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ON TWO PAIRS OF NON-SELF HYBRID MAPPINGS

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Abstract

In this paper we obtain some results on coincidence and common fixed points for two pairs of multi-valued and single-valued non-self mappings in complete convex metric spaces. We improve on previously used methods of proof and obtain results for mappings which are not necessarily compatible and not necessarily continuous, generalizing some known results. In particular, a theorem by Rhoades [19] and a theorem by Ahmed and Rhoades [2] are generalized and improved.

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1. Introduction

Let (X, d) be a complete metric space and $CB(X)$ be the set of all non-empty closed bounded subsets of X. Denote by H the Hausdorff metric induced by the metric d, and for any $x \in X$ and $A \subseteq X$ set $D(x, A) = \inf\{d(x, y) : y \in A\}$.

Extending the Banach contraction principle, Nadler [16] and Markin [15] first initiated the study of fixed point theorems for multi-valued contraction self-mappings. Assad and Kirk [4] first studied fixed point theorems for multi-valued contraction non-self mappings in a complete metrically convex metric space (X, d), using the fact that if K is any non-empty closed subset of X then for each $x \in K$ and $y \notin K$ there exists a point $z \in \partial K$ (the boundary of K) such that $d(x, z) + d(z, y) = d(x, y)$.

In the last four decades several authors have proved some fixed point or common fixed point theorems for self-mappings (see, for example, [5], [12, 13], [15–17], [20]), or for non-self mappings (see, for example, [1–4, 9], [6–8, 10, 11], [14], [18, 19], [21].) Some applications of non-self mappings are given by Assad and Kirk [4] and by Tsachev and Angelov [21].

Recently Ahmed and Rhoades [2] generalized and improved the result of Rhoades [19]. They proved the following theorem.

THEOREM 1.1. *(Ahmed and Rhoades [2]). Let (X, d) be a complete metrically convex metric space and K be a non-empty closed convex subset of X. If mappings $F, G : K \to CB(X)$ and $S, T : K \to X$ satisfy the following condition*

(1.1)

$$H(Fx, Gy) \leq h \max\left\{\frac{d(Sx, Ty)}{a}, D(Sx, Fx), D(Ty, Gy), \frac{D(Sx, Gy) + D(Ty, Fx)}{a+h}\right\}$$

for all x, y in X, where

$$0 < h < \frac{-1+\sqrt{5}}{2}, \quad a \geq 1 + \frac{2h^2}{1+h}$$

and

(i) $\partial K \subseteq SK \cap TK$, $FK \cap K \subseteq TK$, $GK \cap K \subseteq SK$,
(ii) $Tx \in \partial K$ *implies* $Fx \subseteq K$, $Sx \in \partial K$ *implies* $Gx \subseteq K$,
(iii) (F, S) *and* (G, T) *are compatible mappings,*
(iv) F, G, S, T *are continuous on* K,

then there exists a point z in X such that

$$Sz = Tz \in Fz \cap Gz.$$

The purpose of this paper is to generalize the theorem of Rhoades [19] and Theorem 1.1, weakening the condition for the coefficient h and removing the hypotheses of compatibility and continuity for the mappings. We prove coincidence and common fixed point theorems for not necessarily compatible and not necessarily continuous non-self mappings F, G, S and T, which satisfy (1.1) with $0 < h < 2/3$. To accomplish that, we improve on the methods of proof used by Rhoades [19] and Ahmed and Rhoades [2].

2. Results

THEOREM 2.1. *Let (X, d) be a metrically convex metric space and K be a non-empty closed subset of X. If mappings $F, G : K \to CB(X)$ and $S, T : K \to X$*

satisfy the following condition

(2.1)

$$H(Fx, Gy) \le h \max\left\{\frac{d(Sx, Ty)}{a}, D(Sx, Fx), D(Ty, Gy), \frac{D(Sx, Gy) + D(Ty, Fx)}{a+h}\right\}$$

for all x, y in X, where

$$0 < h < \frac{2}{3}, \quad a \ge 1 + \frac{2h^2}{1+h}$$

and

(i) $\partial K \subseteq SK \cap TK$,

(ii) $FK \cap K \subseteq TK$, $GK \cap K \subseteq SK$,

(iii) $Sx \in \partial K$ *implies* $Fx \subseteq K$, $Tx \in \partial K$ *implies* $Gx \subseteq K$

and $S(K)$ and $T(K)$ are complete, then there exist points u and w in K such that

$$Su \in Fu, \quad Tw \in Gw, \quad Su = Tw \quad \text{and} \quad Fu = Gw.$$

PROOF. If $F(K)$, $G(K)$, $S(K)$, $T(K) \subseteq K$ then the Theorem holds without the hypothesis of convexity of X and under a contractive condition weaker than condition (2.1). The proof in that instance is much simpler, since Cases 2 and 3 do not occur. We will give a proof under the hypothesis that each of the mappings F, G, S and T is not necessarily a self-mapping.

Let $x \in \partial K$ be arbitrary. We construct three sequences: $\{x_n\}$ and $\{z_n\}$ in K and a sequence $\{y_n\}$ in X in the following way. Set $z_0 = x$. Since $z_0 \in \partial K$, by (i) there exist points $x_0', x_0'' \in K$ such that $Sx_0' = Tx_0'' = z_0$. Consider the choice $z_0 = Sx_0'$. In this case we denote $x_0 = x_0'$. Now $Sx_0 \in \partial K$ implies $Fx_0 \subseteq K$, so we conclude that $Fx_0 \subseteq K \cap FK$. Then from (ii), $Fx_0 \subseteq TK$. Thus there exists an $x_1 \in K$ such that $Tx_1 \in Fx_0 \subseteq K$. Set $z_1 = y_1 = Tx_1$. Let c be any real number such that

$$1 < c \quad \text{and} \quad ch = q < \frac{2}{3} \tag{2.2}$$

(for example, $c = 1 + 2/3 - h$). Since $y_1 \in Fx_0 \in CB(X)$, from Nadler [16] we know that there exists a point $y_2 \in Gx_1$ such that

$$d(y_1, y_2) \le cH(Fx_0, Gx_1).$$

If $y_2 \in GK \cap K$ then from (ii) we have $y_2 \in SK$ and so there is a point $x_2 \in K$ such that $Sx_2 = y_2 = z_2 \in Gx_1$.

If $y_2 \notin K$ then by z_2 we denote a point in ∂K such that

$$d(y_1, z_2) + d(z_2, y_2) = d(y_1, y_2).$$

There are two possibilities : $d(z_2, y_2) \leq (1/2)d(y_1, y_2)$ or $d(z_2, y_1) < (1/2)d(y_1, y_2)$.

If the first possibility occurs, then we choose $x_2 \in K$ such that $Sx_2 = z_2$. This choice is possible by (i), as $z_2 \in \partial K \subseteq SK$. Now we choose a point $y_3 \in Fx_2$ such that

$$(2.3) \qquad d(y_2, y_3) \leq cH(Gx_1, Fx_2).$$

Since $Fx_2 \in FK \cap K \subseteq SK$, it follows from (ii) that there is a point $x_3 \in K$ such that $Tx_3 = y_3$.

If the second possibility occurs, that is, if $d(z_2, y_1) < (1/2)d(y_1, y_2)$, then by x_2 we denote a point in K such that $Tx_2 = z_2$. This choice is possible because $z_2 \in \partial K \subseteq SK \cap TK$. Since $y_1 \in Fx_0$, we can choose a point $y_3 \in Gx_2 \subseteq K$ such that

$$(2.4) \qquad d(y_1, y_3) \leq cH(Fx_0, Gx_2).$$

Since $y_3 \in Gx \cap K \subseteq SK$, there is a point $x_3 \in K$ such that $Sx_3 = y_3$.

Continuing the foregoing procedure we construct sequences $\{x_n\} \subseteq K$, $\{z_n\} \subseteq K$ and $\{y_n\} \subseteq FK \cup GK$ such that:

(i) $y_n \in Fx_{n-1}$ or $y_n \in Gx_{n-1}$,

(ii) $z_n = Sx_n$ or $z_n = Tx_n$,

(iii) $y_n = z_n$ if and only if $y_n \in K$ and in this case :
if $y_n \in Fx_{n-1}$ then $z_n = Tx_n$ and $y_{n+1} \in Gx_n$ is such that

$$d(y_n, y_{n+1}) \leq cH(Fx_{n-1}, Gx_n),$$

or, if $y_n \in Gx_{n-1}$ then $z_n = Sx_n$ and $y_{n+1} \in Fx_n$ is such that

$$d(y_n, y_{n+1}) \leq cH(Gx_{n-1}, Fx_n),$$

(iv) $y_n \neq z_n$ whenever $y_n \notin K$ and then $z_n \in \partial K$ is such that

$$d(y_{n-1}, z_n) + d(z_n, y_n) = d(y_{n-1}, y_n)$$

and

(iv.i) if $d(z_n, y_n) \leq (1/2)d(y_{n-1}, y_n)$, then
if $y_n \in Fx_{n-1}$ then $z_n = Tx_n$ and $y_{n+1} \in Gx_n$ is such that

$$d(y_n, y_{n+1}) \leq H(Fx_{n-1}, Gx_n)$$

or, if $y_n \in Gx_{n-1}$ then $z_n = Sx_n$ and $y_{n+1} \in Fx_n$ is such that

$$d(y_n, y_{n+1}) \leq H(Gx_{n-1}, Fx_n),$$

(iv.ii) if $d(y_{n-1}, z_n) < (1/2)d(y_{n-1}, y_n)$, then

if $y_n \in Fx_{n-1}$, then $z_n = Sx_n$ and $y_{n+1} \in Fx_n$ is such that

$$d(y_{n-1}, y_{n+1}) \leq cH(Gx_{n-2}, Fx_n),$$

or, if $y_n \in Gx_{n-1}$ then $z_n = Tx_n$ and $y_{n+1} \in Gx_n$ is such that

$$d(y_{n-1}, y_{n+1}) \leq cH(Fx_{n-2}, Gx_n).$$

OBSERVATION. If $z_n \neq y_n$ then $z_n \in \partial K$, which implies $z_{n+1} = y_{n+1} \in K$. This implies that also $z_{n-1} = y_{n-1} \in K$, since otherwise $z_{n-1} \in \partial K$ which implies that $z_n = y_n \in K$.

Now we wish to estimate $d(z_n, z_{n+1})$. If $d(z_n, z_{n+1}) = 0$ for some n then it is easy to show that $z_{n+k} = z_n$ for all $k \geq 1$.

Suppose that $d(z_n, z_{n+1}) > 0$ for all n. From the Observation we conclude that there are three possibilities.

Case 1. Let $z_n = y_n \in K$ and $z_{n+1} = y_{n+1} \in K$. Suppose, without loss of generality, that $z_n = y_n = Tx_n \in Fx_{n-1}$. Then $z_{n+1} = y_{n+1} \in Gx_n$, $z_{n-1} = Tx_{n-1}$ (observe that z_{n-1} is not necessarily equal to y_{n-1}) and points y_n and y_{n+1} are chosen such that $d(y_n, y_{n+1}) \leq cH(Fx_{n-1}, Gx_n)$. Then from (2.1) and (2.2),

$$\begin{aligned}
(2.5)\quad d(y_n, y_{n+1}) &\leq cH(Fx_{n-1}, Gx_n) \\
&\leq q \max\left\{\frac{d(Sx_{n-1}, Tx_n)}{a}, D(Sx_{n-1}, Fx_{n-1}), D(Tx_n, Gx_n),\right. \\
&\qquad \left.\frac{D(Sx_{n-1}, Gx_n) + D(Tx_n, Fx_{n-1})}{a+h}\right\} \\
&\leq q \max\left\{\frac{d(z_{n-1}, z_n)}{a}, d(z_{n-1}, z_n), d(z_n, z_{n+1}), \frac{d(z_{n-1}, z_{n+1})}{a+h}\right\}.
\end{aligned}$$

Note that if (2.1) holds for some $h_1 > 0$ then it also holds for any h such that $h_1 \leq h < 2/3$. Thus we may suppose that $3/5 \leq h < 2/3$, which implies that

$$\frac{1}{a} < \frac{3}{4} \quad \text{and} \quad \frac{1}{a+h} < \frac{1}{2}.$$

Thus from (2.1),

(2.6)

$$H(Fx, Gy) \leq h \max\left\{\frac{3d(Sx, Ty)}{4}, D(Sx, Fx), D(Ty, Gy), \frac{D(Sx, Gy) + D(Ty, Fx)}{2}\right\}.$$

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From (2.5) and (2.6),

$$\begin{aligned} d(y_n, y_{n+1}) &\le q \max\left\{d(z_{n-1}, z_n), d(z_n, z_{n+1}), \frac{d(z_{n-1}, z_n) + d(z_n, z_{n+1})}{2}\right\} \\ &\le q \max\{d(z_{n-1}, z_n), d(z_n, z_{n+1})\}. \end{aligned}$$

Hence, as $z_n = y_n$, $z_{n+1} = y_{n+1}$ and $q < 2/3$,

$$d(y_n, y_{n+1}) \le q d(z_{n-1}, z_n). \tag{2.7}$$

Note that (2.7) holds whenever $y_n = z_n$, without regard to $y_{n+1} = z_{n+1}$ or $y_{n+1} \neq z_{n+1}$.

From (2.7) it immediately follows, since now $y_{n+1} = z_{n+1}$, that

$$d(z_n, z_{n+1}) \le q d(z_{n-1}, z_n). \tag{2.8}$$

Case 2. Let $z_n = y_n \in K$ but $z_{n+1} \neq y_{n+1}$. Then $z_{n+1} \in \partial K$ is such that

$$d(y_n, z_{n+1}) + d(z_{n+1}, y_{n+1}) = d(y_n, y_{n+1}).$$

Thus $d(z_n, z_{n+1}) = d(y_n, z_{n+1}) < d(y_n, y_{n+1})$ and from (2.7),

$$d(z_n, z_{n+1}) \le q d(z_{n-1}, z_n). \tag{2.9}$$

Case 3. Let $z_n \neq y_n$. Then $z_n \in \partial K$, and

$$d(y_{n-1}, z_n) + d(z_n, y_n) = d(y_{n-1}, y_n). \tag{2.10}$$

Also, by the Observation, $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$.

If we suppose that $d(z_n, z_{n+1}) \le d(y_{n-1}, y_n)$ then from (2.7),

$$d(z_n, z_{n+1}) \le q d(z_{n-2}, z_{n-1}), \tag{2.11}$$

so we shall only consider the case

$$d(z_n, z_{n+1}) > d(y_{n-1}, y_n). \tag{2.12}$$

From (2.10) it follows that there are two possibilities:

$$d(z_n, y_n) \le \frac{d(y_{n-1}, y_n)}{2} \tag{2.13}$$

or

$$d(z_n, y_{n-1}) < \frac{d(y_{n-1}, y_n)}{2}. \tag{2.14}$$

Case 3a. Suppose that (2.13) holds and, without loss of generality, assume that $y_n \in Fx_{n-1}$. Then $y_{n-1} = z_{n-1} = Sx_{n-1} \in Gx_{n-2}$ and $y_n \neq z_n = Tx_n$ and, by construction of $\{y_n\}$ (see (iv.i)), $y_{n+1} = z_{n+1} = Sx_{n+1} \in Gx_n$ is such that

$$d(y_n, y_{n+1}) \leq cH(Fx_{n-1}, Gx_n).$$

Thus from (2.6) and (2.2),

(2.15)

$$\begin{aligned} d(y_n, y_{n+1}) &\leq cH(Fx_{n-1}, Gx_n) \\ &\leq q \max\left\{d(Sx_{n-1}, Tx_n), D(Sx_{n-1}, Fx_{n-1}), D(Tx_n, Gx_n),\right. \\ &\qquad \left.\frac{D(Sx_{n-1}, Gx_n) + D(Tx_n, Fx_{n-1})}{2}\right\} \\ &\leq q \max\left\{d(y_{n-1}, z_n), d(y_{n-1}, y_n), d(z_n, z_{n+1}), \frac{d(y_{n-1}, z_{n+1}) + d(z_n, y_n)}{2}\right\}. \end{aligned}$$

Since, from (2.10) and (2.12),

$$\begin{aligned} d(y_{n-1}, z_n) < d(y_{n-1}, y_n) &< d(z_n, z_{n+1}); \\ d(y_{n-1}, z_{n+1}) + d(z_n, y_n) &\leq d(z_n, y_n) + d(y_{n-1}, z_n) + d(z_n, z_{n+1}) \\ &= d(y_{n-1}, y_n) + d(z_n, z_{n+1}) < 2d(z_n, z_{n+1}), \end{aligned}$$

from (2.15) we deduce

$$d(y_n, y_{n+1}) \leq qd(z_n, z_{n+1}). \tag{2.16}$$

Since $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$, from (2.13), (2.16) and (2.7),

$$\begin{aligned} d(z_n, z_{n+1}) \leq d(z_n, y_n) + d(y_n, y_{n+1}) &\leq \frac{d(y_{n-1}, y_n)}{2} + qd(z_n, z_{n+1}) \\ &\leq \frac{q}{2}d(z_{n-2}, z_{n-1}) + \frac{2}{3}d(z_n, z_{n+1}). \end{aligned}$$

Hence

$$d(z_n, z_{n+1}) \leq \left(\frac{3}{2}\right)t\, d(z_{n-2}, z_{n-1}). \tag{2.17}$$

Similarly, if $y_n \in Gx_{n-1}$ then one can show that (2.17) holds.

Case 3b. Consider now the second possibility, namely the case when (2.14) holds. We may suppose, without loss of generality, that $y_n \in Fx_{n-1}$. Then, by construction of $\{x_n\}$ (see (iv.ii)), we now have that $x_n \in K$ is chosen such that $Sx_n = z_n$, and $y_{n+1} = z_{n+1} \in Fx_n$ is such that $d(y_{n-1}, y_{n+1}) \leq cH(Gx_{n-2}, Fx_n)$.

Thus from (2.6),

(2.18)

$$\begin{aligned}
d(y_{n-1}, y_{n+1}) \\
&\le cH(Gx_{n-2}, Fx_n) \\
&\le q \max\left\{\frac{3}{4}d(Tx_{n-2}, Sx_n), D(Tx_{n-2}, Gx_{n-2}), D(Sx_n, Fx_n),\right. \\
&\qquad \left.\frac{D(Tx_{n-2}, Fx_n) + D(Sx_n, Gx_{n-2})}{2}\right\} \\
&\le q \max\left\{\frac{3}{4}d(z_{n-2}, z_n), d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1}), \frac{d(z_{n-2}, y_{n+1}) + d(z_n, y_{n-1})}{2}\right\}.
\end{aligned}$$

Since $y_{n-1} = z_{n-1}$, from (2.14) and (2.7),

$$d(z_n, y_{n-1}) < \frac{1}{2}d(y_{n-1}, y_n) \le \frac{q}{2}d(z_{n-2}, z_{n-1}) < \frac{1}{3}d(z_{n-2}, z_{n-1}),$$

$$d(z_{n-2}, z_n) \le d(z_{n-2}, z_{n-1}) + d(y_{n-1}, z_n) < \frac{4}{3}d(z_{n-2}, z_{n-1}),$$

$$d(z_{n-2}, y_{n+1}) \le d(z_{n-2}, y_{n-1}) + d(y_{n-1}, y_{n+1}).$$

Thus from (2.18),

(2.19)

$$d(y_{n-1}, y_{n+1}) \le q \max\left\{d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1}), \left[\frac{2}{3}d(z_{n-2}, z_{n-1}) + \frac{1}{2}d(y_{n-1}, y_{n+1})\right]\right\}.$$

If we suppose that $d(y_{n-1}, y_{n+1}) > q \max\{d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1})\}$, then from (2.19),

$$d(y_{n-1}, y_{n+1}) \le \frac{2q}{3}d(z_{n-2}, z_{n-1}) + \frac{q}{2}d(y_{n-1}, y_{n+1}) \le \frac{2q}{3}d(z_{n-2}, z_{n-1}) + \frac{1}{3}d(y_{n-1}, y_{n+1})$$

and hence

$$d(y_{n-1}, y_{n+1}) \le q d(z_{n-2}, z_{n-1}),$$

a contradiction. Thus,

$$d(y_{n-1}, y_{n+1}) \le q \max\{d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1})\}. \tag{2.20}$$

Using the triangle inequality together with (2.14), (2.20), (2.7) and the fact that $y_{n+1} = z_{n+1}$ and $y_{n-1} = z_{n-1}$, we obtain

$$\begin{aligned}
d(z_n, y_{n+1}) &\le d(y_{n-1}, z_n) + d(y_{n-1}, y_{n+1}) \le \frac{1}{2}d(y_{n-1}, y_n) + d(y_{n-1}, y_{n+1}) \\
&\le \frac{q}{2}d(z_{n-2}, z_{n-1}) + q \max\{d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1})\} \\
&\le \frac{q}{2}d(z_{n-2}, z_{n-1}) + \max\left\{q\, d(z_{n-2}, z_{n-1}), \frac{2}{3}d(z_n, z_{n+1})\right\}.
\end{aligned}$$

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Hence

$$d(z_n, z_{n+1}) \leq \frac{3q}{2} d(z_{n-2}, z_{n-1}). \tag{2.21}$$

Similarly, if $y_n \in Gx_{n-1}$ then (2.21) holds.

From (2.11), (2.17) and (2.21) we see that in Case 3,

$$d(z_n, z_{n+1}) \leq \frac{3q}{2} d(z_{n-2}, z_{n-1}) \quad \text{for all } n \geq 2. \tag{2.22}$$

From (2.8), (2.9) and (2.22) we conclude that in each of the cases considered we have

$$d(z_n, z_{n+1}) \leq \frac{3q}{2} \max\{d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1})\} \quad \text{for all } n \geq 2.$$

Now, proceeding on the lines of Ćirić [5] and using the fact that $(3/2)q < 1$, it can be shown that the sequence $\{z_n\}$ is a Cauchy sequence and hence converges to some point $z \in K$. Since $S(K)$ and $T(K)$ are complete, there are points u and w in K such that $Su = z$ and $Tw = z$. By the construction of $\{z_n\}$, at least one of its subsequences $\{z_{n(j)}\}$ or $\{z_{n(k)}\}$, defined by $z_{n(j)} = Sx_{n(j)} \in Gx_{n(j)-1}$, or by $z_{n(k)} = Tx_{n(k)} \in Fx_{n(k)-1}$, respectively, is infinite.

Suppose that the sequence $\{z_{n(j)}\}$, with $z_{n(j)} \in Gx_{n(j)-1}$, is infinite. For convenience, denote $z_{n(j)}$, $y_{n(j)}$, $x_{n(j)}$ and $x_{n(j)-1}$ by z_j, y_j, x_j and x_{j-1}, respectively. Then we have $z_j = y_j = Sx_j \in Gx_{j-1}$ and $Tx_{j-1} = z_{j-1} \in K$ (note that $z_{j-1} = y_{j-1} \in Fx_{j-2}$, or $y_{j-1} \neq z_{j-1} \in \partial K$).

Since $z_j \in Gx_{j-1}$, from (2.1) we deduce that

$$\begin{aligned} D(Fu, z_j) &\leq H(Fu, Gx_{j-1}) \\ &\leq h \max\left\{\frac{d(Su, Tx_{j-1})}{a}, D(Su, Fu), D(Tx_{j-1}, Gx_{j-1}),\right. \\ &\quad \left.\frac{D(Su, Gx_{j-1}) + D(Tx_{j-1}, Fu)}{a+h}\right\} \\ &\leq h \max\left\{\frac{d(z, z_{j-1})}{a}, D(z, Fu), d(z_{j-1}, z_j), \frac{d(z, z_j) + D(z_{j-1}, Fu)}{a+h}\right\}. \end{aligned}$$

Taking the limit as $j \to \infty$, we get $D(Fu, z) \leq hD(z, Fu)$ and hence $D(Fu, z) = 0$. Since Fu is closed, $z \in Fu$. Thus we have proved that $Su \in Fu$.

Similarly, if $\{z_{n(k)}\}$, with $z_{n(k)} \in Fx_{n(k)-1}$, is infinite then $Tw \in Gw$.

Consider now the case when $\{z_{n(k)}\}$, with $z_{n(k)} \in Fx_{n(k)-1}$, is finite. In this case we will construct a new infinite sequence $\{u_n\}$ in $FK \cap K$ such that $\lim_{n\to\infty} u_n = z$. Since $\{z_{n(k)}\}$ is finite, it follows that there is some n_0 such that $z_n \in K \backslash FK$ for all $n > n_0$.

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Hence there is an infinite subsequence $\{z_{n(i)}\}$ of $\{z_n\}$, defined by $z_{n(i)} \in \partial K$. For convenience, we denote $x_{n(i)}, z_{n(i)}, z_{n(i)\pm 1}$ and $z_{n(i)-2}$ by $x_i, z_i, z_{i\pm 1}$ and z_{i-2}, respectively. Now, parallel with the already considered points $x_i \in K$, $z_i = Tx_i$, $z_{i+1} = y_{i+1} \in Gx_i$ (see Case 3a), we choose a new point $x_i \in K$ such that $Sx_i = z_i \in \partial K$ and a point $u_{i+1} \in Fx_i \subseteq K$ (see Case 3b) such that

$$d(z_{i-1}, u_{i+1}) \leq cH(Gx_{i-2}, Fx_i).$$

This choice is possible because $z_{i-1} \in Gx_{i-2}$.

As in Case 3b (see (2.20)), it can be shown that

$$d(z_{i-1}, u_{i+1}) \leq q \max\{d(z_{i-2}, z_{i-1}), d(z_i, u_{n+1})\}.$$

Hence, as $z_n \to z$, it follows that $u_i \to z$. From (2.1),

$$\begin{aligned} D(u_{i+1}, Gw) &\leq H(Fx_i, Gw) \\ &\leq h \max\left\{\frac{d(z_i, z)}{a}, d(z_i, u_{i+1}), D(z, Gw), \frac{D(z_i, Gw) + d(z, u_{i+1})}{a+h}\right\}. \end{aligned}$$

Taking the limit as $i \to \infty$, we get $D(z, Gw) \leq hD(z, Gw)$ and hence $D(z, Gw) = 0$. Therefore

$$z = Tw \in Gw.$$

Now from (2.1) we have $H(Fu, Gw) = 0$. Hence $Fu = Gw$. Thus we have proved that

$$Su \in Fu, \quad Tw \in Gw, \quad Su = Tw \quad \text{and} \quad Fu = Gw. \qquad \square$$

We note that a coincidence of F and S and a coincidence of G and T need not be the same point, even if F, G, S and T are single-valued self mappings. The following example shows this.

EXAMPLE 1. Let X be the Euclidean space $[0, \infty)$ with the usual metric. Define F, G, S and T on X as follows :

$$Fx = x^2 + 7/64, \quad Gx = x^3 + 7/64, \quad Sx = 8x^2 \quad \text{and} \quad Tx = 8x^3.$$

Then

$$d(Fx, Gy) = |x^2 - y^3| \leq \frac{8|x^2 - y^3|}{4} = \frac{d(Sx, Ty)}{4}.$$

Thus (2.1) holds for all $x, y \in X$. Also the other hypotheses (i) and (ii) are satisfied. It is easy to see that $F(1/8) = S(1/8) = 1/8$ and $G(1/4) = T(1/4) = 1/8$. Therefore F and S have a coincidence at the point $u = 1/8$ and G and T have a coincidence at the point $w = 1/4$ and $F(1/8) = G(1/4)$.

In Theorem 2.1, if we set $S = T = id$ (where id is the identity mapping), we obtain the following result.

COROLLARY 2.2. *Let (X, d) be a complete and metrically convex metric space and K be a non-empty closed subset of X. If mappings $F, G : K \to CB(X)$ satisfy*

$$H(Fx, Gy) \le h \max\left\{\frac{d(x, y)}{a}, D(x, Fx), D(y, Gy), \frac{D(x, Gy) + D(y, Fx)}{a + h}\right\}$$

for all x, y in X, where

$$0 < h < \frac{2}{3} \quad \text{and} \quad a \ge 1 + \frac{2h^2}{1 + h},$$

and $x \in \partial K$ implies $Fx \subseteq K$ and $Gx \subseteq K$, then there exists a point z in K such that $z \in Fz \cap Gz$ and $Fz = Gz$.

In Corollary 2.2, if we set $G = F$, we obtain the following generalization of the Rhoades result in [19].

COROLLARY 2.3. *Let (X, d) be a complete metrically convex metric space and K a non-empty closed subset of X. Let $F : K \to CB(X)$ be a non-self mapping satisfying*

$$H(Fx, Fy) \le h \max\left\{\frac{d(x, y)}{a}, D(x, Fx), D(y, Fy), \frac{D(x, Fy) + D(y, Fx)}{a + h}\right\}$$

for all x, y in X, where

$$0 < h < \frac{2}{3} \quad \text{and} \quad a \ge 1 + \frac{2h^2}{1 + h}.$$

If $Fx \subseteq K$ for each $x \in \partial K$, then F has a fixed point.

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