Canad. Math. Bull.Vol. 36 (1), 1993 pp. 15–21

DESCRIPTIONS OF THE CHARACTERISTIC SEQUENCE OF AN IRRATIONAL

TOM C. BROWN

ABSTRACT. Let α be a positive irrational real number. (Without loss of generality assume $0 < \alpha < 1$.) The *characteristic sequence* of α is

$$f(\alpha) = f_1 f_2 \cdots, \text{ where } f_n = [(n+1)\alpha] - [n\alpha].$$

We make some observations on the various descriptions of the characteristic sequence of α which have appeared in the literature. We then refine one of these descriptions in order to obtain a very simple derivation of an arithmetic expression for $[n\alpha]$ which appears in A. S. Fraenkel, J. Levitt, and M. Shimshoni [17]. Some concluding remarks give conditions on n which are equivalent to $f_n = 1$.

1. **Introduction.** Let α be an irrational real number, $0 < \alpha < 1$. The *characteristic sequence* associated with α is the sequence

$$f(\alpha) = f_1 f_2 \cdots, \text{ where } f_n = [(n+1)\alpha] - [n\alpha], \quad n \geq 1.$$

(This terminology is due to E. B. Christoffel [10].)

Note that $f(\alpha)$ is a sequence of 0's and 1's, and that $[k\alpha] = f_1 + f_2 + \cdots + f_{k-1}$, $k \geq 2$. (If desired, one can allow $\alpha > 1$ by setting $f_n = [(n+1)\alpha] - [n\alpha] - [\alpha]$. In this case, $[k\alpha] = f_1 + f_2 + \cdots + f_{k-1} + k[\alpha]$.)

Facts 1, 2, 3 below are explicit descriptions, which have already appeared in the literature, of the sequence $f(\alpha)$, and each can be used to generate arbitrarily long initial segments of $f(\alpha)$. Fact 2 is undoubtedly the easiest of the three to use for this purpose. Fact 4 is concerned with the sequence $f(\alpha)$ when α is a quadratic irrational.

2. **Statements of Facts 1–4.**

DEFINITION 1. Let α be an irrational real number with $0 < \alpha < 1$. Let $[0, a_1, a_2, \ldots]$ be the simple continued fraction for α, that is,

$$\alpha = [0, a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

Let

$$\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n], \quad n \geq 1,$$

Partially supported by NSERC.
Received by the editors July 15, 1991; revised March 3, 1992.
AMS subject classification: 10L10.
© Canadian Mathematical Society 1993.

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

and let

$$X_n = f_1 f_2 \cdots f_{q_n}, \quad n \geq 1.$$

Thus X_n is the initial segment of $f(\alpha)$ of length q_n.

It is standard to define $p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1$, so that

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1.$$

This fact will be used throughout the remainder of the paper.

DEFINITION 2. For each $n \geq 1$, define the irrational number α_n by $\alpha = [0, a_1, a_2, \ldots, a_n + \alpha_n]$.

DEFINITION 3. Let G be the free group generated by the symbols 0, 1. Thus the elements of G are blocks or words in the symbols 0, 1 (and their inverses), and the identity element of G is the empty word. We define, for each $t \geq 1$, homomorphisms k_t and h_t from G into G by setting $k_t(0) = 0^{t-1}1$, $k_t(1) = 0^t 1$ and $h_t(0) = 0^{t-1}1$, $h_t(1) = 0^{t-1}10$. Thus $k_t(w_1 w_2 \cdots w_n) = k_t(w_1)k_t(w_2)\cdots k_t(w_n)$, and similarly for h_t. Furthermore, we extend k_t and h_t to act on infinite binary sequences by setting $k_t(w_1 w_2 \cdots) = k_t(w_1)k_t(w_2)\cdots$, and similarly for h_t.

FACT 1. For each $m \geq 1$, define

$$c_m = k_{a_1} \circ k_{a_2} \circ \cdots \circ k_{a_m}(0),$$

where $\circ$ denotes composition. Then for each $n \geq 1$,

$$f(\alpha) = (c_1 c_2 \cdots c_n)\Big(k_{a_1} \circ k_{a_2} \circ \cdots \circ k_{a_n}\big(f(\alpha_n)\big)\Big).$$

FACT 2. For each $n \geq 2$,

$$X_n = X_{n-1}^{a_n} X_{n-2},$$

where $X_0 = 0$ and $X_1 = 0^{a_1 - 1}1$. (Here $X_{n-1}^{a_n}$ denotes $X_{n-1}X_{n-1}\cdots X_{n-1}$, with a_n repetitions. If $a_1 = 1$, then $X_1 = 1$.)

FACT 3. For each $n \geq 1$,

$$f(\alpha) = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_n}\big(f(\alpha_n)\big).$$

FACT 4. Let h be the homomorphism from G to G defined by $h(0) = X_m$, $h(1) = X_m X_{m-1}$, and extend h to act also on infinite binary sequences. Then if α is purely periodic with period m, that is, $\alpha = [0, a_1, \ldots, a_m, a_1, \ldots, a_m, a_1, \ldots, a_m, \ldots]$, then $h\big(f(\alpha)\big) = f(\alpha)$. (As pointed out by J. Shallit [34], this can be deduced quickly from Fact 2 by showing by induction on n that $h(X_n) = X_{m+n}$ for all $n \geq 0$.)

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

3. **Historical remarks.** D. H. Fowler has argued persuasively ([11], [12], [13]) that sequences essentially equivalent to the sequence $f(\alpha)$ were studied in ancient times, in fact prior to the development of the Eudoxan theory of proportions. (Fowler notes that the sequence of occurrences of two periodic events is closely related to $f(\alpha)$, where α is the ratio of the periods of the two events.) (See also C. Series [32], R. C. Riddell [30] and W. R. Knorr [21].)

The earliest known explicit reference to characteristic sequences seems to be in the 1772 work of the astronomer J. Bernoulli [2], where he gave without proof a description of the sequence $g(\alpha) = g_1g_2\cdots$, where $g_n = [(n+1)\alpha+1/2]-[n\alpha+1/2]$. (A recipe for transforming $f(\alpha)$ into $g(\alpha)$ is given in Venkov [37].) In 1875 E. B. Christoffel [10] asserted without proof (essentially) Fact 1 above, which was proved by A. A. Markov [24] in 1882. In the meantime, H. J. S. Smith [35] in 1876 proved Fact 2 above.

Papers by M. Morse and G. Hedlund [26], G. Hedlund [19], H. Cohn [7], and F. Mignosi [25] deal with other aspects of characteristic sequences. (In most of these papers the adjective "Sturmian" is used rather than "characteristic.") For example, Morse and Hedlund show that for each n, the sequence $f(\alpha)$ contains exactly $n+1$ distinct blocks of length n, and they calculate exactly the minimum length $m(n)$ such that every block of length $m(n)$ in $f(\alpha)$ contains all of the distinct blocks of length n. Mignosi shows that $f(\alpha)$ contains k identical consecutive blocks for arbitrarily large k, if and only if the simple continued fraction for α has unbounded partial quotients.

K. B. Stolarsky [36], A. S. Fraenkel, M. Mushkin, and U. Tassa [18], J. Rosenblatt [31], and J. Shallit [34] gave new proofs of Fact 2. (Fact 2 appears implicitly in the first three of these four papers, explicitly in the fourth.)

S. Ito and S. Yasutomi [20], and K. Nishioka, I. Shiokawa, and J. Tamura [27], gave extensive results dealing with generalized characteristic sequences $h(\alpha,\beta) = h_1h_2\cdots$, where $h_n = [(n+1)\alpha+\beta] - [n\alpha+\beta]$, including Fact 2 and Fact 4.

Fact 3 appears in Brown [5].

Finally, we mention that Fact 4 was proved by no fewer than *six* independent parties: H. Cohn [7, 1974] (implicitly), S. Ito and S. Yasutomi [20, 1990], F. Laubie [22, 1991], J. Shallit [34, 1991], K. Nishioka, I. Shiokawa, and J. Tamura [27, 1992], and T. Brown [5, 1991].

Extensive lists of references can be found in [18], [29], and [36].

4. **Relations among Facts 1–3, and an extension of Fact 2.** Facts 1–3 are related to each other by Theorem 1 below. It will be seen during the proof of Theorem 1 that each of Facts 1 and 3 implies Fact 2.

LEMMA 1. *Define words $Y_0, Y_1, \ldots$ by $Y_0 = 0$, $Y_1 = 0^{a_1-1}1$, $Y_n = Y_{n-1}^{a_n}Y_{n-2}$, $n \geq 2$. Also define words $Z_0, Z_1, \ldots$ by $Z_0 = 0$, $Z_1 = 0^{a_1-1}1$, $Z_n = Z_{n-1}^{a_n-1}Z_{n-2}Z_{n-1}$, $n \geq 2$. Then for each $n \geq 2$,*

$$Z_n = (Y_{n-1}\cdots Y_1)^{-1}Y_n(Y_{n-1}\cdots Y_1).$$

PROOF. Induction on n.

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

THEOREM 1. *For each $n \geq 1$,*

$$c_1c_2\cdots c_n = X_nX_{n-1}\cdots X_1,$$
$$h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(0) = X_n,$$
$$h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(1) = X_nX_{n-1}.$$

PROOF. From Lemma 1, it follows that $Z_1Z_2\cdots Z_n = Y_nY_{n-1}\cdots Y_1$ for $n \geq 1$. Next, by induction on m, $c_m = Z_m$ for $m \geq 1$, so we now have $c_1c_2\cdots c_n = Y_nY_{n-1}\cdots Y_1$. By Fact 1, $c_1c_2\cdots c_n$ is an initial segment of $f(\alpha)$, so that Y_n is an initial segment of $f(\alpha)$. Finally, by induction on n, Y_n has length q_n, therefore $Y_n = X_n$, and finally $c_1c_2\cdots c_n = X_nX_{n-1}\cdots X_1$, $n \geq 1$.

(Also, note that since $Y_n = X_n$, it follows from the definition of Y_n that $X_n = X_{n-1}^{a_n}X_{n-2}$, $n \geq 2$, so that Fact 1 implies Fact 2.)

Next, with Y_n defined as above, it follows by induction that $h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(0) = Y_n$ and $h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(1) = Y_nY_{n-1}$, $n \geq 1$. Using $Y_n = X_n$ gives $h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(0) = X_n$ and $h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(1) = X_nX_{n-1}$, $n \geq 1$.

(Also, note that from Fact 3 and $h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_n}(0) = Y_n$, $h_{a_1}\circ\cdots\circ h_{a_n}(1) = Y_nY_{n-1}$, one can conclude that Y_n is an initial segment of $f(\alpha)$. Therefore (since Y_n has length q_n) $Y_n = X_n$, and by the definition of Y_n, $X_n = X_{n-1}^{a_n}X_{n-2}$, $n \geq 2$. Thus Fact 3 implies Fact 2.)

COROLLARY. *Fact 1 implies Fact 2, and Fact 3 implies Fact 2.*

Now we extend Fact 2 by using the "Zeckendorff number system". (See A. Ostrowski [28, 1922], C. G. Lekkerkerker [23, 1952], A. M. Yaglom and I. M. Yaglom [38, 1967], A. S. Fraenkel, J. Levitt, and M. Shimshoni [17, 1972], and E. Zeckendorff [39, 1972]. These systems of numeration have been generalized by A. S. Fraenkel and I. Borosh [16, 1973], A. S. Fraenkel [14, 1985], J. Shallit [33, 1988], and A. S. Fraenkel [15, 1989].)

Our extension (Theorem 2 below) enables one to give an exact formula for $\sum_{k=1}^{m}[k\alpha]$, for arbitrary m. This has been done in [6].

The next Lemma is well-known. (See for example A. S. Fraenkel [14, p. 111].) Recall that α is irrational, $\alpha = [0, a_1, a_2, \ldots]$, $\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n]$, $n \geq 1$, $q_{-1} = 0$, $q_0 = 1$, and $q_n = a_nq_{n-1} + q_{n-2}$, $n \geq 1$.

LEMMA 2. *Let $t \geq 1$. Then every m, $0 \leq m \leq q_t - 1$, has a unique representation in the form $m = z_tq_{t-1} + \cdots + z_2q_1 + z_1q_0$, where*

(1) $0 \leq z_1 \leq a_1 - 1$,

(2) $0 \leq z_i \leq a_i$, $\quad 2 \leq i \leq t$,

(3) $z_i = a_i \Rightarrow z_{i-1} = 0$, $\quad 2 \leq i \leq t$.

We then write $m = (z_t, \ldots, z_2, z_1)_\alpha$, and we call this the Zeckendorff representation of m.

THEOREM 2. *Let $m \geq 1$ be given, and let $m = (z_t, \ldots, z_2, z_1)_\alpha$. Then the initial segment of $f(\alpha)$ of length m is*

$$f_1f_2\cdots f_m = X_{t-1}^{z_t}\cdots X_1^{z_2}X_0^{z_1}.$$

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

(Note that for any i, X_i^0 is the empty word.)

PROOF. We use induction on m. For $1 \leq m \leq q_1 - 1 = a_1 - 1$ (if $a_1 > 1$), $m = m_{q_0}$ and $f_1 \cdots f_m = 0^m = X_0^m$.

Suppose the result is true for all $m < q_t$ for some fixed $t \geq 1$, and now let $q_t \leq m < q_{t+1}$, say $m = z_{t+1}q_t + r$, where $1 \leq z_{t+1} \leq a_{t+1}$ and $0 \leq r < q_t$.

If $r = 0$, then since $m = z_{t+1}q_t$ and $X_t^{a_{t+1}}X_{t-1} = X_{t+1}$ is an initial segment of $f(\alpha)$, it follows that $f_1 \cdots f_m = X_t^{z_{t+1}}$.

If $1 \leq r < q_t$, then by hypothesis we have $r = (z_t, \ldots, z_2, z_1)_\alpha$ and $f_1 \cdots f_r = X_{t-1}^{z_t} \cdots X_1^{z_2} X_0^{z_1}$.

Now there are two cases, depending on whether $z_{t+1} = a_{t+1}$ or $z_{t+1} < a_{t+1}$.

If $z_{t+1} = a_{t+1}$, then since $m = a_{t+1}q_t + r < q_{t+1} = a_{t+1}q_t + q_{t-1}$, we have $r < q_{t-1}$, so that $f_1 \cdots f_r$ is an initial segment of X_{t-1} (and $z_t = 0$). Then since $X_t^{a_{t+1}}X_{t-1} = X_{t+1}$ is an initial segment of $f(\alpha)$, we have $f_1 \cdots f_m = X_t^{a_{t+1}} f_1 \cdots f_r = X_t^{z_{t+1}} X_{t-1}^{z_t} \cdots X_1^{z_2} X_0^{z_1}$, and $m = (z_{t+1}, \ldots, z_2, z_1)_\alpha$.

The remaining case is $m = z_{t+1}q_t + r$, where $1 \leq z_{t+1} < a_{t+1}$ and $1 \leq r < q_t$. Then $f_1 \cdots f_m$ is an initial segment of $X_t^{z_{t+1}} X_t$, which is an initial segment of $X_t^{a_{t+1}}$ (which is an initial segment of $f(\alpha)$). Thus again $f_1 \cdots f_m = X_t^{z_{t+1}} f_1 \cdots f_r = X_t^{z_{t+1}} X_{t-1}^{z_t} \cdots X_1^{z_2} X_0^{z_1}$, and $m = (z_{t+1}, \ldots, z_2, z_1)_\alpha$.

This finishes the induction step and the proof of Theorem 2.

5. **A simple arithmetic expression for** $[m\alpha]$. Recall once again that X_n is the initial segment of $f(\alpha)$ of length q_n, where $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$.

LEMMA 3. *For each $n \geq 0$, the number of 1's in X_n is p_n.*

PROOF. This is a simple induction on n, using Fact 2 and $p_n = a_n p_{n-1} + p_{n-2}$.

The following theorem appears, in a somewhat more complex form, in A. S. Fraenkel, J. Levitt, and M. Shimshoni [17]. The proof given here is simpler.

THEOREM 3. *For $m \geq 1$, let $m - 1 = (z_t, \ldots, z_2, z_1)_\alpha$. Then $[m\alpha] = z_t p_{t-1} + \cdots + z_2 p_1 + z_1 p_0$.*

PROOF. We have $[m\alpha] = f_1 + f_2 + \cdots + f_{m-1}$, and $f_1 + f_2 + \cdots + f_{m-1}$ equals the number of 1's in W_{m-1}, the initial segment of $f(\alpha)$ of length $m - 1$. Since $W_{m-1} = X_{t-1}^{z_t} \cdots X_1^{z_2} X_0^{z_1}$, and the number of 1's in each X_i is p_i, the number of 1's in W_{m-1} is $z_t p_{t-1} + \cdots + z_2 p_1 + z_1 p_0$.

6. **Remarks and questions.** An interesting algorithm (which exploits the complementarity of the sequences $[n(1+\alpha)]$ and $[n(1+1/\alpha)]$) for calculating the terms of the sequence $[n(1+\alpha)]$ is given I. G. Connell [8].

J. Rosenblatt [31] obtained a recursive property of the "hit sequence" $h(\alpha) = h_0 h_1 h_2 \cdots$, where for $k \geq 0$, h_k is the *number* of non-negative integers m such that $[m\alpha] = k$. His property is similar to Fact 2 above. The connection between $h(\alpha)$ and $f(\alpha)$ is the following. If $\alpha = 1/(a_1 + \alpha_1) < 1$, then $h(\alpha)(m) = f(\alpha_1)(m) + a_1$, $m \geq 1$,

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

where $h(\alpha)(m), f(\alpha_1)(m)$ denote the m-th terms of the sequences $h(\alpha), f(\alpha_1)$ respectively. If $\alpha > 1$, then $h(\alpha)(m) = f(1/\alpha)(m), m \geq 1$.

D. Crisp, W. Moran, A. Pollington, and P. Shiue [9] have found a necessary and sufficient condition on α for $f(\alpha)$ to be invariant under a substitution of the type $0 \longrightarrow B_1$, $1 \longrightarrow B_2$, thus answering the natural question concerning the converse of the statement of Fact 2.

Perhaps there is a simple way to show that Fact 1 implies Fact 3 and vice versa. If not, perhaps the two can be put together in an interesting way.

Peter G. Anderson [1] has observed that if u_n denotes the least significant digit in the Zeckendorff representation of n, $n \geq 0$, and if $Y_n = u_0 u_1 \cdots u_{q_n-1}$, $n \geq 0$, then the sequence Y_n, $n \geq 2$, satisfies the same recurrence as does the sequence $X_n = f_1 f_2 \cdots f_{q_n}$, $n \geq 2$. It follows easily that if $\alpha = [0, a_1, a_2, \ldots]$ with $a_1 > 1$, then $f_n = 1$ if and only if $u_{n-1} = a_1 - 1, n \geq 1$. If $\alpha = [0, 1, a_2, \ldots,]$ it follows in the same way that $f_n = 0$ if and only if $v_{n-1} = a_2$, where v_{n-1} is the second least significant digit in the Zeckendorff representation of $n - 1$.

Finally, we remark that $f_n = 1$ if and only if the Zeckendorff representation of n terminates in an odd number of zeros. This follows from the fact that $f_n = 1$ if and only if $n = [k/\alpha]$ for some k (see the last paragraph of [18] or Lemma 1 of [5]), together with the characterization of the values of $[k/\alpha]$, $k \geq 1$, given in [17]. (See also Property 1 in [15, 1989].) One can also deduce this result directly, using Anderson's observation.

The author is grateful to Jeffrey Shallit for references [7], [9], [10], [20], [22], [25], [27], [28], and [35], and to the referee for references [9], [10], [15], [20], [23], [35], for an exact reference to [2], and for several helpful remarks.

References

1. P. G. Anderson, personal communication.

2. J. Bernoulli III, *Sur une nouvelle espece de calcul*, Recueil pour les Astronomes, **1** Berlin, 1772, 255–284.

3. J.-P. Borel and F. Laubie, *Construction de mots de Christoffel*, C. R. Acad. Sci. Paris (I) **313**(1991), 483–485.

4. ———, *Quelques mots sur la droite projective reelle*, Seminaire de Theorie des Nombres de Bordeaux, to appear.

5. T. C. Brown, *A characterization of the quadratic irrationals*, Canad. Math. Bull. **34**(1991), 36–41.

6. ———, *Sums of fractional parts of integer multiples of an irrational*, preprint.

7. H. Cohn, *Some direct limits of primitive homotopy words and of Markoff geodesics*. In: Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies **79**, Princeton University Press, Princeton, 1974, 81–98.

8. Ian G. Connell, *Some properties of Beatty sequences II*, Canad. Math. Bull. **3**(1960), 17–22.

9. D. Crisp, W. Moran, A. Pollington and P. Shiue, *Substitution invariant cutting sequences*, preprint.

10. E. B. Christoffel, *Observatio Arithmetica*, Annali di Matematica **6**(1875), 145–152.

11. D. H. Fowler, *Ratio in early Greek mathematics*, Bull. AMS New Series **1**(1979), 807–846.

12. ———, *Book II of Euclid's Elements and a pre-Eudoxan theory of ratio*, Archive for Hist. of Exact Sci. **22**(1980), 5–56.

13. ———, *Anthyphairetic ratio and Eudoxan proportion*, Archive for Hist. of Exact Sci. **24**(1981), 69–72.

14. A. S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly **92**(1985), 105–114.

15. ———, *The use and usefulness of numeration systems*, Information and Computation **81**(1989), 46–61.

16. A. S. Fraenkel and I. Borosh, *A generalization of Whythoff's game*, J. Combin. Theory, (A) **15**(1973), 175–191.

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press

17. A. S. Fraenkel, J. Levitt and M. Shimshoni, *Characterization of the set of values of* $f(n) = [n\alpha]$, $n = 1, 2, \ldots$, Discrete Math. **2**(1972), 335–345.

18. A. S. Fraenkel, M. Mushkin and U. Tassa, *Determination of* $[n\theta]$ *by its sequence of differences*, Canad. Math. Bull. **21**(1978), 441–446.

19. G. A. Hedlund, *Sturmian minimal sets*, Amer. J. Math. **66**(1954), 605–620.

20. S. Ito and S. Yasutomi, *On continued fractions, substitutions and characteristic sequences* $[nx + y] - [(n-1)x + y]$, Japan J. Math. **16**(1990), 287–306.

21. W. R. Knorr, *The Evolution of the Euclidean Elements*, Reidel, Dordrecht, 1975.

22. F. Laubie, *Prolongements homographiques de substitutions de mots de Christoffel*, C. R. Acad. Sci. Paris (I) **313**(1991), 565–567.

23. C. G. Lekkerkerker, *Representation of natural numbers as a sum of Fibonacci numbers*, Simon Stevin **29**(1952), 190–195.

24. A. A. Markoff, *Sur une question de Jean Bernoulli*, Math. Ann. **19**(1882), 27–36.

25. F. Mignosi, *Infinite words with linear subword complexity*, Theor. Comput. Sci. **65**(1989), 221–242.

26. M. Morse and G. A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. **62**(1940), 1–42.

27. K. Nishioka, I. Shiokawa and J. Tamura, *Arithmetical properties of a certain power series*, J. Number Theory, to appear.

28. A. Ostrowski, *Bemerkungen zur Theorie der Diophantischen Approximation*, Abh. Math. Sem. Hamburg **1**(1922), 77–98.

29. H. Porta and K. B. Stolarsky, *Half-silvered mirrors and Wythoff's game*, Canad. Math. Bull. **33**(1990), 119–125.

30. R. C. Riddell, *Eudoxan mathematics and the Eudoxan spheres*, Archive for Hist. of Exact. Sci. **20**(1979), 1–19.

31. J. Rosenblatt, *The sequence of greatest integers of an arithmetic progression*, J. London Math. Soc. (2) **17**(1978), 213–218.

32. C. Series, *The geometry of Markoff numbers*, Math. Intelligencer **7**(1985), 20–29.

33. J. Shallit, *A generalization of automatic sequences*, Theor. Comp. Sci. **61**(1988), 1–16.

34. ———, *Characteristic words as fixed points of homomorphisms*, Univ. of Waterloo, Dept. of Computer Science, Tech. Report CS-91-72, 1991.

35. H. J. S. Smith, *Note on continued fractions*, Messenger of Math. **6**(1876), 1–14.

36. K. B. Stolarsky, *Beatty sequences, continued fractions, and certain shift operators*, Canad. Math. Bull. **19**(1976), 473–482.

37. B. A. Venkov, *Elementary Number Theory*, Translated and edited by Helen Alderson, Wolters-Noordhoff, Groningen, 1970, 65–68.

38. A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, (trans. J. McCawley, Jr.), (revis. and ed. B. Gordon) **2**, Holden-Day, San Francisco, 1967.

39. E. Zeckendorff, *Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Royale Sci. Liege **42**(1972), 179–182.

40. E. C. Zeeman, *An algorithm for Eudoxan and anthipairetic ratios*, preprint.

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, British Columbia
V5A 1S6

https://doi.org/10.4153/CMB-1993-003-6 Published online by Cambridge University Press