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SOME RESULTS ON VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS IN THE UNIT DISK

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1. Let C and D be the unit circle and the open unit disk respectively. We shall use $\rho(z,z')$ to represent the non-Euclidean distance [3, p. 263] between the two points z and z' in D, and $X(w,w')$ to represent the chordal distance between the two points w and w' on the Riemann Sphere Ω. If $z' \in D$, $r>0$, $D(z',r)=\{z: \rho(z,z')<r\}$. We denote by $R(\theta)$ the radius at $e^{i\theta}$ in D and $R(\theta,\phi)$, $-\pi/2<\phi<\pi/2$, the chord at $e^{i\theta}$ making an angle ϕ with the radius $R(\theta)$. If $f(z)$ is a meromorphic function in D, $F(f)$ and $F(f,w)$ will represent the set of all Fatou points [3, p. 264] of $f(z)$ and the set of all Fatou points of $f(z)$ for which the corresponding Fatou values [3, p. 264] are w respectively. A sequence of points $\{z_n\}$ in D, tending to C, is said to be a sequence of p-points of $f(z)$ (see Gavrilov [7] and Gauthier [6]) if for each $r>0$ and each subsequence $\{z_m\}$ of $\{z_n\}$, $f(z)$ assumes every value on Ω, infinitely often, perhaps except two, in the set $\bigcup_{m=1}^{\infty} D(z_m,r)$. Finally, we denote by $V(S,\bar{\mu}\leq M)$, $0\leq M\leq\infty$, the classes of holomorphic functions unbounded in D but bounded on a monotone spiral S with $\bar{\mu}S\leq M$ [13, p. 160].

Remark 1. If $f(z)$ belongs to the class $V(S,\bar{\mu}\leq M)$, then there exists [15, p. 431] a spiral S' in D such that $f(z)$ tends to infinity on S' as $|z|\to 1$. Obviously, $\bar{\mu}S'\leq M$.

The notion of a normal meromorphic function in D was first defined by Noshiro [11, p. 149]. This idea was first formulated by Yosida [16, p. 227]. The latter defined the class of normal meromorphic functions in the finite

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complex plane. Lehto and Virtanen [9, p. 53], later in 1957, defined the notion of a normal function in a domain G, which is similar to that given by Yosida when G is the finite complex plane, and the same as that given by Noshiro when $G = D$.

Noshiro [11, p. 154] decomposed the class of normal meromorphic functions into two categories, the classes of normal meromorphic functions of the first kind and those of the second. (See also [2, §2, p. 15].) He obtained in his paper [11, p. 155] the following remarkable result: If $f(z)$ is a normal meromorphic function of the first kind in D, then there exists a finite positive number r_0 such that for each z in D, $f(z)$ assumes every value on Ω, at least once, in the set $D(z, r_0)$.

In the second section of this paper, we shall generalize the above idea of Noshiro's in decomposing the set of all meromorphic functions in D into two categories, the classes of meromorphic functions of the first kind and those of the second kind. Roughly speaking, meromorphic functions of the second kind behave mildly, while those of the first kind are comparatively wild in the sense that the values which they assume do not attain any limit on sequences of non-Euclidean disks of fixed non-Euclidean diameter in D tending to C. We shall obtain, in this section, some necessary and sufficient conditions for a meromorphic function in D to be of the first kind.

Section three deals with value distribution and boundary behavior of meromorphic functions of the first kind in D. Among other theorems, we shall prove a theorem similar to that of Noshiro [11, p.155, Theorem 7].

In section four, we generalize the notion of sequences of p-points to that of sequences of pseudo-p-points. We shall prove, among many other theorems, two identity theorems with this new definition.

Finally, in section five, we obtain two Plessner type theorems, one for Tsuji functions and the other for Tsuji functions which are also of the first kind.

2. Noshiro [11, p. 154] gave the following definition:
A normal meromorphic function $f(z)$ in D is said to be of the first kind if the family of functions $\{f \circ S(z)\}$ admits no constant limit in D, where $S(z)$ represents any 1—1 conformal mapping of D onto itself. A normal function in D which is not of the first kind is said to be of the second kind.

We shall generalize this notion to general meromorphic functions.

DEFINITION 1. A meromorphic function $f(z)$ in D is said to be of the second kind if there exists a sequence of points $\{z_n\}$ in D, $|z_n| \to 1$, such that the sequence

$$\left\{ f_n(z) = f\left(\frac{z_n - z}{1 - \bar{z}_n z} \right) \right\}, \tag{1}$$

tends uniformly to a constant limit in some closed neighborhood of $z = 0$.

A meromorphic function $f(z)$ in D is said to be of the first kind if it is not of the second kind.

Remark 2. Since $f(z)$ is merely assumed to be meromorphic in D, for any arbitrary sequence of points $\{z_n\}$ in D tending to C, the sequence of functions (1), unlike the case when $f(z)$ is normal in D, may not converge in any neighborhood of $z = 0$.

It is obvious that every normal function of the first kind in D is a meromorphic function of the first kind. By a result of Bagemihl [1, Theorem 1, p. 3], no normal holomorphic function in D is of the first kind. But there are surely examples of holomorphic functions of the first kind. In fact, every function which belongs to the class $V(S, \bar{\mu} = 0)$ is a function of the first kind. Because for every sequence of non-Euclidean disks $\{D_n\}$ in D, of fixed non-Euclidean diameter, tending to C, for sufficiently large n, D_n would intersect both S and S' (see Remark 1). Since $f(z)$ is bounded on S and tends to infinity on S', it can not tend to any limit in $\bigcup_{n=1}^{\infty} D_n$, as $|z| \to 1$, i.e. the sequence of functions (1), where z_n is the non-Euclidean center of D_n, for $n = 1, 2, \cdots$, does not tend to a constant limit in any neighborhood of $z = 0$. In other words, $f(z)$ is of the first kind.

THEOREM 1. *A meromorphic function $f(z)$ in D is of the first kind if, and only if, for each $r_0 > 0$*

$$\liminf_{\substack{|a| \to 1 \\ |a| < 1}} \left\{ \iint_{\rho(z,0) < r_0} \frac{|f_a'(z)|^2}{(1 + |f_a(z)|^2)^2} r \, dr \, d\theta \right\} = \delta_1(r_0) > 0, \tag{2}$$

where $f_a(z) = f\left(\dfrac{a - z}{1 - \bar{a} z} \right)$.

Proof. The condition is sufficient, for if $f(z)$ belonged to the second kind, then there exists a sequence of points $\{z_n\}$ in D, $|z_n| \to 1$, such that the sequence of functions (1) tends to a constant limit in each compact sub-

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set of a neighborhood $D(0, r_1)$ of $z = 0$. It is easy to see that the sequence of real-valued functions

$$\left\{\frac{|f_n'(z)|}{1+|f_n(z)|^2}\right\}, \tag{3}$$

tends to zero there too. This implies that for all $0 < r < r_1$,

$$\lim \iint_{\rho(z,0)<r} \frac{|f_n'(z)|^2}{(1+|f_n(z)|^2)^2} r\, dr\, d\theta = 0. \tag{4}$$

This contradicts (2).

The condition is also necessary. Indeed, suppose on the contrary, that $f(z)$ is of the first kind and that there exists $r > 0$, with a sequence of points $\{z_n\}$ in D, $|z_n| \to 1$, such that the sequence of functions (1) has the property that the equation (4) holds. We shall show that $\{z_n\}$ is a sequence of p-points of $f(z)$. For if this were not the case, then there exists a finite positive number t such that $f(z)$ omits three distinct values in the set $\bigcup_{m=1}^{\infty} D(z_m, t)$, where $\{z_m\}$ is an infinite subsequence of $\{z_n\}$. Hence, the sequence of functions

$$\left\{f_m(z) = f\left(\frac{z_m - z}{1 - \bar{z}_m z}\right)\right\}, \tag{5}$$

omits three distinct values in the set $D(0, t)$, and thus forms a normal family there. This implies that the sequence of functions (5) possesses an infinite subsequence $\{f_p(z)\}$ such that $\lim f_p(z) = g(z)$, where $g(z)$ is a meromorphic function in $D(0, t)$. Since we supposed that $f(z)$ is of the first kind, $g(z) \not\equiv$ constant. On the other hand,

$$\lim \frac{|f_p'(z)|}{1+|f_p(z)|^2} = \frac{|g'(z)|}{1+|g(z)|^2}, \tag{6}$$

uniformly in each compact subset of $D(0, t)$, so that

$$\lim \iint_{\rho(0,z)<r} \frac{|f_p'(z)|^2}{(1+|f_p(z)|^2)^2} r\, dr\, d\theta = \iint_{\rho(0,z)<r} \frac{|g'(z)|^2}{(1+|g(z)|^2)^2} r\, dr\, d\theta, \tag{7}$$

$=$ Area of image on Ω of the set $\{z\colon \rho(z,0) < r\}$ under the mapping $g(z)$ (which is positive). This contradicts (4). Thus $\{z_n\}$ should be a sequence of p-points of $f(z)$. However, if this were true, then (4) can not hold again. This contradiction completes our proof.

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THEOREM 2. *A meromorphic function $f(z)$ in D is of the first kind if, and only if, for each $r>0$,*

$$\liminf_{\substack{|a|\to 1\\ |a|<1}} \left\{ \max_{\rho(z,a)<r} (1-|z|^2) \frac{|f'(z)|}{1+|f(z)|^2} \right\} = \delta_2(r) > 0. \tag{8}$$

Proof. By a similar argument as we employed in Theorem 1, we see that $f(z)$ is of the second kind if, and only if, there exists a sequence of points $\{z_n\}$, $|z_n| \to 1$, such that the sequence of functions (1) would tend to a constant limit in some neighborhood $D(0, r_0)$ of $z=0$, for some $r_0>0$. Moreover, the sequence of real-valued functions (3) also tend to zero there. Thus, if $r<r_0$, for any sequence of points $\{w_n\}$ in D, $w_n \in D(z_n, r/2)$, for $n=1, 2, \cdots$, there exists $t>0$, such that $D(w_n, t) \subseteq D(z_n, r)$, for $n=1, 2, \cdots$.

Consider the sequence of functions

$$\left\{ g_n(z) = f\left(\frac{w_n - z}{1-\bar{w}_n z}\right) \right\}. \tag{9}$$

Using the same argument as above, we have that the sequence of real-valued functions

$$\left\{ \frac{|g_n'(z)|}{1+|g_n(z)|^2} \right\}, \tag{10}$$

tends to zero uniformly in each compact subset of $D(0, t)$. An easy calculation shows that for $n=1, 2, \cdots$,

$$(1-|z|^2)\frac{|g_n'(z)|}{1+|g_n(z)|^2} = (1-|w|^2)\frac{|f_n'(w)|}{1+|f_n(w)|^2}. \tag{11}$$

(e.g. see [9, Theorem 2]), where $z = \dfrac{w_n - w}{1-\bar{w}_n w}$. Thus, we have

$$\lim (1-|w_n|^2)\frac{|f_n'(w_n)|}{1+|f_n(w_n)|^2} = \lim \frac{|g_n'(0)|}{1+|g_n(0)|^2} = 0. \quad \text{i.e.,}$$

$f(z)$ is of the second kind if, and only if,

$$\liminf_{\substack{|z|\to 1\\ |z|<1}} \left\{ \max_{\rho(z,a)<r} (1-|z|^2) \frac{|f'(z)|}{1+|f(z)|^2} \right\} = 0, \tag{12}$$

for some $r>0$. This is equivalent to the conclusion of our theorem.

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COROLLARY 2. 1. *If $f(z)$ is a meromorphic function in D, and tends to a limit w as $|z| \to 1$, in the set $D_0 = \bigcup_{n=1}^{\infty} D(z_n, r)$, $r > 0$, where $\{z_n\}$ is a sequence of points in D with $|z_n| \to 1$, then*

$$\lim_{\substack{z \in D_0 \\ |z| \to 1}} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} = 0. \tag{13}$$

Remark 3. The above corollary generalizes a classic result that if the limit w in the above theorem is a finite complex number, then we have

$$\lim_{\substack{z \in D_0 \\ |z| \to 1}} (1 - |z|^2) |f'(z)| = 0. \tag{14}$$

3. In this section, we will discuss the value distribution and boundary behavior of meromorphic functions of the first kind.

THEOREM 3. *Suppose that $f(z)$ is a meromorphic function in D belonging to the first kind and that B is a boundary path in D for which $\lim_{\substack{|z| \to 1 \\ z \in B}} f(z) = a$, where a is finite or infinite. If $\{z_n\}$ is a sequence of points in D, $|z_n| \to 1$, such that $\lim \rho(z_n, B) \leq M$, where M is a finite positive number, then for each $M' > M$, $f(z)$ assumes every value on Ω infinitely often, except perhaps two, in the union of the non-Euclidean disks $\bigcup_{n=1}^{\infty} D(z_n, M')$.*

Proof. Suppose, on the contrary, that $f(z)$ assumes some three distinct values in $\bigcup_{n=1}^{\infty} D(z_n, M')$ at most a finite number of times. We may, indeed, without loss of generality, assume that $f(z)$ omits the three values there.

We shall use arguments similar to those of [9, pp. 52-53] and [3, Theorem 3, p. 266]. Consider the sequence of functions (1) corresponding to the above given sequence of points $\{z_n\}$. It is easy to see that (1) forms a normal family in $D(0, M')$. Thus, there exists a subsequence of function $\{f_m(z)\}$ of (1) such that $\{f_m(z)\}$ tends to a meromorphic function $g(z)$, uniformly on each compact subset of $D(0, M')$. Let $B_m = B \cap D(z_m, M')$, for $m = 1, 2, \cdots$. Then for sufficiently large m, $B_m \neq \phi$, because $\lim \rho(z, B) \leq M < M'$. Let also $B_n^* = S_n(B_n) \subseteq D(0, M')$, for $n = 1, 2, \cdots$, where

$$S_n(z) = \frac{z_n - z}{1 - \bar{z}_n z}. \tag{15}$$

It is now easy to see (e.g. see [9, pp. 52-53] and [3, p. 266]) that

$B^* = \limsup B_m^*$ is a non-degenerated continuum (a connected compact set consisting of more than one point) in $D(0, M')$. Thus for each $t \in B^*$, there exists an infinite sequence of points $\{t_{m_k}\}$, $t_{m_k} \in B_{m_k}^*$, for $k = 1, 2, \cdots$, such that $t_{m_k} \to t$ as $k \to \infty$. Now if we let $t_n' = S_n(t_n)$, then $t_n' \in B_n^*$, and $\lim |t_n'| = 1$. Thus for each $t \in B^*$, $g(t) = \lim f_{m_k}(t_{m_k}) = \lim f(t_{m_k}') = a$, i.e. $g(z) \equiv a$. In other words, the sequence of functions $\{f_m(z)\}$ tends to the constant a uniformly in each compact subset of $D(0, M')$, which contradicts our assumption that $f(z)$ belongs to the first kind.

COROLLARY 3. 1. *If $f(z)$ is a meromorphic function of the first kind in D and B is any boundary path in D such that $\lim_{\substack{|z|\to 1 \\ z\in B}} f(z) = a$, where a is finite or infinite, then B is a strong-p-path of $f(z)$.*†

THEOREM 4. *Suppose that $f(z)$ is meromorphic and is of the first kind in D, $\{z_n\}$ is a sequence of points in D, $|z_n| \to 1$, and $\{r_n\}$ any sequence of positive numbers such that $r_n \uparrow \infty$. If A_n denotes the set of values that $f(z)$ omits in the set $D(z_n, r_n)$, for $n = 1, 2, \cdots$, then for any infinite subsequence $\{A_m\}$ of $\{A_n\}$, $\liminf A_m$*†† *consists of at most two points.*

Proof. Suppose, on the contrary, that there exists an infinite subsequence $\{A_m\}$ of $\{A_n\}$, such that $\liminf A_m$ consists of three distinct values w_1, w_2, and w_3. Then there exists a subsequence of sets $\{A_k\}$ of $\{A_m\}$ such that $\lim A_k \supseteq \{w_1, w_2, w_3\}$, and in fact, without loss of generality, we may assume for each $k = 1, 2, 3, \cdots$, that A_k is a set consisting of at least three points. We shall refer to the sequence of sets $\{A_k\}$ as $\{A_n\}$ again.

Consider the sequence of functions (9) corresponding to the above sequence of points $\{z_n\}$. Then $f(D(z_n, r_n)) = f_n(D(0, r_n))$. We shall first show that there exists a subsequence $\{f_k(z)\}$ of $\{f_n(z)\}$ such that $\{f_k(z)\}$ tends to a meromorphic function $g(z)$, uniformly on each compact subset of D. Moreover $g(z)$ omits the three values w_1, w_2, w_3 in D.

Since $r_n \uparrow \infty$, for $n = 1, 2, \cdots$, the function $f_n(z)$ omits the set of values A_n in $D(0, r_1)$. Since $\lim A_n \supseteq \{w_1, w_2, w_3\}$, by a theorem of Montel [10, p. 73, §38], $\{f_n(z)\}$ forms a normal family in $D(0, r_1)$. Thus there exists a subsequence of functions $\{f_{n(1)}(z)\}$ of $\{f_n(z)\}$, where $n(1) \geq n$, for $n = 1, 2,$

† A boundary path in D is said to be a strong-p-path of a meromorphic function $f(z)$ if every sequence of points on it tending to C is a sequence of p-points of $f(z)$.

†† For definition, see Topology by Hocking and Young (Addison-Wesley) pp. 100–101.

$\cdots$, such that $\{f_{n(1)}(z)\}$ tends to a meromorphic function $g_1(z)$, uniformly on each compact subset of $D(0, r_1)$. Since $f(z)$ is of the first kind, by Hurwitz's theorem, $g_1(z)$ is not identically constant and in fact omits the three values w_1, w_2, w_3 in $D(0, r_1)$.

Now, for $n(1) \geq 2$, the function $f_{n(1)}(z)$ omits the set of values $A_{n(1)}$ in $D(0, r_2)$. Thus by the same argument, there exists a subsequence of functions $\{f_{n(2)}(z)\}$ of $\{f_{n(1)}(z)\}$ such that $\{f_{n(2)}(z)\}$ tends to a non-constant meromorphic function $g_2(z)$, uniformly on each compact subset of $D(0, r_2)$. Moreover, $g_2(z)$ omits the three values in $D(0, r_2)$. Since $\{f_{n(2)}(z)\}$ is a subsequence of $\{f_{n(1)}(z)\}$, we obtain $g_2(z) \equiv g_1(z)$ in $D(0, r_1)$.

In general, for each $q > 0$, there exists an infinite subsequence of functions $\{f_{n(q+1)}(z)\}$ of $\{f_{n(q)}(z)\}$ such that $\{f_{n(q+1)}(z)\}$ tends to a non-constant meromorphic function $g_{q+1}(z)$, uniformly on each compact subset of $D(0, r_{q+1})$. Moreover, $g_{q+1}(z) \equiv g_q(z)$ in $D(0, r_q)$, and $g_{q+1}(z)$ omits the three values w_j, $j = 1, 2, 3$, in $D(0, r_{q+1})$.

We now define the function $g(z)$ as follows: $g(z) = g_q(z)$ if $z \in D(0, r_q)$. It is easy to see that $g(z)$ also omits the three values w_j, $j = 1, 2, 3$. For if it were not the case, then $g(z)$ would assume, say, w_1 at a point z_0 in D. Suppose $z_0 \in D(0, r_q)$, for some r_q, then $g(z_0) = g_q(z_0) = w_1$, which contradicts the fact that $g_q(z)$ omits w_1 in $D(0, r_q)$. By the diagonal method, we can extract a subsequence of functions $\{f_k(z)\}$ of $\{f_n(z)\}$ such that $\{f_k(z)\}$ tends to $g(z)$ uniformly on each compact subset of D. We shall refer to $\{f_k(z)\}$ as $\{f_n(z)\}$ again.

Since $g(z)$ omits three distinct values in D, by a theorem of Bagemihl [1, Theorem 1, p. 3], the set of Fatou points of $g(z)$ is dense on C. Let $e^{i\theta}$ be a Fatou point of $g(z)$ with Fatou value w, and let A be a symmetric Stolz angle at $e^{i\theta}$ so large that at each point t on $R(\theta)$, $D\left(t, \frac{1}{2}\right) \subseteq A$. We now choose a sequence of positive numbers $\{R_k\}$ such that

1) $R_k \uparrow 1$,

2) For each $k = 1, 2, \cdots$, $|g(z) - w| \leq \frac{1}{2k}$, if $z \in A$ $\{z: |z| \geq R_k\}$. (16)

For each $k = 1, 2, \cdots$, choose R_k' such that

1) $0 < R_k < R_k' < 1$,

2) in each set $B_k = \{z: |R_k| < z < |R_k'|\}$, we can find a point t_k on $R(\theta)$

such that $D\left(t_k, \frac{1}{2}\right) \subseteq B_k$,

3) $B_k \subseteq D(0, r_{m(k)})$ for some integer $m(k)$.

On the other hand, since $\{f_n(z)\}$ tends to $g(z)$ uniformly on each compact subset of D, for each $k = 1, 2, \cdots$, there exists a natural number $n(k)$, such that for all $n \geq n(k)$ and all $z \in B_k$,

$$|f_n(z) - g(z)| \leq \frac{1}{2} k. \tag{17}$$

Let $p(k) = \max\{m(k), n(k)\}$, and consider the following functions:

$$z = S_{p(k)}(t) = \frac{z_{p(k)} - t}{1 - \bar{z}_{p(k)} t}, \tag{18}$$

$$f_{p(k)}(z) = f\ S_{p(k)}(z). \tag{19}$$

And let

$$z'_{p(k)} = S_{p(k)}(t_k), \tag{20}$$

i.e.

$$D\left(z'_{p(k)}, \frac{1}{2}\right) = S_{p(k)}\left(D\left(t_k, \frac{1}{2}\right)\right). \tag{21}$$

Consider also the functions

$$z = T_{p(k)}(\zeta) = \frac{z'_{p(k)} - \zeta}{1 - \bar{z}'_{p(k)} \zeta} \tag{22}$$

$$h_{p(k)}(\zeta) = f(T_{p(k)}(\zeta)). \tag{23}$$

We shall show that $f_{p(k)}(\zeta)$ tends to the constant limit w in $D(0, 1/2)$ in the ζ-plane and this will contradict the fact $f(z)$ is of the first kind in D.

To this end, let $a \in D(0, 1/2)$ in the ζ-plane and let

$$z^*_{p(k)} = T_{p(k)}(a), \tag{24}$$

$$t^*_k = S^{-1}_{p(k)}(z^*_{p(k)}), \tag{25}$$

then $\rho(z^*_{p(k)}, z'_{p(k)}) = \rho(a, 0) < 1/2$, i.e. $z^*_{p(k)} \in D(z'_{p(k)}, 1/2)$, and $\rho(t^*_k, t_k) = \rho(z^*_{p(k)}, z'_{p(k)}) < 1/2$, i.e. $t^*_k \in D(t_k, 1/2)$ or $t^*_k \in A \cap \{z: |z| \geq R_k\} \cap B_{p(k)}$.

Now, $h_{p(k)}(a) = f(T_{p(k)}(a)) = f(z^*_{p(k)}) = f(S_{p(k)}(t^*_k)) = f_{p(k)}(t^*_k)$. Hence,

$$|h_{p(k)}(a) - w| = |f_{p(k)}(t^*_k) - w| \leq |f_{p(k)}(t^*_k) - g(t^*_k)| + |g(t^*_k) - w|. \tag{26}$$

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The first absolute value of the last expression in equality (26) is less than $\frac{1}{2}k$ by (17) and the last term is less than $\frac{1}{2}k$ by (16). Thus, $|h_{p(k)}(a)-w|\leq 1/k$, for each $k=1, 2, \cdots$, i.e. $\lim h_{p(k)}(a)=w$, for each $a\in D(0,1/2)$ in the $\mathcal{Z}$-plane and that completes our proof.

COROLLARY 4. 1. *Suppose that $f(z)$ is a meromorphic function of the first kind in D. If $\{z_n\}$ is a sequence of points in D, $|z_n|\to 1$, and $\{r_n\}$ is a sequence of positive numbers such that $r_n\uparrow\infty$, then $f(z)$ assumes every value on Ω infinitely often, except perhaps two, in the set $\bigcup_{n=1}^{\infty} D(z_n, r_n)$.*

Remark 4. It is natural to ask whether meromorphic functions of the first kind in D possess the stronger property like the one proved by Noshiro [11, Theorem 7, p. 155] for normal functions of the first kind in D. Obviously, we can not obtain an equally strong theorem, for as we mentioned in Remark 2 that every holomorphic function $f(z)$ in the class $V(S, \bar{\mu}=0)$ is of the first kind and omits the values infinity.

4. With the help of Theorem 4, we shall generalize the notion of sequences of p-points as follows;

DEFINITION 2. A sequence of points $\{z_n\}$ in D, $|z_n|\to 1$, is called a sequence of pseudo-p-points of a meromorphic function $f(z)$ in D if for each sequence of positive number $\{r_n\}$, $r_n\uparrow\infty$, $f(z)$ assumes every value on Ω infinitely often, except perhaps two, in the set $\bigcup_{m=1}^{\infty} D(z_m, r_m)$, for each subsequence $\{z_m\}$ *of* $\{z_n\}$.

With this definition, we can restate Corollary 4. 1 as follows:

COROLLARY 4. 1a. *If $f(z)$ is a meromorphic function of the first kind in D, then every sequence of points $\{z_n\}$ in D, $|z_n|\to 1$, is a sequence of pseudo-p-points.*

THEOREM 5. *Let $f(z)$ be a meromorphic function in D and let $\{z_n\}$ be a sequence of points in D, $|z_n|\to 1$, which possesses no subsequence of pseudo-p-points of $f(z)$. Then there exists no sequence of p-points of $f(z)$ in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$ for any $r<\infty$.*

Proof. Trivial.

THEOREM 6. *Let $f(z)$ be meromorphic in D and let $\{z_n\}$ be a sequence of points in D which possesses no subsequence of pseudo-p-points of $f(z)$. If $\lim f(z_n)=w$,*

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where w is an omitted value of $f(z)$, then $f(z)$ tends to w as $|z| \to 1$, uniformly in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$, for each $r < \infty$.

Proof. Suppose, on the contrary, that there exists a sequence of points $\{w_m\}$ in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$. Suppose further that $w_m \in D(z_m, r)$, for $m = 1, 2, \cdots$, where $\{z_m\}$ is a subsequence of $\{z_n\}$. By a theorem of Gavrilov [7, Theorem 5], there exists a sequence of p-points of $f(z)$ in the set $\bigcup_{m=1}^{\infty} D(z_m, r)$, which contradicts our assumption that $\{z_n\}$ possesses no subsequence of pseudo-p-points. (See also Theorem 5.)

COROLLARY 6. 1. *Let $f(z)$ be a holomorphic function in D, and let $\{z_n\}$ be a sequence of points in D which possesses no subsequence of pseudo-p-points of $f(z)$. Then, for each $r < \infty$, we have:*

1) *If $\lim f(z_n) = \infty$, then $f(z)$ tends to infinity as $|z| \to 1$, uniformly in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$.*
2) *If $|f(z_n)| < K$, for some finite constant K, and for $n = 1, 2, \cdots$, then $f(z)$ is bounded on the set $\bigcup_{n=1}^{\infty} D(z_n, r)$. Here, the bound is dependent on r.*

DEFINITION 3. A boundary path B is called a pseudo-p-path of a meromorphic function $f(z)$ in D if there exists a sequence of pseudo-p-points on B. It is called a strong pseudo-p-path of $f(z)$ if every sequence of points on B tending to C is a sequence of pseudo-p-points of $f(z)$.

THEOREM 7. *Let $f(z)$ be a meromorphic function in D, and B be a boundary path in D but not a pseudo-p-path of $f(z)$. If $\lim f(z) = w$, where w is finite or infinite, then $f(z)$ tends to w uniformly in the set $\{z: \rho(z, B) < M\}$ for each $M < \infty$.*

Proof. By a theorem of Gauthier [6, (2. 8), p. 13], $f(z)$ tends to w uniformly in the set $\{z: \rho(z, B) < M\}$, for each $M < M_0(f)$, where $0 \leqq M_0(f) \leqq \infty$. Moreover, if $M_0(f) < \infty$, $f(z)$ possesses a sequence of p-points on the set $\{z: \rho(z, B) = M_0(f)\}$. Thus, under our assumption that B is not a pseudo-p-path and by Theorem 5, it is easy to see that $f(z)$ tends to w uniformly in the set $\{z: \rho(z, B) < M\}$ for each $M < \infty$.

THEOREM 8. *Let $f(z)$ be a holomorphic function in D, and let $\{z_n\}$ be a sequence of points in D satisfying:*

1) $\overline{\lim}\, \rho(z_n, z_{n+1}) < M$,

2) *the limit points of* $\{z_n\}$ *on* C *contains an open arc* $\overset{\frown}{\alpha\beta}$ *of* C,

3) *no subsequence of* $\{z_n\}$ *is a sequence of pseudo-p-points of* $f(z)$,

4) $\lim f(z_n) = w$, *where* $w \neq \infty$.

If, in addition, there exist two points $e^{i\theta_1}$ *and* $e^{i\theta_2}$ *on* $\overset{\frown}{\alpha\beta}$ *such that* $f(z)$ *possesses radial limits along* $R(\theta_1)$ *and* $R(\theta_2)$, *then* $f(z) \equiv w$.

Proof. Consider the boundary path B by joining the consecutive points z_n to z_{n+1} with a non-Euclidean straight line. By Theorem 7 and condition 1), we have $\lim_{\substack{|z|\to 1 \\ z\in B}} f(z) = w$. Since the radii $R(\theta_1)$ and $R(\theta_2)$ intersect B infinitely often, the radial limits of $f(z)$ at both $e^{i\theta_1}$ and $e^{i\theta_2}$ are w. Since $w \neq \infty$, by the Maximum Modulus Principle, we see that $f(z)$ is bounded by w in the sector determined by the radii $R(\theta_1)$ and $R(\theta_2)$. By an easy modification of Fatou's Theorem (e.g. see [12, p. 5]), we see that the set of Fatou points of $f(z)$ on $\overset{\frown}{\alpha\beta}$ has positive Lebesgue measure and their corresponding Fatou values are w. By the standard Luzin and Privalov's uniqueness theorem [12, p. 72], we have $f(z) \equiv w$.

THEOREM 9. *Let* $f(z)$ *be meromorphic in* D *and omit two distinct values* w_1 *and* w_2 *there, and let* $\{z_n\}$ *be a sequence of points in* D *satisfying the conditions:*

1) $\overline{\lim}\, \rho(z_n, z_{n+1}) < M$,

2) *the limit points of* $\{z_n\}$ *on* C *contain an open arc* $\overset{\frown}{\alpha\beta}$ *of* C,

3) *no subsequence of* $\{z_n\}$ *is a sequence of pseudo-p-points of* $f(z)$

4) $\lim f(z_n) = w$.

If, in addition, there exist two points $e^{i\theta_1}$ *and* $e^{i\theta_2}$ *on* $\overset{\frown}{\alpha\beta}$ *such that* $f(z)$ *possesses radial limits along* $R(\theta_1)$ *and* $R(\theta_2)$, *then*

1) $w \neq w_1$, $w \neq w_2$

and

2) $f(z) \equiv w$.

Proof. Without loss of generality, we may suppose that $w_1 = 0$, and $w_2 = \infty$, for otherwise, we may consider the function $g(z) = (f(z) - w_1)/$

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$(f(z)-w_2)$ instead. By the same argument as we used in Theorem 8 with both Maximum and Minimum Modulus Principles, we can easily show that $f(z)\equiv w$, and since w_1 and w_2 are omitted values, we have $w\neq w_1$ and $w\neq w_2$.

5. Tsuji [13] had defined a sub-class of meromorphic functions which satisfy the following condition:

$$\sup_{r<1}\int_0^{2\pi}\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\,r\,d\theta<\infty. \tag{27}$$

Collingwood and Piranian in one of their papers [5, p. 246] reviewed the properties of the functions in the subclass defined by Tsuji and called them Tsuji functions. We now start to prove two theorems for Tsuji functions and Tsuji functions which are also of the first kind.

THEOREM 10. *If $f(z)$ is a Tsuji function in D, then $C=E\cup F\cup G$, where* (1) *meas $(E)=0$,* (2) *F is the set of Fatou points of $f(z)$, and* (3) *every chord at each point of G is a p-path*† *of $f(z)$.*

Proof. Since $f(z)$ is a Tsuji function, then $C=A\cup B$ [14, p. 53], where meas $(A)=0$, and for each point $e^{i\theta}$ of B, $f(z)$ tends to the same asymptotic limit $w(\theta)$ along almost all chords at $e^{i\theta}$.

Let $E=A\cup A'$, where A' is the set of points on C which are neither Plessner points nor Fatou points. By the Plessner Theorem [12, p. 70], meas $(A')=0$, and thus meas $(E)=0$. Let F be the set of Fatou points of $f(z)$ on C and let $G=C-(E\cup F)$. Since $G\subseteq B$, we see that for each point $e^{i\theta}$ on G and for almost all ϕ, $-\pi/2<\phi<\pi/2$, $f(z)$ approaches an asymptotic value $w(\theta)$ on the chords $R(\theta,\phi)$. By a theorem of Gauthier [6, (2. 8), p. 13], $f(z)$ tends to $w(\theta)$ uniformly in the set $\{z\colon \rho(z,R(\theta,\phi))<M\}$, for all $M<M_0(f)$, where $0\leq M_0(f)\leq\infty$. Moreover, if $M_0(f)<\infty$, $f(z)$ possesses a sequence of p-points on the set $\{z\colon \rho(z,R(\theta,\phi))=M_0(f)\}$.

If $M_0(f)=\infty$, then $e^{i\theta}$ is a Fatou point of $f(z)$, if $0<M_0(f)<\infty$, then $e^{i\theta}$ is not a Fatou point nor a Plessner point of $f(z)$. Thus if $e^{i\theta}\in G$, $M_0(f)=0$, i.e. almost all chords at each point of G are p-paths. Since these chords are dense in any Stolz angle at $e^{i\theta}$, by another theorem of Gauthier

† A boundary path in D is said to be a p-path of a meromorphic function of $f(z)$ if it contains a sequence of p-points of $f(z)$.

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[6, (2. 1), p. 10], it is easy to see that every chord at each point of G is a p-path of $f(z)$.

THEOREM 11. *If $f(z)$ is a Tsuji function which is also of the first kind in D, then every chord at almost all points of C is a strong-p-path of $f(z)$. (See also Corollary* 3. 1.)

Proof. By Theorem 10, $C = E \cup F \cup G$, where $\text{meas}(E) = 0$. Since $f(z)$ is of the first kind, $F = \phi$, so that $\text{meas}(G) = 2\pi$, moreover, at each point $e^{i\theta}$ of G, there exists a set $H(\theta) \subseteq (-\pi/2, \pi/2)$, such that

1) $\text{meas}(H) = \pi$.

2) for each $\phi \in H$, $f(z)$ attains an asymptotic limit $w(\theta)$ on $B(\theta, \phi)$.

By the Corollary of Theorem 3, we have for each $\phi \in H$, $R(\theta, \phi)$ is a strong-p-path. By a theorem of Gauthier [6, (2. 1), p. 10], since H is dense in $(-\pi/2, \pi/2)$, it is easy to see that for each $e^{i\theta} \in G$, $R(\phi, \theta)$ is a strong-p-path of $f(z)$. (Note that $\text{meas}(G) = 2\pi$.) And this completes our proof.

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