TOPOLOGY OF THE NEGATIVE ENERGY-MANIFOLD OF THE KEPLER MOTION

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1. Introduction

The motion of a particle of mass m according to the central force of Newton, is denoted by

$$m \frac{dx}{dt} = y, \qquad \frac{dy}{dt} = - \frac{K}{|x|^3} x, \tag{1}$$

where K is a constant. x=0 corresponds to singular points of this equations. The domain of (1), denoted by $\mathfrak{X}=(R^3-\{0\})\times R^3$, is called the phase space of the Kepler motion. In the sequel we set m=K=1 for simplicity and also transform the independent variable from t to s by $dt = |x|\,ds$ $(x\neq 0)$, then the Kepler motion in the phase space $\mathfrak{X}$ is written as

$$\frac{dx}{ds} = |x|\,y, \qquad \frac{dy}{ds} = - \frac{x}{|x|^2}. \tag{2}$$

Further, we shall confine the following discussion to the case of the negative energy value, except the preliminary discussion.

Moser [1] investigated the Kepler motion, describing the problem in an n-dimensional space, and showed that: (i) The energy surface with a negative constant is homeomorphic to the unit tangent bundle of the n-sphere S^n, punctured at one point which corresponds to the collision states. (In particular for n=2, the unit tangent bundle of the sphere S^2 is homeomorphic to the 3-dimensional projective space P^3.) (ii) The orbit space of the Kepler flow on an energy surface is homeomorphic to S^2 for n=2, and to $S^2\times S^2$ for n=3.

In the investigation of Moser, however, the problem of the reconstruction of the energy-manifold on the basis of the orbit space was not treated. The goal of this note is to fill up this gap. To this end, we consider first the Kepler flow on the Kustaanheimo-Stiefel's parametric space, by making use of theKustaanheimo-Stiefel's transformation (abbreviated as KS-map in the sequel) (Kustaanheimo-Stiefel [2]), and discuss the topology of the relevant energy-manifold, and then pull back the results to the phase space.

R. L. Duncombe (ed.), Dynamics of the Solar System, 45–48.

https://doi.org/10.1017/S0074180900012481 Published online by Cambridge University Press

2. Preliminaries

We collect here the important properties with respect to the KS-map (cf. Stiefel-Scheifele [3]), and define the notions convenient in what follows.

Def. 1. If we set: $i=\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$, $j=\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$, $k=\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)$,

and construct iu, ju, ku from a 4-vector $u=(u_1, u_2, u_3, u_4)'$, where prime denotes "transposed", and if any point of R^3 is automatically supplemented to a point of R^4 with fourth-component of value zero, then the KS-map $T_1: R^4 \to R^3$ is defined by $u \mapsto x = L(u)u$, where $L(u)=(u, iu, ju, ku)$.

Def. 2. For any $u \in R^4$ $(u \neq 0)$, $\mathcal{L}_u$ is defined by $\mathcal{L}_u = \{v \in R^4 \mid L(u)v = L(v)u\}$.

Def. 3. For any $u \in R^4$ $(u \neq 0)$, a map $T_u: \mathcal{L}_u \to R^3$ is defined by $v \mapsto y = (2/|u|^2)L(u)v$.

Def. 4. In $R^4 \times R^4$ we call a set $\mathcal{P} = \{(u,v) \mid u \in (R^4 - \{0\}),\ v \in \mathcal{L}_u\}$ the KS-space of Kustaanheimo-Stiefel.

Def. 5. We define a C^∞-surjection $T: \mathcal{P} \to \mathfrak{X}$; $(u,v) \mapsto (x,y)$ by $x = T_1(u) = L(u)u$, $y = T_u(v) = (2/|u|^2)L(u)v$. We call T the enlarged KS-map.

Lemma 1. For any $u, \bar{u} \in R^4$ $(u, \bar{u} \neq 0)$ and $v \in \mathcal{L}_u$, $\bar{v} \in \mathcal{L}_{\bar{u}}$, a necessary and sufficient condition for $L(u)u = L(\bar{u})\bar{u}$, and $T_u(v) = T_{\bar{u}}(\bar{v})$ is that there exists $\chi \pmod{2\pi}$ such that $\bar{u} = \exp(-\chi k)u$, $\bar{v} = \exp(-\chi k)v$.

Lemma 2. The system of differential equations on $\mathcal{P}$:

$$\frac{du}{ds} = v, \qquad \frac{dv}{ds} = \frac{1}{|u|^2}\left[|v|^2 - \frac{1}{2}\right]u \tag{3}$$

transforms to the Kepler motion (2) on the phase space $\mathfrak{X}$ by the enlarged KS-map T.

Lemma 3. The system of differential equations (3) on the KS-space has the following integrals:

(i) $L(u)kv - L(v)ku = C$, $\quad L(u)v - L(v)u = C_0$, $\quad L(u)iv - L(v)iu = C_1$, $L(u)jv - L(v)ju = C_2$.

The first one corresponds to the integral of angular momentum and the constant C_0 of the second one must be zero for the consistency of the theory (cf. Def. 4 and 2).

(ii) $\dfrac{2}{|u|^2}\left[|v|^2 - \dfrac{1}{2}\right] = h$; this corresponds to the integral of energy.

Taking the energy-integral into account, the system of differential equations (3) reduces to the following system:

$$\frac{du}{ds} = v, \qquad \frac{dv}{ds} = \frac{h}{2}u, \tag{4}$$

on each energy-manifold P_h^* in KS-space, where

$$P_h^* = \{(u,v)\ \ u \neq 0,\ v \in \mathcal{L}_u,\ \frac{2}{|u|^2}\left[|v|^2 - \frac{1}{2}\right] = h\}. \tag{5}$$

Since, for a given constant h, the functions on the right-hand sides of (4) have no singular points on $R^4 \times R^4$, we shall extend its domain from

(5) to

$$P_h = \{(u,v)\,|\,2|v|^2 - h|u|^2 = 1, \text{ and } v\in\mathcal{L}_u \text{ if } u\neq 0\}. \tag{6}$$

Hence P_h is a union of P_h^* and P_{col}, which is defined as:

$$P_{col} = \{(u,v)\,|\,u=0,\ |v|^2 = \tfrac{1}{2}\}. \tag{7}$$

Def. 6. Any two points (u,v) and $(\bar{u},\bar{v})$ on P_h are call orbit-equivalent each other, if they are on a same solution curve of the flow (4) on P_h, and are called KS-equivalent each other, if there exists such χ(mod. 2π) that $\bar{u}=\exp(-\chi k)u$, $\bar{v}=\exp(-\chi k)v$.

3. Results

In what follows we shall restrict ourselves to a discussion of a negative energy-manifold with an energy constant $h = -2\omega^2$. And we here introduce an important diffeomorphism H on $R^4\times R^4$ after Stiefel-Scheifele ([3], § 45) by

$$u = \tfrac{1}{2}(\xi+\eta), \qquad v = -\tfrac{1}{2}\omega k(\xi-\eta). \tag{8}$$

By the transformation (8), the Kepler flow (4) on P_h is transformed into

$$\frac{d\xi}{ds} = -\,\omega k\xi, \qquad \frac{d\eta}{ds} = \omega k\eta. \tag{9}$$

In the (ξ,η)-space we can easily consider the topological structure of the corresponding energy-manifold induced by the diffeomorphism H. In the sequel we will state the main results without proof.

We define a C^∞-surjection Π by $(\xi,\eta)\to\Pi(\xi,\eta)=(L(\xi)\xi, L(\eta)\eta)$ in (ξ,η)-space, which defines a C^∞-projection of $S^3\times S^3$ onto $S^2\times S^2$. With this Π we obtain the following:

Prop. 1. P_h is diffeomorphic to $S^3\times S^3$.

Prop. 2. $S^3\times S^3$ is a bundle space of a fibre bundle with base space $S^2\times S^2$, fibre $S^1\times S^1$ and structure group $SO(2)\times SO(2)$.

Let us consider a quotient space of $S^3\times S^3$ by the KS-equivalence relation, induced in (ξ,η)-space by the same way as § 2, Def. 6. We denote it by $(S^3\times S^3)/KS\!\sim$, and let Υ be a canonical projection of $S^3\times S^3$ onto $(S^3\times S^3)/KS\!\sim$. Since Π is compatible with the KS-equivalence relation, there exists uniquely a C^∞-surjection Π^* of $(S^3\times S^3)/KS\!\sim$ onto $S^2\times S^2$ such that $\Pi = \Pi^*\circ\Upsilon$. By the projection Π any points which are orbit- or KS-equivalent each other, are mapped onto a same point. The set of equivalence classes $S^2\times S^2$ corresponds to the orbit space of the Kepler motion.

Prop. 3. $((S^3\times S^3)/KS\!\sim,\ \Pi^*,\ S^2\times S^2,\ S^1,\ SO(2))$ is a fibre bundle.

On the basis of Prop. 2 and 3 we obtain the following theorems.

Theo. 1. P_h is diffeomorphic to a bundle space $S^3\times S^3$ of a fibre bundle: $(S^3\times S^3,\ \Pi,\ S^2\times S^2,\ S^1\times S^1,\ SO(2)\times SO(2))$.

Theo. 2. $P_h/KS\sim$ is diffeomorphic to a bundle space $(S^3 \times S^3)/KS\sim$ of a fibre bundle: $((S^3 \times S^3)/KS\sim, \Pi^*, S^2 \times S^2, S^1, SO(2))$.

Let $P_{h,c}$ be a set of all (u,v) with energy $h=-2\omega^2$ and with angular momentum C:

$$P_{h,c} = \{(u,v) \,|\, L(u)kv - L(v)ku = C,\ 2|v|^2 + 2\omega^2 |u|^2 = 1, \text{ and } v \in \mathcal{L}_u \text{ if } u \neq 0\}.$$

Theo. 3. If $|C| \neq 1/2\omega, \neq 0$, $P_{h,c}/KS\sim$ is diffeomorphic to $S^1 \times S^1$, if $|C| = 1/2\omega$, $P_{h,c}/KS\sim$ is diffeomorphic to S^1, and if C=0, $P_{h,c}/KS\sim$ is diffeomorphic to $S^2 \times S^1$.

Theo. 4. The energy-manifold $X_h \subset \mathfrak{X}$ of the Kepler motion is homeomorphic to a subset $(P_h \setminus P_{col})/KS\sim$ of $P_h/KS\sim$.

In a planar case, $P_h/KS\sim$ is diffeomorphic to a projective space P^3 and Theo. 2 should be rewritten as:

Theo. 2'. In the planar case $P_h/KS\sim$ is diffeomorphic to a bundle space P^3 of a fibre bundle with base space S^2, fibre S^1 and structure group SO(2).

References

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