case 2. Suppose that Λ is not admissible for T with respect to G, and that all points of Λ in T lie on the two opposite edges $\sigma_i = P'P$, $\sigma_i' = QQ'$, say. We prove that $P'P$, QQ' lie on adjacent lattice planes. We may assume, without loss of generality, that $P'P$ contains a point R' of Λ, with $R' \neq P'$, $R' \neq P$. Take R' to be the nearest such point to P and set $R' = O$. Let R be the point of Λ on QQ' closest to Q. Then Λ is clearly admissible for $OPQR$ with respect to its vertices, and we may assign co-ordinates

$$(x, y, z) = xP + yQ + zR, \qquad [u, v, w] = uP + vQ + wS$$

as in (28) to (33) of Case 1. Since $O = R' \neq P'$, we have $P'' = [-1, 0, 0] \in \Lambda$ and, from our hypothesis, Λ is admissible for $OP''QR$ with respect to $\{O, P'', Q, R\}$, So, as in (36) of Case 1, applied now to $OPQR$ and $OP''QR$, we have

$$(37) \qquad \begin{cases} a \equiv 1 \text{ or } b \equiv 1 \text{ or } \quad a + b \equiv 0 \pmod{d}, \text{ and} \\ -a \equiv 1 \text{ or } b \equiv 1 \text{ or } -a + b \equiv 0 \pmod{d}, \end{cases}$$

where $P = [1, 0, 0]$, $P'' = [-1, 0, 0]$, $Q = [0, 1, 0]$, $R = [a, b, d]$. Note that if $b \equiv 1 \pmod{d}$, the planes $y + z = 0$ and $y + z = 1$ through OP and QR are adjacent, and since $OP \subset P'P$, $QR \subset QQ'$, our result follows. We thus conclude our proof of Theorem 1 by showing that (37) implies that $b \equiv 1 \pmod{d}$. Firstly, if either $a \equiv 1$ and $-a + b \equiv 0$, or $-a \equiv 1$ and $a + b \equiv 0$, then $b \equiv 1 \pmod{d}$. Next, if $a \equiv 1$ and $a \equiv -1$, then $d = 2$, $b \equiv 1 \pmod{d}$ (if $b \equiv 0$, $\frac{1}{2}(P + R) \in OPQR \cap \Lambda$). Finally, if $a + b \equiv -a + b \equiv 0$, we have $2a \equiv 0 \pmod{d}$. If $d = 2e$ is even, then e divides a, b, and d, so that

$$e^{-1}R \in OPQR \cap \Lambda.$$

Hence $e = 1$, $d = 2$, and $b \equiv 1 \pmod{d}$ as before. If d is odd, then $a \equiv b \equiv 0 \pmod{d}$, $d^{-1}R \in \Lambda$, and so $d = 1$. But when $d = 1$, it is trivial that $b \equiv 1 \pmod{d}$. This completes our proof of Theorem 1.

5. Remark. There is a strikingly simple relation between the problems for I and II, when $n = 3$. If T is the tetrahedron with vertices at O, P_1, P_2, P_3 and K is the corresponding octahedron with vertices $\pm P_1$, $\pm P_2$, $\pm P_3$, then Λ is admissible for K with respect to $G = \{O, \pm P_1, \pm P_2, \pm P_3\}$ if and only if *all* three pairs of opposite edges of T belong to adjacent lattice planes of Λ.

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University of Toronto and
University of British Columbia