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BOUNDARY BEHAVIOR OF THE BERGMAN METRIC

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Abstract. Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$. We give sufficient conditions for the Bergman metric to go to infinity uniformly at some boundary point, which is stated by the existence of a Hölder continuous plurisubharmonic peak function at this point or the verification of property (P) (in the sense of Coman) which is characterized by the pluricomplex Green function.

§1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$, and let $K_\Omega(z)$ be the Bergman kernel function on Ω. The Bergman metric is defined by

$$B_\Omega(z;X)=\left(\sum_{j.k=1}^{n}\frac{\partial^2\log K_\Omega(z)}{\partial z_j\partial\bar{z}_k}X_j\bar{X}_k\right)^{1/2}$$

where $X=\sum_{j=1}^n X_j\partial/\partial z_j\in T^{1,0}(\mathbf{C}^n)$. Recently, it was proved independently by Blocki-Pflug [2] and Herbort [16] that the Bergman metric of an arbitrary bounded hyperconvex domain is complete. This is a partial but satisfactory answer to the old problem of Kobayashi [19]: Which bounded pseudoconvex domain is Bergman complete? This problem has been extensively studied (see also [3], [17], [21], [22]). However, the Bergman completeness does not guarantee that the Bergman metric tends to infinity uniformly at the boundary. A polydisc is the most simple example. Notice that such a domain does not admit a plurisubharmonic (psh for short) peak function at each boundary point. Recall that a function ρ is called a psh peak function at a boundary point w_0 of a domain Ω if ρ is psh in Ω, continuous on $\overline{\Omega}$ with $\rho(w_0)=0$ and $\rho(z)<0$ for all $z\in\overline{\Omega}\backslash\{w_0\}$. It is natural to ask whether the Bergman metric of a bounded pseudoconvex domain go to infinity uniformly at a boundary point if there is a psh peak function at

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this point. The main purpose of this paper is to give the following partial result:

THEOREM 1. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let $w_0 \in \partial\Omega$. Suppose that there exists a psh peak function ρ for Ω at w_0 which is Hölder continuous at w_0, that is, there exist constants $c, \gamma > 0$ such that $\rho(z) \geq -c|z - w_0|^\gamma$ holds for all $z \in \overline{\Omega}$. Then one has*

$$\inf_{0 \neq X \in T^{1,0}(\mathbf{C}^n)} B_\Omega(z; X)/|X| \to \infty \tag{1}$$

as $z \to w_0$ in Ω.

As an application of Theorem 1, we obtain following

COROLLARY 1. *Let Ω be a bounded domain in $\mathbf{C}$ with $\overline{\Omega}^\circ = \Omega$. Then the Bergman metric can not be extended beyond Ω.*

It is well known that the Bergman kernel function and the Bergman metric of $\Omega \backslash A$ can be extended through A if A is an complex analytic subvariety of dimension$\leq n - 1$. For any bounded pseudoconvex Ω in $\mathbf{C}^n$ with $\overline{\Omega}^\circ = \Omega$, Pflug [23] proved that the Bergman kernel function can not be extended through any boundary point. However, this phenomenon dose not always occur for the Bergman metric in the case $n \geq 2$. For example, the Hartogs triangle $\Omega = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| < |z_2| < 1\}$ satisfies $\overline{\Omega}^\circ = \Omega$; nevertheless, it is well-known that the Bergman metric can be continued across the origin. It would be interesting to find sufficient conditions for the Bergman metric not being extended through the boundary.

When the psh peak function in Theorem 1 satisfies certain growth condition, one can even obtain a quantitative estimate for the Bergman metric:

THEOREM 2. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$. Suppose there exist positive constants c, α, γ such that for each $p \in \partial\Omega$ there is a psh peak function ρ_p for Ω at p verifying*

$$|z - p|^\alpha \leq -\rho_p(z) \leq c|z - p|^\gamma$$

for all $z \in \overline{\Omega}$. Then there exist a number $\tau > 0$ and a constant $C > 0$ such that

$$B_\Omega(z; X) \geq C|X|/\delta_\Omega^\tau(z),$$

where $\delta_\Omega(z)$ is the euclidean boundary distance of z.

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Remarks. (1) The assumptions are satisfied for pseudoconvex domains with real analytic boundary [9] or more generally, pseudoconvex domains of finite type in $\mathbf{C}^n$ [4]. In both cases such an estimate has also been obtained by Diederich et. al. [10] and McNeal [20];

(2) We are not going to determine the exponent τ precisely.

The domains in Theorem 1 or 2 are quite general. We provide in the last section some examples of non-smooth pseudoconvex domains on which the hypothesis of Theorem 1 or 2 is satisfied. The defining equation is $|z_1|^{2/\alpha_1} + |z_2|^{2/\alpha_2} + \cdots + |z_n|^{2/\alpha_n} + \varphi(z) < 0$ where $\alpha_j > 0,\ j = 1, 2, \ldots, n$ and φ is a Hölder continuous psh function in $\mathbf{C}^n$.

In the proof of the Bergman completeness for hyperconvex domains, the boundary behavior of the pluricomplex Green function plays an important role. Motivated by this fact, we shall prove that (1) also holds under a hypothesis characterized by the pluricomplex Green function. We begin with some notions.

DEFINITION. The pluricomplex Green function of Ω is defined by

$$g_\Omega(z, w) = \sup\{\varphi(z) : \varphi \text{ is psh on } \Omega,\ \varphi \le 0,\ \varphi(z) \le \log|z - w| + O(1)\ \}.$$

This definition was firstly given by Klimek [18]. It coincides with the (negative) Green function for the Laplace operator in one dimensional case. The function $g_\Omega(\cdot, w)$ is a negative psh function in Ω and it has a logarithmic pole at w. It is also invariant under biholomorphic mappings. If Ω is hyperconvex, then $g_\Omega(z, w) \to 0$ as $z \to \partial\Omega$ and it is continuous on $\overline{\Omega} \times \Omega$. This result is due to Demailly [8]. The pluricomplex Green function plays a similar role in the pluripotential theory as the Green function in the classical potential theory.

Coman introduced in [5] the following

DEFINITION. We say that Ω has property (P) at $w_0 \in \partial\Omega$ if for every compact set $K \subset \overline{\Omega}\backslash\{w_0\}$ one has $g_\Omega(z, w) \to 0$ as $w \to w_0$, uniformly for $z \in K \cap \Omega$.

We shall show

THEOREM 3. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let $w_0 \in \partial\Omega$. Suppose that Ω has property (P) at w_0. Then* (1) *holds.*

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This class of domains contains the following cases (cf. [5]):

(1) w_0 is a local weak peak point for Ω, i.e., there are a neighborhood U of w_0 and a holomorphic map $h : \Omega \cap U \to \Delta$, where Δ is the unit disc in the complex plane, so that $\lim_{z\to w_0} |h(z)| = 1$ and $\limsup_{z\to q} |h(z)| < 1$, for every $q \in \partial\Omega, q \neq w_0$;

(2) g_Ω is symmetric, that is, $g_\Omega(z, w) = g_\Omega(w, z)$ for all $z, w \in \Omega$ (this case involves all bounded domains in $\mathbf{C}$, convex domains and homogeneous domains in $\mathbf{C}^n$ [8]), and there exists a psh peak function ρ for Ω at w_0;

(3) There is a psh peak function ρ for Ω at w_0 such that

(i) ρ is Hölder continuous at w_0;
(ii)

$$\begin{aligned} N_\rho(r) &:= \max\left\{\frac{\log|\rho(z)|}{\log|z-w_0|} : z \in \overline{\Omega},\ r \le |z-w_0| \le 1/2\right\} \\ &= O(\log\log(1/r)) \end{aligned}$$

as $r \to 0$.

The proofs of the theorems are based on the L^2-estimates of the $\bar\partial$-operator firstly introduced by Donnelly-Fefferman [13] and then generalized by Diederich-Ohsawa [11], [12] and Berndtsson [1]. The proof of the main theorem is inspired especially by Diederich-Ohsawa [11].

§2. Some L^2-estimates for the $\bar\partial$-operator

In this section, we recall some powerful L^2-tools for the $\bar\partial$-equation. The following brilliant result is due to Donnelly-Fefferman [13]:

PROPOSITION 4. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let φ be psh in Ω. Let ψ be psh and assume that $\partial\bar\partial\psi \ge \partial\psi\bar\partial\psi$ holds in the distribution sense (equivalently, the function $e^{-\psi}$ is plurisuperharmonic). Then for any $\bar\partial$-closed $(0,1)$-form g in Ω there is a solution to the equation $\bar\partial u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi} dV_n \le C \int_\Omega |g|^2_{\partial\bar\partial\psi} e^{-\varphi} dV_n.$$

Here $C > 0$ is a numerical constant and dV_n denotes the Lebesgue measure in $\mathbf{C}^n$.

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Important generalizations of this result have been done by Diederich-Ohsawa [11], [12] and Berndtsson [1]. We use the version of Berndtsson:

PROPOSITION 5. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let φ, ψ be as in Porposition 4. Let $0 < \nu < 1$. Then for any $\bar{\partial}$-closed $(0,1)$-form g in Ω there is a solution u to the equation $\bar{\partial}u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi+\nu\psi} dV_n \leq \frac{4}{\nu(1-\nu)^2} \int_\Omega |g|^2_{\partial\bar{\partial}\psi} e^{-\varphi+\nu\psi} dV_n.$$

§3. Proof of Theorem 3

We begin with some notions. Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let φ be a psh function on Ω. By $L^2(\Omega)$ we denote all the square-integrable functions on Ω. The L^2 norm is denoted by $\|\cdot\|_\Omega$. Let $H^2(\Omega)$ denote the subspace of holomorphic functions in $L^2(\Omega)$. We define for any measurable function on Ω:

$$\|f\|^2_{\Omega,\varphi} := \int_\Omega |f|^2 e^{-\varphi} dV_n.$$

We say that $f \in L^2(\Omega, \varphi)$ if $\|f\|_{\Omega,\varphi} < \infty$. Now we are going to prove Theorem 3. Without loss of generality, we may assume $w_0 = 0$ and $\Omega \subset \{|z| < 1\}$. By the hypothesis of the theorem, for each $0 < \epsilon < 1$ there exists a number $0 < \delta << \epsilon$ so that

$$g_\Omega(z, w) \geq -1$$

for all $z, w \in \Omega$ with $|z| \geq \epsilon$ and $|w| \leq \delta$. Let $\chi : \mathbf{R} \to [0,1]$ be a C^∞ function satisfying $\chi \equiv 1$ on $(-\infty, 1)$ and $\chi \equiv 0$ on $(1 + \log 2, \infty)$. We put

$$\begin{aligned}
\psi(z) &= -\log(-\log|z|) \\
\eta_\epsilon(z) &= \chi(\psi(z) + \log(-\log\epsilon) + 1) \\
\varphi(z) &= 2(n+1)g_\Omega(z, w) + \frac{1}{2}\psi(z).
\end{aligned}$$

Now fix $w \in \Omega$ with $|w| \leq \delta$ for a moment. Let $X \in T^{1,0}(\mathbf{C}^n)$. Since the Bergman metric is biholomorphically invariant, we may assume $X = |X|\partial/\partial z_1$ for the sake of simplicity. Since ψ is psh on Ω and satisfies $\partial\bar{\partial}\psi \geq \partial\psi\bar{\partial}\psi$, there is, according to Proposition 4, a solution to the $\bar{\partial}$-equation

$$\bar{\partial}u_\epsilon = (z_1 - w_1)\frac{K_\Omega(z, w)}{K^{1/2}_\Omega(w)}\bar{\partial}\eta_\epsilon(z)$$

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together with the estimate

$$\begin{aligned}
&\int_\Omega |u_\epsilon|^2 e^{-2(n+1)g_\Omega(z,w)} dV_n \\
&\le C \int_\Omega |z_1 - w_1|^2 |K_\Omega(z,w)|^2 / K_\Omega(w) |\bar\partial \eta_\epsilon|_{\partial\bar\partial\psi} e^{-2(n+1)g_\Omega(z,w)} dV_n \\
&\le C \sup|\chi'|^2 \int_{\Omega\cap\{\epsilon\le|z|\le\epsilon^{1/2}\}} |z-w|^2 |K_\Omega(z,w)|^2 / K_\Omega(w) e^{-2(n+1)g_\Omega(z,w)} dV_n \\
&\le C e^{2(n+1)} \sup|\chi'|^2 \int_{\Omega\cap\{|z|\le\epsilon^{1/2}\}} |z-w|^2 |K_\Omega(z,w)|^2 / K_\Omega(w) dV_n \\
&\le C_1 \epsilon
\end{aligned}$$

where C_1 only depends on n, $\sup|\chi'|$. It follows that $u_\epsilon(w) = 0$ and $\partial u_\epsilon/\partial z_1(w) = 0$ since $g_\Omega(z,w) \sim \log|z-w|$ as $z \to w$. Let

$$f_\epsilon(z) = (z_1 - w_1)\eta_\epsilon(z) K_\Omega(z,w) / K_\Omega^{1/2}(w) - u_\epsilon(z).$$

Then f_ϵ is holomorphic on Ω. Since $|w| < \delta < \epsilon$, one has $\eta_w(w) = 1$. Hence, $f_\epsilon(w) = 0$, $Xf_\epsilon(w) = |X| K_\Omega^{1/2}(w)$ and

$$\begin{aligned}
\int_\Omega |f_\epsilon|^2 dV_n &\le 2\int_\Omega |(z_1 - w_1)\eta_\epsilon(z) K_\Omega(z,w)/K_\Omega^{1/2}(w)|^2 dV_n + 2\int_\Omega |u_\epsilon|^2 dV_n \\
&\le C_2\epsilon + 2\int_\Omega |u_\epsilon|^2 e^{-2(n+1)g_\Omega(z,w)} dV_n \\
&\le C_3\epsilon
\end{aligned}$$

because $g_\Omega(z,w) < 0$ on Ω. Here the positive constants only depend on n and $\sup|\chi'|$. We set $h_\epsilon = f_\epsilon / \|f_\epsilon\|_\Omega$. From the well-known formula of the Bergman metric

$$B_\Omega(w;X) = K_\Omega^{-1/2}(w) \sup\{|Xf(w)| : f \in H^2(\Omega),\ f(w) = 0 \text{ and } \|f\|_\Omega \le 1\},$$

we obtain

$$B_\Omega(w;X) \ge \frac{|Xh_\epsilon(w)|}{K_\Omega^{1/2}(w)} = \frac{|Xf_\epsilon(w)|}{K_\Omega^{1/2}(w)\|f_\epsilon\|_\Omega} \ge C_3^{-1/2}\epsilon^{-1/2}|X|$$

for any $w \in \Omega \cap \{|w| < \delta\}$. This proves the theorem.

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§4. Proofs of Theorems 1, 2

We assume $c = 1$ for the sake of simplicity (otherwise, we replace ρ by ρ/c). Put $\phi = -\log(-\rho)$. Without loss of generality, we may assume $\rho(z) > -1/2$ on Ω and $\operatorname{diam}\Omega \le e^{-1}$. This implies $\phi > \log 2$ on Ω. Choose a C^∞ cut-off function (we still denote χ for the sake of simplicity) so that $\chi|_{(-\infty,0)} \equiv 1$, $\chi|_{(1,\infty)} \equiv 0$, $\sup|\chi'| \le 2$ and $\sup|\chi''| \le 1$. Let $0 < \epsilon << 1$ be an arbitrary fixed number. We define on Ω a function as follows:

$$g_{k,w}(z) = \chi\left(\frac{1}{\log k}(-\log\phi(z) + \log(-\log\epsilon)) + 1\right)\log|z-w|$$

for any $k > 1$ and $w \in \Omega$. The following lemma is similar to Proposition 2.1 in Diederich-Ohsawa [11].

LEMMA 6. *There exists a constant $k_0 > 1$ (only depends on n, γ) so that for any $w \in \Omega$ with $|w - w_0| < \frac{1}{2}\epsilon^{k_0/\gamma}$ the following holds*

(i) *$g_{k_0,w} \sim \log|z-w|$ near w;*

(ii) *$g_{k_0,w}(z) + \frac{1}{8(n+1)}(\phi(z) - \log(-\log|z-w|))$ is a psh function on Ω.*

Proof. We assume $|w - w_0| < \frac{1}{2}\epsilon^{k/\gamma}$ where k will be determined later. Then one has

$$\rho(w) \ge -|w-w_0|^\gamma > -\frac{1}{2^\gamma}\epsilon^k.$$

Since

$$\{z \in \Omega : g_{k,w}(z) = \log|z-w|\} \supset \{z \in \Omega : \rho(z) > -\epsilon^k\},$$

one has $g_{k,w} \sim \log|z-w|$ near w. By a straightforward computation, we obtain the following equality in the distribution sense

$$\begin{aligned}\partial\bar\partial g_{k,w} = {} & \frac{\log|z-w|}{\phi\log k}\left(\chi''(\,\cdot\,)\frac{\partial\phi\bar\partial\phi}{\phi\log k} + \chi'(\,\cdot\,)\frac{\partial\phi\bar\partial\phi}{\phi} - \chi'(\,\cdot\,)\partial\bar\partial\phi\right)\\ & - \frac{\chi'(\,\cdot\,)\log|z-w|}{\phi\log k}\left(\partial\phi\frac{\bar\partial\log|z-w|}{\log|z-w|} + \bar\partial\phi\frac{\partial\log|z-w|}{\log|z-w|}\right)\\ & + \chi(\,\cdot\,)\partial\bar\partial\log|z-w|.\end{aligned}$$

Observe that

$$\begin{aligned}\operatorname{supp}\chi'(\,\cdot\,) &\subset \{z \in \Omega : \rho(z) \le -\epsilon^k\}\\ &\subset \{z \in \Omega : |z-w_0| \ge \epsilon^{k/\gamma}\}.\end{aligned}$$

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It follows that

$$|z-w| \geq |z-w_0| - |w-w_0| \geq \frac{1}{2}|z-w_0|$$

on $\operatorname{supp}\chi'(\,\cdot\,)$ since $|w-w_0| < \frac{1}{2}\epsilon^{k/\gamma}$. Hence

$$\begin{aligned} |\phi(z)| &= |\log(-\rho(z))| \\ &\geq \gamma|\log|z-w_0|| \\ &\geq \gamma|\log(2|z-w|)| \end{aligned}$$

holds there. By Cauchy-Schwarz's inequality, the following inequality holds in the distribution sense

$$\pm 2\mathrm{Re}\,\frac{\partial\phi\bar{\partial}\log|z-w|}{\log|z-w|} \geq -\partial\phi\bar{\partial}\phi - \partial\log(-\log|z-w|)\bar{\partial}\log(-\log|z-w|).$$

Then we can find, by neglecting the semipositive term $\chi(\,\cdot\,)\partial\bar{\partial}\log|z-w|$, a constant $C' > 0$ (only depends on γ) so that

$$\partial\bar{\partial}g_{k,w}(z) \geq -\frac{C'}{\log k}(\partial\bar{\partial}\phi + \partial\bar{\partial}(-\log(-\log|z-w|)))$$

provided $k \geq e$, because $\phi > \log 2$ on Ω, $\partial\bar{\partial}\phi \geq \partial\phi\bar{\partial}\phi$ and

$$\partial\bar{\partial}(-\log(-\log|z-w|)) \geq \partial\log(-\log|z-w|)\bar{\partial}\log(-\log|z-w|).$$

To complete the proof we only need to take k_0 so large that $C'/\log k_0 \leq \frac{1}{8(n+1)}$.

Proof of Theorem 1. Let $0 < \epsilon << 1$ and $w \in \Omega$ with $g_{k_0,w}$ be as above. We denote by $g_w = g_{k_0,w}$ for the sake of simplicity. Let

$$\delta = \delta(\epsilon) := \sup_{z\in\overline{\Omega},\rho(z)\geq-\epsilon} |z-w_0|.$$

Then $\delta > 0$ and $\delta \to 0$ as $\epsilon \to 0$ since ρ is a psh peak function. Let $\kappa : \mathbf{R} \to [0,1]$ such that $\kappa|_{(-\infty,1-\log 2)} = 1$ and $\kappa|_{(1,\infty)} = 0$. We put

$$\begin{aligned} \eta_w &= \kappa(-\log(-\log|z-w_0|) + \log(-\log\delta^{1/2}) + 1) \\ \psi_w &= \frac{1}{2}(\phi - \log(-\log|z-w_0|)) \\ \varphi_w &= 2(n+1)g_w - \frac{1}{4}\log(-\log|z-w|) + \frac{1}{2}\psi_w. \end{aligned}$$

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By Lemma 6, φ_w is psh on Ω. Obviously one has $\partial\bar\partial\psi_w \geq \partial\psi_w\bar\partial\psi_w$ and $\partial\bar\partial\psi_w \geq \frac{1}{2}\partial\log(-\log|z-w_0|)\bar\partial\log(-\log|z-w_0|)$. This implies

$$|\bar\partial\eta_w|_{\partial\bar\partial\psi_w} \leq \sqrt{2}\sup|\kappa'|.$$

Notice that

$$\begin{aligned}\operatorname{supp} g_w &\subset \{z\in\Omega : \rho(z)\geq -\epsilon\}\\ &\subset \{z\in\Omega : \eta_w(z)=1\}.\end{aligned}$$

In particular, one has $\eta_w(w)=1$. Let $X=|X|\partial/\partial z_1$. We apply Proposition 5 with $\nu=1/2,\ \varphi=\varphi_w,\ \psi=\psi_w$ to solve the $\bar\partial$-equation

$$\bar\partial u_w = (z_1-w_1)K_\Omega(z,w)/K_\Omega^{1/2}(w)\bar\partial\eta_w$$

on Ω with the estimate

$$\begin{aligned}&\int_\Omega |u_w|^2 e^{-2(n+1)g_w+\frac{1}{4}\log(-\log|z-w|)}dV_n\\ &\leq 32\int_{\Omega_t}|z_1-w_1|^2|K_\Omega(z,w)/K_\Omega^{1/2}(w)|^2|\bar\partial\eta_w|^2_{\partial\bar\partial\psi_w}\\ &\quad e^{-2(n+1)g_w+\frac{1}{4}\log(-\log|z-w|)}dV_n\\ &\leq C_1\int_{\Omega\cap\{|z-w_0|<\delta^{1/2}\}}|z-w|^2(-\log|z-w|)^{1/4}\\ &\quad |K_\Omega(z,w)/K_\Omega^{1/2}(w)|^2dV_n\\ &\leq C_2\delta^{1/2}\end{aligned}$$

because $\operatorname{supp} g_w\cap\operatorname{supp}\bar\partial\eta_w=\emptyset$. Here $C_1,C_2>0$ are constants only depend on $\sup|\kappa'|$. Since

$$2(n+1)g_w-\frac{1}{4}\log(-\log|z-w|)<0$$

on Ω and

$$2(n+1)g_w-\frac{1}{4}\log(-\log|z-w|)<2(n+1)\log|z-w|$$

near w, the function $f_w=(z_1-w_1)\eta_wK_\Omega(z,w)/K_\Omega^{1/2}(w)-u_w$ is holomorphic on Ω and satisfies

$$f_w(w)=u_w(w)=0,\ Xf_w(w)=|X|K_\Omega^{1/2}(w)$$

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and

$$\begin{aligned}\|f_w\|_\Omega &\le \|(z_1-w_1)\eta_w K_\Omega(\,\cdot\,,w)/K_\Omega^{1/2}(w)\|_\Omega + \|u_w\|_\Omega\\ &\le C_3\delta^{1/2} + \|u_w\|_{\Omega,2(n+1)g_w-\frac{1}{4}\log(-\log|z-w|)}\\ &\le C_4\delta^{1/4}.\end{aligned}$$

This implies

$$B_\Omega(w;X) \ge C_4^{-1}\delta^{-1/4}|X|$$

for any $w \in \Omega$ with $|z-w_0| < \frac{1}{2}\epsilon^{k_0/\gamma}$. This proves the theorem.

Proof of Theorem 2. It suffices to modify the proof of Theorem 1 slightly. Let $w \in \Omega$ be fixed for a moment. Choose a boundary point w' so that $|w-w'| = \delta_\Omega(w)$. We take $\epsilon = (3\delta_\Omega(w))^{\gamma/k_0}$. We replace the w_0 by w', ρ by $\rho_{w'}$ and repeat the argument as in the proof of Theorem 1. Clearly one has $|w-w'| < \frac{1}{2}\epsilon^{k_0/\gamma}$ and

$$\delta = \sup_{z\in\overline{\Omega},\rho_{w'}(z)\ge-\epsilon} |z-w'| \le c'\delta_\Omega^{\frac{\gamma}{\alpha k_0}}(w)$$

because $-\rho_{w'}(z) \ge |z-w'|^\alpha$. We obtain finally the following

$$B_\Omega(w;X) \ge C\delta_\Omega^{-\frac{\gamma}{4\alpha k_0}}(w)|X|$$

for suitable constant $C > 0$. To complete the proof, we only need to take $\tau = \frac{\gamma}{4\alpha k_0}$.

Proof of the corollary. Let $w_0 \in \partial\Omega$ be an arbitrary point. Then there exist a sequence of points $w_k \in \mathbf{C}\backslash\overline{\Omega}$, $k = 1,2,\ldots$ which converge to w_0 as $k \to \infty$. Let $w_k' \in \partial\Omega$ be the nearest point to w_k for each k. It follows that $\Omega \subset \mathbf{C}\backslash\Delta(w_k,r_k)$ where $r_k = |w_k - w_k'|$ and $\Delta(x,r)$ denotes the disc which is centered at x with radius r. Since $\partial\Delta(w_k,r_k)$ is C^∞, Ω admits a Hölder continuous psh peak function at w_k' for each k. According to Theorem 1, the Bergman metric goes to infinity uniformly at w_k. This implies that the Bergman metric can not be continuously extended across w_0.

§5. Examples and remarks

1. Let φ be a Hölder continuous psh function in $\mathbf{C}^n$. We consider the following domain

$$\Omega = \{z \in \mathbf{C}^n : r(z) = |z_1|^{2/\alpha_1} + |z_2|^{2/\alpha_2} + \cdots + |z_n|^{2/\alpha_n} + \varphi(z) < 0\}$$

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where $\alpha_j > 0,\ j = 1, 2, \ldots, n$. If $\varphi \equiv -1$, then Ω is a well-known Reinhardt domain which has been studied by many authors (cf. [6], [7], [14]). Now let p be any boundary point. Without loss of generality, we may assume $p_j \neq 0$ if $1 \leq j \leq l$ and $p_j = 0$ if $j > l$ for some positive integer l. We take an open neighborhood U of p so that z_j does not vanish there for $1 \leq j \leq l$. For each $z \in \Omega \cap U$ one has

$$\begin{aligned} -r(z) + |z_j|^{2/\alpha_j} &> |z_j^{1/\alpha_j}|^2 \\ &= |z_j^{1/\alpha_j} - p_j^{1/\alpha_j}|^2 + 2\mathrm{Re}\,\bar{p}_j^{1/\alpha_j}(z_j^{1/\alpha_j} - p_j^{1/\alpha_j}) + |p_j|^{2/\alpha_j} \end{aligned}$$

for $1 \leq j \leq l$. Put

$$\rho_j(z) = \begin{cases} r(z) - |z_j|^{2/\alpha_j} + 2\mathrm{Re}\,\bar{p}_j^{1/\alpha_j}(z_j^{1/\alpha_j} - p_j^{1/\alpha_j}) + |p_j|^{2/\alpha_j} & \text{if } 1 \leq j \leq l \\ r(z) - |z_j|^{2/\alpha_j} & \text{if } j > l \end{cases}$$

Notice that

$$r(z) - |z_j|^{2/\alpha_j} = |z_1|^{2/\alpha_1} + \cdots + |z_{j-1}|^{2/\alpha_{j-1}} + |z_{j+1}|^{2/\alpha_{j+1}} + \cdots + |z_n|^{2/\alpha_n} + \varphi(z).$$

It follows that ρ_j is psh on $\Omega \cap U$ for each j and satisfies

$$\rho_j(z) \leq \begin{cases} -|z_j^{1/\alpha_j} - p_j^{1/\alpha_j}|^2 & \text{if } 1 \leq j \leq l \\ -|z_j|^{2/\alpha_j} & \text{if } j > l. \end{cases}$$

Let $\rho = \sum_{j=1}^n \rho_j$. Then ρ is a psh peak function for $\Omega \cap U$ at p and clearly, it is Hölder continuous in U. To get the global psh peak function, we take $\tilde{\rho} = \max\{\rho, -\delta\}$ for suitable fixed constant $\delta > 0$, and extend ρ as constant $-\delta$ where it is not defined. Thus Theorem 1 applies.

2. Let Ω be defined as above. Furthermore, we assume $0 < \alpha_j \leq 1$ for all $1 \leq j \leq n$. Let p be any boundary point. Note that for any $z \in \Omega$ one has

$$(-r(z) + |z_j|^{2/\alpha_j})^{\alpha_j} > |z_j|^2 = |z_j - p_j|^2 + 2\mathrm{Re}\,\bar{p}_j(z_j - p_j) + |p_j|^2$$

if $1 \leq j \leq l$. Set

$$\rho_{j,p} = |p_j|^2 + 2\mathrm{Re}\,\bar{p}_j(z_j - p_j) - (-r(z) + |z_j|^{2/p_j})^{\alpha_j}.$$

Clearly $\rho_{j,p}$ is psh since $0 < \alpha_j \leq 1$, and also $-\rho_{j,p}(z) > |z_j - p_j|^2$ holds on Ω. It follows that the function $\rho_p := \sum_{j=1}^n \rho_{j,p}$ is a psh peak function which satisfies the hypothesis of Theorem 2 because r is Hölder continuous in $\mathbf{C}^n$. Then Theorem 2 applies.

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Remarks. If we let $\varphi \equiv -1$ in the above example, then Ω is convex and the function defined by

$$\tilde{r}(z) = \left(\sup_{p \in \partial\Omega} (\rho_p(z) + |z-p|^2) \right)^*$$

(where $*$ denotes the uppersemicontinuous regularization) is a psh function on Ω with the following holds

(i) $-C\delta_\Omega^\alpha(z) \leq \tilde{r}(z) < 0$ where $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$;

(ii) $\partial\bar{\partial}\tilde{r} \geq \partial\bar{\partial}|z|^2$ holds in the distribution sense.

Due to a result of Sibony [24], we may bound the Kobayashi metric from below by $C|X|/\delta_\Omega^{\alpha/2}(z)$. Combining the well-known fact that the Caratheodory metric coincides with the Kobayashi metric for convex domains [8] together with the fact that the Bergman metric is always no less than the Caratheodory metric [15], we immediately obtain a sharp estimate for the Bergman metric

$$B_\Omega(z; X) \geq C|X|/\delta_\Omega^{\alpha/2}(z).$$

3. In [11], Diederich-Ohsawa obtained a quantitative estimate for the Bergman distance for bounded pseudoconvex domains in $\mathbf{C}^n$, on which there exists a bounded C^∞ strictly psh exhaustion function ρ on Ω such that

$$\frac{1}{c_1}\delta_\Omega^{1/c_2}(z) \leq -\rho(z) \leq c_1\delta_\Omega^{c_2}(z)$$

holds for suitable constants $c_1, c_2 > 0$. With the method we used in proving Theorem 1, the above condition can be weakened to only assume the existence of a bounded psh exhaustion function which is Hölder continuous on $\overline{\Omega}$.

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