Can. J. Math. Vol. **50** (6), 1998 pp. 1273–1297.

MEAN CONVERGENCE OF LAGRANGE INTERPOLATION FOR EXPONENTIAL WEIGHTS ON $[-1,1]$

D. S. LUBINSKY

ABSTRACT. We obtain necessary and sufficient conditions for mean convergence of Lagrange interpolation at zeros of orthogonal polynomials for weights on $[-1,1]$, such as

$$w(x)=\exp\Big(-(1-x^2)^{-\alpha}\Big),\quad \alpha>0$$

or

$$w(x)=\exp\Big(-\exp_k(1-x^2)^{-\alpha}\Big),\quad k\ge 1,\ \alpha>0,$$

where $\exp_k=\exp\Big(\exp\Big(\cdots\exp(\,)\cdots\Big)\Big)$ denotes the k-th iterated exponential.

1**. Introduction and results.** There is a vast literature on mean convergence of Lagrange interpolation at zeros of orthogonal polynomials. For weights on $[-1,1]$, most of the positive results deal with generalized Jacobi weights—see [12], [13], [17], [20] for some recent references. The broad spectrum of results have applications ranging from approximation theory to number theory and numerical analysis—see [18] for some of these, notably for the insights that Lagrange interpolation provides on the orthogonal polynomials themselves.

In this paper, we consider the analogous problem for exponential weights w^2 on $[-1,1]$, such as

$$w_{0,\alpha}(x)=\exp\Big(-(1-x^2)^{-\alpha}\Big),\quad \alpha>0 \tag{1}$$

or

$$w_{k,\alpha}(x)=\exp\Big(-\exp_k(1-x^2)^{-\alpha}\Big),\quad k\ge 1,\ \alpha>0, \tag{2}$$

where

$$\exp_k=\exp\Big(\exp\Big(\cdots\exp(\cdots)\cdots\Big)\Big)$$

denotes the k-th iterated exponential. These are the first positive results on mean convergence associated with weights that vanish strongly at ± 1. The corresponding question for exponential weights on $\mathbb{R}$ has been considered in [3], [4], [10], [16].

Our results are based on the estimates of [8], which involve the following class of weights: In its definition, we use the notation $\sim$. We write

$$f(t)\sim g(t)$$

if there exist positive constants C_1 and C_2 such that for the relevant range of t,

$$C_1\le f(t)/g(t)\le C_2.$$

Received by the editors December 20, 1997; revised April 16, 1998.
AMS subject classification: Primary: 41A05; secondary: 42C99.

Similar notation is used for sequences and sequences of functions.

DEFINITION 1.1. Let $w := e^{-Q}$, where $Q\colon (-1,1) \longrightarrow \mathbb{R}$ is even, and twice continuously differentiable in $(-1,1)$. Assume moreover, that $Q^{(j)} \geq 0$ in $(0,1)$, $j = 1,2$, and that the function

$$T(t) := 1 + tQ''(t)/Q'(t), \quad t \in (-1,1) \setminus \{0\}$$

is increasing in $(0,1)$ with

$$T(0+) := \lim_{t \longrightarrow 0+} T(t) > 1$$

and for t close enough to 1,

$$T(t) \sim Q'(t)/Q(t)$$

while for some $A > 2$ and t close enough to 1,

$$T(t) \geq \frac{A}{1-t^2}.$$

Then we write $w \in \mathcal{W}$.

We note that the last inequality is (1.34) in [8, p. 9] and is needed for the bounds on the orthogonal polynomials there. In particular, it implies

$$\lim_{t \longrightarrow 1-} Q(t) = \infty,$$

which is required in Definition 1.1 in [8]. The weights $w_{k,\alpha}$, $k \geq 0$, $\alpha > 0$ are the archetypal elements of $\mathcal{W}$.

Associated with the weight w^2 (note that we write the weight as a square), we can define orthonormal polynomials

$$p_n(x) = p_n(w^2, x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

satisfying

$$\int_{-1}^{1} p_n p_m w^2 = \delta_{mn}.$$

We denote the zeros of p_n by

$$-1 < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < 1.$$

The Lagrange interpolation polynomial to a function $f\colon (-1,1) \longrightarrow \mathbb{R}$ at $\{x_{jn}\}$ is denoted by $L_n[f]$. Thus, if $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$, then $L_n[f] \in \mathcal{P}_{n-1}$ satisfies

$$L_n[f](x_{jn}) = f(x_{jn}), \quad 1 \leq j \leq n.$$

The Gauss quadrature rule for w^2 has the form

$$\int_{-1}^{1} Pw^2 = \sum_{j=1}^{n} \lambda_{jn} P(x_{jn}), \quad P \in \mathcal{P}_{2n-1},$$

where the Christoffel numbers λ_{jn} are positive.

In analysis of exponential weights, an important role is played by the Mhaskar-Rahmanov-Saff number a_n, the positive root of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\,dt. \tag{3}$$

One of its features is the Mhaskar-Saff identity [11]

$$\|Pw\|_{L_\infty[-1,1]} = \|Pw\|_{L_\infty[-a_n,a_n]}, \quad P \in \mathcal{P}_n.$$

For $w \in \mathcal{W}$, $a_n \longrightarrow 1$ as $n \longrightarrow \infty$. We also need the quantity

$$\delta_n := \big(nT(a_n)\big)^{-2/3}. \tag{4}$$

One may think of δ_n as the spacing $x_{1n} - x_{2n}$ between the largest and second largest zeros of p_n. For

$$w = w_{0,\alpha}, \quad T(a_n) \sim n^{\frac{1}{\alpha+\frac{1}{2}}};\ \delta_n \sim n^{-\frac{2\alpha+3}{2\alpha+1}\frac{2}{3}}$$

[8, p. 8] so as $\alpha \longrightarrow 0$, we have roughly speaking, $\delta_n \longrightarrow n^{-2}$, the spacing between the largest and second largest zeros of the orthogonal polynomials for generalized Jacobi weights. By contrast, for

$$w = w_{k,\alpha},\ k \geq 1, \quad T(a_n) \sim \Big(\prod_{j=1}^{k-1} \log_j n\Big)(\log_k n)^{1+\frac{1}{\alpha}}$$

where

$$\log_j = \log\Big(\log\big(\cdots \log(\cdots)\big)\Big)$$

denotes the j-th iterated logarithm [8, p. 11]. Thus for $w_{0,\alpha}$, $T(a_n)$ grows like a power of n, while for $w_{k,\alpha}$, $k \geq 1$, it grows slower than any power of n. This difference plays a role in describing our convergence results for $L_n[f]$. The final piece of notation needed to state our result is the function

$$g_n(x) := \left|1 - \frac{|x|}{a_n}\right| + \delta_n, \quad x \in (-1,1). \tag{5}$$

Our Lagrange interpolation results depend on the following converse quadrature sum estimate, which is an analogue of the classical Marcinkiewicz inequality for trigonometric polynomials:

THEOREM 1.2. *Let $w \in \mathcal{W}$, $1 < p < \infty$ and*

$$\frac{1}{4} - \frac{1}{p} < \Delta < \min\Big\{\frac{5}{4} - \frac{1}{p}, \frac{3}{4} + \frac{1}{2p}\Big\}. \tag{6}$$

Then for $n \geq 1$ and $P \in \mathcal{P}_{n-1}$,

$$\|Pwg_n^\Delta\|_{L_p[-1,1]} \leq C\Big(\sum_{k=1}^{n} \lambda_{kn} w^{-2}(x_{kn})|Pwg_n^\Delta|^p(x_{kn})\Big)^{1/p}. \tag{7}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

Here C is independent of P and n.

The upper bound on Δ in (6) is probably not sharp, but this is largely irrelevant to this paper: it is the lower bound on Δ in (6), which is sharp. Following is our main result, which requires (7) only for Δ close to $\frac{1}{4}-\frac{1}{p}$.

THEOREM 1.3. *Let $w \in \mathcal{W}$, $1 < p < \infty$ and $\Delta \in \mathbb{R}$. The following are equivalent:*

(a) There exists C independent of f and n such that for $n \geq 1$, and measurable $f\colon(-1,1) \longrightarrow \mathbb{R}$,

$$\|L_n[f]wg_n^{\Delta}\|_{L_p[-1,1]} \leq C\|fw\|_{L_\infty[-1,1]}. \tag{8}$$

(b)

$$\Delta > \frac{1}{4}-\frac{1}{p}. \tag{9}$$

The disadvantage of the above result is that the weighting factor g_n in the left-hand side of (8) depends on n. In analogous questions for generalized Jacobi weights on $[-1,1]$, one can effectively take $g_n(x) = g(x) = 1-|x|$, but not here.

We note too that there is no advantage to be gained by placing a factor g_n^r, no matter what choice for r, in $\|fw\|_{L_\infty[-1,1]}$. Indeed one needs (9) if (8) is to hold merely for f that vanish outside any fixed non-empty subinterval of $(-1,1)$.

To avoid weighting factors that depend on n, we consider separately $p < 4$ and $p \geq 4$: for the former case, we do not need a weighting factor:

THEOREM 1.4. *Let $w \in \mathcal{W}$ and $1 < p < 4$. Let $f\colon(-1,1) \longrightarrow \mathbb{R}$ be Riemann integrable in each compact subinterval of $(-1,1)$ and assume that for some $\alpha < \frac{1}{p}$,*

$$\lim_{|x|\to 1-} (fw)(x)(1-x^2)^{\alpha} = 0. \tag{10}$$

Then

$$\lim_{n\to\infty} \|(L_n[f]-f)w\|_{L_p[-1,1]} = 0. \tag{11}$$

We note that one may replace $(1-x^2)^{\alpha}$ by $(1-x^2)^{1/p}|\log(1-x^2)|^{\alpha}$, where $\alpha < \frac{1}{p}$ (and so on). The weighting factor is more complex for $p > 4$:

THEOREM 1.5. *Let $w \in \mathcal{W}$, $p \geq 4$, $\Delta \in \mathbb{R}$.*

(a) Let

$$\Delta > \frac{1}{4}-\frac{1}{p}. \tag{12}$$

Let $f\colon(-1,1) \longrightarrow \mathbb{R}$ be Riemann integrable in each compact subinterval of $(-1,1)$ and assume that for some $\alpha < \frac{1}{p}$, (10) holds. Then

$$\lim_{n\to\infty} \|(L_n[f]-f)w[1+Q^{2/3}T]^{-\Delta}\|_{L_p[-1,1]} = 0. \tag{13}$$

(b) Conversely, if (13) holds for each $f\colon[-1,1] \longrightarrow \mathbb{R}$ that is continuous and vanishes outside $[-\frac{1}{2},\frac{1}{2}]$, it is necessary that

$$\Delta \geq \frac{1}{4}-\frac{1}{p}. \tag{14}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

If $p = 4$, (12) is also necessary.

If instead of (13) we considered

$$\lim_{n\to\infty} \Big\| (L_n[f] - f) w \big(\log(2+Q)\big) [1 + Q^{2/3}T]^{-\Delta} \Big\|_{L_p[-1,1]} = 0$$

then (12) is necessary and sufficient for all $p \geq 4$. In fact we could replace $\log(2 + Q)$ by any slowly growing function with limit ∞ at 1. However, this would make an awkward weighting factor even more awkward!

The closest relative of this situation for weights on $\mathbb{R}$ is the so-called Erdős weights considered by S. B. Damelin and the author [3], [4]. There the weighting factor used was equivalent to $(1 + Q^{2/3})^{-\Delta}$. This was possible as T there grows slower than Q^{ε} for any $\varepsilon > 0$. However, as we have noted above, in the present situation, T may grow faster than Q, unless Q grows fast enough near 1. Indeed, for $w_{0,\alpha}$,

$$T(x) \sim \frac{1}{1-x^2};\ Q(x) = \frac{1}{(1-x^2)^{\alpha}}, \quad x \longrightarrow 1- .$$

By contrast for $w_{k,\alpha}$, $k \geq 1$, for each $\varepsilon > 0$,

$$T(x) = O\Big(\log Q(x)\Big)^{1+\varepsilon}, \quad x \longrightarrow 1- .$$

For weights such as the latter we can then drop the T in (13).

Further justification for the choice of $1 + Q^{2/3}T$ is provided by:

THEOREM 1.6. *Let $w \in \mathcal{W}$ and $p \geq 4$. Let $U\colon (-1,1) \longrightarrow \mathbb{R}$ be measurable and satisfy*

$$\lim_{x\to 1-} U(x) [1 + Q^{2/3}(x)T(x)]^{\frac{1}{4}-\frac{1}{p}} = \infty. \tag{15}$$

Then there exists continuous $f\colon (-1,1) \longrightarrow \mathbb{R}$ such that f vanishes outside $[-\frac{1}{2}, \frac{1}{2}]$ and

$$\limsup_{n\to\infty} \|L_n[f]wU\|_{L_p[-1,1]} = \infty. \tag{16}$$

Our proof of Theorem 1.2 uses König's method [6], [7], adjusted so as to work for all $1 < p < \infty$. It is interesting to note that this is the third method we tried, the first two failed to yield sharp results: we began by applying the method that has proved successful for Freud weights on $\mathbb{R}$ ([3], [10], [16]). This failed because the requisite results on mean convergence of orthonormal expansions are not available, and moreover, the use of a Lebesgue function type estimate yielded an extra factor of $\log n$. The second method tried was Nevai's from [17], but that failed as it needs polynomials R_n of degree $\leq n$ such that

$$R_n \sim w \quad \text{in } [-a_n, a_n].$$

These are not available.

This paper is organised as follows: In Section 2, we state extra notation, and state some technical lemmas. In Section 3, we state lemmas needed specifically for Theorem 1.2 and in Section 4, we prove Theorem 1.2. In Section 5, we prove the remaining results.

2. **Technical estimates.** In the sequel, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences.

The Lagrange interpolation polynomial $L_n[f]$ admits the representation

$$L_n[f] = \sum_{j=1}^{n} f(x_{jn})\ell_{jn}(x)$$

where the *fundamental polynomials* ℓ_{jn} in turn admit the representation

$$\ell_{jn}(x) = \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}.$$

We set

$$x_{0n} := x_{1n}(1 + \delta_n); \quad x_{n+1,n} := x_{nn}(1 + \delta_n) \tag{17}$$

and

$$I_{jn} := (x_{jn}, x_{j-1,n}); \ |I_{jn}| := x_{j-1,n} - x_{jn}, \quad 1 \le j \le n.$$

We also define the characteristic functions

$$\chi_{jn}(x) := \begin{cases} 1, & x \in I_{jn} \\ 0, & x \notin I_{jn} \end{cases}, \quad 1 \le j \le n.$$

In describing spacing of zeros and related quantities, the function

$$\phi_n(x) := \max\left\{ \sqrt{g_n(x)}, \frac{1}{T(a_n)\sqrt{g_n(x)}} \right\} \tag{18}$$

plays an important role. (Recall g_n was defined at (5)). In the sequel, we assume that $w \in \mathcal{W}$ without further mention. First we record all our estimates relating specifically to orthogonal polynomials:

LEMMA 2.1. *(a) For $n \ge 1$,*

$$\left| 1 - \frac{x_{1n}}{a_n} \right| \le C\delta_n. \tag{19}$$

(b) Uniformly for $n \ge 1$ and $1 \le j \le n$,

$$\lambda_{jn} w^{-2}(x_{jn}) \sim |x_{j\pm1,n} - x_{jn}| \sim \frac{1}{n}\phi_n(x_{jn}). \tag{20}$$

(c) Uniformly for $n \ge 1$ and $1 \le j \le n$, and $x \in [x_{j+1,n}, x_{j-1,n}]$,

$$g_n(x) \sim g_n(x_{jn}); \quad \phi_n(x) \sim \phi_n(x_{jn}). \tag{21}$$

(d) Uniformly for $n \ge 1$ and $1 \le j \le n$,

$$\frac{1}{|p_n' w|(x_{jn})} \sim (x_{jn} - x_{j+1,n}) g_n(x_{jn})^{1/4}. \tag{22}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

(e) *Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in (-1,1)$,*

$$|\ell_{jn}(x)|w^{-1}(x_{jn})w(x) \sim (x_{jn} - x_{j+1,n})g_n(x_{jn})^{1/4}\left|\frac{p_n(x)w(x)}{x - x_{jn}}\right|. \tag{23}$$

(f) *Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in (-1,1)$,*

$$|\ell_{jn}(x)|w^{-1}(x_{jn})w(x) \leq C. \tag{24}$$

(g) *Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$ and $x \in [x_{j+1,n}, x_{jn}]$,*

$$\ell_{jn}(x)w^{-1}(x_{jn})w(x) + \ell_{j+1,n}(x)w^{-1}(x_{j+1,n})w(x) \sim 1. \tag{25}$$

(h) *Uniformly for $n \geq 1$ and $x \in (-1,1)$,*

$$|p_n w|(x) \leq C g_n(x)^{-1/4}. \tag{26}$$

(i) *Let $0 < p \leq \infty$. For $n \geq 1$,*

$$\|p_n w\|_{L_p[-1,1]} \sim \begin{cases} 1, & p < 4, \\ \big(\log(n+1)\big)^{1/4}, & p = 4, \\ \delta_n^{\frac{1}{p}-\frac{1}{4}}, & p > 4 \end{cases} \tag{27}$$

(j) *Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$ and $x \in (x_{j+1,n}, x_{jn})$,*

$$|p_n w|(x) \sim \frac{g_n(x_{jn})^{-1/4}}{x_{jn} - x_{j+1,n}} \min\{|x - x_{jn}|, |x - x_{j+1,n}|\}. \tag{28}$$

PROOF. (a) This is Corollary 1.4(i) in [8, p. 9].
(b) This follows from Theorem 1.2 and Corollary 1.4 (ii) in [8].
(c) This follows from (10.12) in [8, p. 111].
(d) This follows from Corollary 1.5(iii) in [8, p. 11] and Corollary 1.4(ii) in [8, p. 9].
(e) This is a consequence of (d) and the formula for ℓ_{jn}.
(f) This is Lemma 12.2(b) in [8, p. 134].
(g) It is a special case of the result of [9] that in $[x_{j+1,n}, x_{jn}]$,

$$\ell_{jn}(x)w^{-1}(x_{jn})w(x) + \ell_{j+1,n}(x)w^{-1}(x_{jn})w(x) \geq 1.$$

An inequality in the other direction follows from (f).
(h) This follows from Corollary 1.5 (i), (ii) in [8, p. 10].
(i) This follows from Corollary 1.5(ii) and Theorem 1.8 in [8, p. 12].
(j) This follows easily from (e) and (g) and the fact that $|I_{jn}| \sim |I_{j\pm1,n}|$. ∎

Next we record estimates involving Q and a_n. We note that we may define a_u by (3) even for all $u > 0$ (and not just for integers n).

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

LEMMA 2.2. *(a) For* $j = 0, 1, 2$, *and* $u \geq C$,

$$Q^{(j)}(a_u) \sim uT(a_u)^{j-\frac{1}{2}}. \tag{29}$$

(b) Let $\alpha, \beta > 0$. *Then uniformly for* $j = 0, 1, 2$, *and* $u \geq C$,

$$T(a_{\alpha u}) \sim T(a_{\beta u}); \quad Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}). \tag{30}$$

(c) There exist $C, \varepsilon > 0$ *such that*

$$T(a_u) \leq Cu^{2-\varepsilon}, \quad u \geq C. \tag{31}$$

(d) There exists $C > 0$ *such that for* $\frac{1}{2} \leq \frac{u}{v} \leq 2$,

$$\left|1 - \frac{a_u}{a_v}\right| \sim \frac{1}{T(a_u)}\left|1 - \frac{u}{v}\right|. \tag{32}$$

Moreover, if $\alpha > 0$, *there exists* $C > 0$ *such that for* $u \geq C$,

$$\left|1 - \frac{a_{\alpha u}}{a_u}\right| \sim \frac{1}{T(a_u)}. \tag{33}$$

PROOF. This is part of Lemma 3.2 in [8, p. 24], except (32), which follows by integrating (3.9) in [8, p. 24]. ■

In the proof of the necessity parts of the theorems, we shall need:

LEMMA 2.3. *Let* $0 < p < \infty$, $0 < A < B < \infty$. *Let* $\xi: (-1, 1) \longrightarrow (0, \infty)$ *be an even function with the following property: Uniformly for* $n \geq 1$, $1 \leq j \leq n$,

$$A \leq \frac{\xi(x)}{\xi(x_{jn})} \leq B, \quad x \in [x_{j+1,n}, x_{jn}]. \tag{34}$$

For $n \geq 1$, *let* $\mathcal{I}_n$ *be a subinterval of* (x_{nn}, x_{1n}) *containing at least two zeros of* p_n. *Then*

$$\|p_n w\xi\|_{L_p(\mathcal{I}_n)} \geq C\|g_n^{-1/4}\xi\|_{L_p(\mathcal{I}_n)}. \tag{35}$$

The constant C *is independent of* n, $\mathcal{I}_n$, ξ *but depends on* A, B *in (34).*

PROOF. We note first that if $1 \leq j \leq n-1$, (28) and (34) give

$$\int_{x_{j+1,n}}^{x_{jn}} |p_n w\xi|^p \sim \left(\frac{g_n(x_{jn})^{-1/4}}{x_{jn} - x_{j+1,n}}\right)^p \xi(x_{jn})^p \int_{x_{j+1,n}}^{x_{jn}} \min\{|x - x_{jn}|, |x - x_{j+1,n}|\}^p \, dx$$
$$\sim g_n(x_{jn})^{-p/4}\xi(x_{jn})^p(x_{jn} - x_{j+1,n}) \sim \int_{x_{j+1,n}}^{x_{jn}} g_n^{-p/4}\xi^p$$

by (21) and (34). Adding over those j for which $[x_{j+1,n}, x_{jn}] \subset \mathcal{I}_n$ gives the result: Note that terms over adjacent intervals are of the same size up to $\sim$. Thus if the endpoints of $\mathcal{I}_n$ do not coincide with zeros of p_n, the small intervals around these endpoints are of the same size as an adjacent $[x_{j+1,n}, x_{jn}] \subset \mathcal{I}_n$. Of course, as $\mathcal{I}_n$ contains at least two zeros, there is such an adjacent interval. ■

Our final lemma in this section is a restricted range inequality:

LEMMA 2.4. *Let* $0 < p \le \infty$ *and* $\Delta \in \mathbb{R}$. *Let* $s > 0$. *Then there exists* n_0 *such that for* $n \ge n_0$ *and* $P \in P_n$,

$$\|Pwg_n^{\Delta}\|_{L_p[-1,1]} \le C\|Pwg_n^{\Delta}\|_{L_p[-a_n(1-s\delta_n),a_n(1-s\delta_n)]}. \tag{36}$$

PROOF. We claim that we can find n_0 and for $n \ge n_0$, polynomials $R_{n,\Delta}$ of degree $O(\delta_n^{-1/2})$ such that

$$R_{n,\Delta} \sim g_n^{\Delta} \text{ in } [-a_n, a_n]; \quad R_{n,\Delta} \ge Cg_n^{\Delta} \text{ in } [-1,1]. \tag{37}$$

Once we have such polynomials, we note that for $P \in P_n$, $P_nR_{n,\Delta}$ has degree

$$m = m(n) = n + O(\delta_n^{-1/2}) = n\left(1 + O\left(\frac{T(a_n)}{n^2}\right)^{1/3}\right) = n\bigl(1 + o(1)\bigr).$$

Then from (32),

$$1 - \frac{a_n}{a_m} \le \frac{C}{T(a_n)}\left(1 - \frac{n}{m}\right) \le C\delta_n,$$

so

$$a_m \le a_n(1 + C\delta_n); \quad \delta_m \ge C\delta_n.$$

In particular for a given s, we can choose $t > 0$ so large that for large enough n,

$$a_m(1 - t\delta_m) \le a_n(1 - s\delta_n).$$

By Theorem 1.7 in [8, p. 12], given $K > 0$, we have for $n \ge n_1(K)$,

$$\begin{aligned}
\|Pwg_n^{\Delta}\|_{L_p[-1,1]} &\le C\|PwR_{n,\Delta}\|_{L_p[-1,1]}\\
&\le C\|PwR_{n,\Delta}\|_{L_p[-a_m(1-t\delta_m),a_m(1-t\delta_m)]}\\
&\le C\|PwR_{n,\Delta}\|_{L_p[-a_n(1-s\delta_n),a_n(1-s\delta_n)]}.
\end{aligned}$$

So we have (36).

We now turn to the proof of (37). For this purpose, we use Christoffel functions for the classical Jacobi weights. First let $\beta \in [-\frac{1}{4}, 0)$, and

$$u(x) := (1 - x^2)^{-\beta-\frac{1}{2}}, \quad x \in (-1,1).$$

Its Christoffel function $\lambda_\ell(u,x)$ satisfies [15, p. 108],

$$\ell^{-1}\lambda_\ell^{-1}(u,x) \sim (|1 - x^2| + \ell^{-2})^{\beta}, \quad x \in [-1,1].$$

Moreover, if $p_j(u,x)$ is the j-th orthonormal polynomial for u, its zeros all lie in $(-1,1)$, so

$$\lambda_\ell^{-1}(u,x) = \sum_{j=0}^{\ell-1} p_j^2(u,x)$$

is increasing in $(1,\infty)$ while $(|1-x^2|+\ell^{-2})^\beta$ is decreasing there. Thus

$$\ell^{-1}\lambda_\ell^{-1}(u,x)\geq C(|1-x^2|+\ell^{-2})^\beta,\quad |x|>1.$$

Then defining $\ell=\ell(n):=$ greatest integer $\leq \delta_n^{-1/2}$, and

$$R_{n,\beta}(x):=\ell^{-1}\lambda_\ell^{-1}\Big(u,\frac{x}{a_n}\Big)$$

we see that

$$R_{n,\beta}\sim g_n^\beta\quad\text{in } [a_{-n},a_n];\ R_{n,\beta}\geq g_n^\beta\quad\text{in } [-1,1].$$

So we have $R_{n,\Delta}$ satisfying (37) for $\Delta=\beta\in[-\frac{1}{4},0)$. Now for any $\Delta\in\mathbb{R}$, we can write $\Delta=k\beta+2j$, where k,j are non-negative integers, and $\beta\in[-\frac{1}{4},0)$. We can then set

$$R_{n,\Delta}(x):=R_{n,\beta}^k(x)\bigg(\Big(1-\frac{x^2}{a_n^2}\Big)^2+\delta_n^2\bigg)^j$$

and easily see that (37) holds. ■

3**. Lemmas for Theorem 1.2.** In this section, we present three lemmas required specifically for the proof of Theorem 1.2. The first involves the Hilbert transform

$$H[g](x):=\lim_{\varepsilon\to0+}\int_{|t-x|\geq\varepsilon}\frac{g(t)}{x-t}\,dt.$$

If $g\in L_1(\mathbb{R})$, then g exists a.e. Moreover, a famous theorem of M. Riesz asserts that H is a bounded operator on L_p for $1<p<\infty$. We need a modification of M. Riesz' theorem that is essentially due to Muckenhoupt:

LEMMA 3.1. *Let $1<p<\infty$ and*

$$-\frac{1}{p}<r<1-\frac{1}{p}. \tag{38}$$

Then for $b,c\in\mathbb{R}$ and $g\in L_p[-1,1]$,

$$\Big\|H[g](x)\Big|\big|b-|x|\big|+c\Big|^r\Big\|_{L_p(\mathbb{R})}\leq C\Big\|g(x)\Big|\big|b-|x|\big|+c\Big|^r\Big\|_{L_p(\mathbb{R})}, \tag{39}$$

where C is independent of g, b and c.

PROOF. The result is a special case of general results on A_p weights, and the idea already appears in [14], [6]. See also [17, p. 676]. However, since the result is not formally stated anywhere, we give the proof. We shall use the notation

$$H[f(y)](x)=H[f](x)$$

to indicate the variable y of the function whose Hilbert transform is being taken. Now we see that (39) follows if we can show that

$$\left\|H\left[\frac{h(y)}{\Big|\big|b-|y|\big|+c\Big|^r}\right](x)\Big|\big|b-|x|\big|+c\Big|^r\right\|_{L_p(\mathbb{R})}\leq C\|h\|_{L_p(\mathbb{R})}.$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

(For, one can then set $h(x) = g(x)\Big|\big|b - |x|\big| + c\Big|^r$.) We shall assume that $b \geq 0$, the case $b < 0$ is similar. By a variable substitution $y = bt$, $x = bs$, $c = bd$ we see that it suffices to show that

$$\tau := \left\| H\left[\frac{h(t)}{\Big|\big|1 - |t|\big| + d\Big|^r}\right](s)\Big|\big|1 - |s|\big| + d\Big|^r \right\|_{L_p(\mathbb{R})} \leq C\|h\|_{L_p(\mathbb{R})}, \tag{40}$$

with C independent of h and d. (This works for $b \neq 0$, see below for $b = 0$). It suffices to prove (40) for non-negative h. (In the general case, we write $h = \max\{0, h\} - \max\{0, -h\}$ and use the fact that the L_p norms of each of these factors on the right is no larger than the corresponding norm for h.) We may also assume that h vanishes in $(-\infty, 0)$. (For in the general case, we write $h = h\chi_{[0,\infty)} + h\chi_{(-\infty,0)}$, where χ denotes characteristic function, and use the fact that each of the components on the right has L_p norm no larger than that of h. We also use a reflection $t \longrightarrow -t$ in handling the second term). So suppose now $h \geq 0$ and has support in $[0, \infty)$. Then we estimate τ above by

$$\begin{aligned}\tau &\leq 2^{1/p}\Bigg(\left\| H\left[\frac{h(t)}{\big||1 - t| + d\big|^r}\right](s)\big||1 - s| + d\big|^r \right\|_{L_p[0,\infty)} \\ &\qquad + \left\| H\left[\frac{h(t)}{\big||1 - t| + d\big|^r}\right](s)\big||1 + s| + d\big|^r \right\|_{L_p(-\infty,0]}\Bigg) \\ &=: 2^{1/p}(\tau_1 + \tau_2).\end{aligned}$$

We attend first to the term τ_2, which is more difficult because of differing factors $1 - t$ and $1 + s$. Now if $r \geq 0$, then for $s \in (-\infty, 0]$,

$$\big(|1 + s| + d\big)^r \leq \big(|1 - s| + d\big)^r$$

so that

$$\tau_2 \leq \left\| H\left[\frac{h(t)}{\big||1 - t| + d\big|^r}\right](s)\big||1 - s| + d\big|^r \right\|_{L_p(-\infty,0]}.$$

On the other hand if $r < 0$ then for $t \in [0, \infty)$,

$$\frac{1}{\big||1 - t| + d\big|^r} \leq \frac{1}{\big||1 + t| + d\big|^r}.$$

Since the integrand in

$$H\left[\frac{h(t)}{\big||1 - t| + d\big|^r}\right](s)$$

is of one sign for $s < 0$, we deduce that

$$\tau_2 \leq \left\| H\left[\frac{h(t)}{\big||1 + t| + d\big|^r}\right](s)\big||1 + s| + d\big|^r \right\|_{L_p(-\infty,0]}.$$

In summary, if $\sigma := -\mathrm{sign}\,(r)$, then

$$\tau_2 \le \left\| H\left[\frac{h(t)}{\big||1+\sigma t|+d\big|^r}\right](s)\big||1+\sigma s|+d\big|^r\right\|_{L_p(-\infty,0]}.$$

Now in τ_2 we make the substitutions

$$1+\sigma t = |d|v; \quad 1+\sigma s = |d|u$$

and in τ_1, we make the substitutions

$$1-t = |d|v; \quad 1-s = |d|u.$$

We see then that it suffices to show that

$$\left\| H\left[\frac{g(v)}{\big||v|\pm 1\big|^r}\right](s)\big||v|\pm 1\big|^r\right\|_{L_p(\mathbb{R})} \le C\|g\|_{L_p(\mathbb{R})}$$

with C independent of $g \in L_p(\mathbb{R})$. For the factor $\big||v|+1\big|$, this is a well known result of Muckenhoupt [14, p. 308]. For the factor $\big||v|-1\big|$, we may proceed as above to show that it suffices to consider factors of the form $|v \pm 1|$, which can be reduced to the factor $|v|$; this and the case $b=0$ or $d=0$ reduces to Muckenhoupt's inequality [14, p. 308]

$$\left\| H\left[\frac{h(v)}{|v|^r}\right](s)|v|^r\right\|_{L_p(\mathbb{R})} \le C\|h\|_{L_p(\mathbb{R})}. \qquad \blacksquare$$

We shall also need an operator inequality of König, involving

$$\|h\|_{L_p(d\mu)} := \left(\int_\Omega |h|^p \, d\mu\right)^{1/p}$$

where (Ω, μ) is a measure space and h is μ-measurable.

LEMMA 3.2. *Let* $1 < p < \infty$ *and* $q := \frac{p}{p-1}$. *Let* (Ω, μ) *be a measure space and* $S, R\colon \Omega^2 \longrightarrow \mathbb{R}$. *For* μ*-measurable* f, *define*

$$J[f](u) := \int_\Omega S(u,v) f(v)\, d\mu(v). \tag{41}$$

Assume that

$$\sup_u \int_\Omega |S(u,v)|\,|R(u,v)|^q \, d\mu(v) \le N; \tag{42}$$

$$\sup_v \int_\Omega |S(u,v)|\,|R(u,v)|^{-p} \, d\mu(u) \le N; \tag{43}$$

Then J *is a bounded operator from* $L_p(d\mu)$ *to* $L_p(d\mu)$, *more precisely,*

$$\|J\|_{L_p(d\mu)\longrightarrow L_p(d\mu)} \le N. \tag{44}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

PROOF. See [4, Lemma 2.5, p. 745] for a full proof. ∎

König's method involves replacing $\frac{1}{x-x_{jn}}$ by $H[\chi_{I_{jn}}]$. This is achieved with the aid of the following lemma:

LEMMA 3.3. *For* $1 \le j \le n$, *let*

$$\tau_{jn}(x) := (p_n w)(x)\left[\frac{1}{x-x_{jn}} - \frac{H[\chi_{jn}](x)}{|I_{jn}|}\right]. \tag{45}$$

Let

$$f_{jn}(x) := \min\left\{\frac{1}{|I_{jn}|}, \frac{|I_{jn}|}{(x-x_{jn})^2}\right\} g_n(x)^{-1/4}. \tag{46}$$

Then uniformly for $n \ge 1$ *and* $1 \le j \le n$ *and* $x \in [x_{nn}, x_{1n}]$,

$$|\tau_{jn}(x)| \le C f_{jn}(x). \tag{47}$$

PROOF. The idea already appears in [6], [7] and the proof is very similar to that in [4], but we include the details. Note first that

$$H[\chi_{jn}](x) = \log\left|\frac{x-x_{jn}}{x_{j-1,n}-x}\right| = -\log\left|1 - \frac{|I_{jn}|}{x-x_{jn}}\right|. \tag{48}$$

We consider two ranges of x:

(I) $|x-x_{jn}| \ge 2|I_{jn}|$

Using the inequality

$$|\log(1-t)+t| \le t^2, \quad |t| \le \frac{1}{2}$$

we see that

$$\begin{aligned}\left|\frac{1}{x-x_{jn}} - \frac{H[\chi_{jn}](x)}{|I_{jn}|}\right| &= \frac{1}{|I_{jn}|}\left|\frac{|I_{jn}|}{x-x_{jn}} + \log\left[1 - \frac{|I_{jn}|}{x-x_{jn}}\right]\right| \\ &\le \frac{|I_{jn}|}{(x-x_{jn})^2}.\end{aligned}$$

Then our bound (26) for p_n gives (47) for this range of x.

(II) $|x-x_{jn}| < 2|I_{jn}|$

From (28) and (21) we obtain if $2 \le j \le n$,

$$\begin{aligned}|p_n w|(x) &\le C\frac{g_n(x)^{-1/4}}{|I_{jn}|}\min\{|x-x_{jn}|, |x-x_{j-1,n}|\} \\ &\le C f_{jn}(x)\min\{|x-x_{jn}|, |x-x_{j-1,n}|\}.\end{aligned}$$

(We also use the fact that if k is fixed, $|I_{jn}| \sim |I_{j\pm k,n}|$ uniformly in j, n). For $j = 1$, this holds with the minimum replaced by $|x-x_{1n}|$. Then for $2 \le j \le n$, the first identity in (48) shows that

$$|\tau_{jn}(x)| \le C f_{jn}(x)\left[1 + \min\{|x-x_{jn}|, |x-x_{j-1,n}|\}\frac{1}{|I_{jn}|}\left|\log\left|\frac{x-x_{jn}}{x_{j-1,n}-x}\right|\right|\right]. \tag{49}$$

Since

$$|I_{jn}| \geq C \max\{|x - x_{jn}|, |x - x_{j-1,n}|\}$$

we see that with

$$u := \left|\frac{x - x_{jn}}{x_{j-1,n} - x}\right|,$$

we obtain for both signs of the exponent,

$$|\tau_{jn}(x)| \leq C f_{jn}(x)[1 + u^{\pm 1}|\log u^{\pm 1}|].$$

As either u or u^{-1} lies in $[0, 1]$ and $t \log t$ is bounded for $t \in [0, 1]$, we obtain (47). It remains to handle the case $j = 1$. Note that for $x \in [x_{nn}, x_{1n}]$, (it is only here we need this restriction), with $|x - x_{1n}| \leq 2|I_{1n}|$, we have

$$|x - x_{0n}| \sim \delta_n.$$

(See (17)). Then instead of (49), we obtain

$$|\tau_{1n}(x)| \leq C f_{1n}(x)\left[1 + C\frac{|x - x_{1n}|}{\delta_n}\left|\log\left[\sigma\frac{|x - x_{1n}|}{\delta_n}\right]\right|\right],$$

where $\sigma \sim 1$ independently of x, j, n. As $|x - x_{1n}| \leq C\delta_n$, the boundedness of $t \log t$ in any finite subinterval of $[0, \infty)$ again gives the result. ∎

4. **Proof of Theorem 1.2.** Throughout we assume that $w \in \mathcal{W}$, that the hypotheses of Theorem 1.2 hold, and assume the notation of Section 2, as well as (45), (46). We shall break the proof of Theorem 1.2 into several steps:

STEP 1. Express PW as a sum of two terms.

Let $P \in \mathcal{P}_{n-1}$. For $1 \leq k \leq n$, set

$$y_{kn} := \frac{(Pw)(x_{kn})}{(p_n' w)(x_{kn})}$$

and recall that

$$\lambda_{kn} w^{-2}(x_{kn}) \sim x_{k-1,n} - x_{kn} = |I_{kn}|.$$

We write

$$\begin{aligned}(Pw)(x) &= (L_n[P]w)(x)\\ &= (p_n w)(x) \sum_{k=1}^{n} y_{kn}\left[\frac{1}{x - x_{kn}} - \frac{H[\chi_{kn}](x)}{|I_{kn}|}\right]\\ &\quad + (p_n w)(x) H\left[\sum_{k=1}^{n} y_{kn}\frac{\chi_{kn}}{|I_{kn}|}\right](x) =: J_1(x) + J_2(x).\end{aligned} \tag{50}$$

Note that in view of the behaviour of the smallest and largest zeros (see (19)) and the restricted range inequality Lemma 2.4, we have for some C independent of P and n,

$$\|Pwg_n^{\Delta}\|_{L_p[-1,1]} \leq C\|Pwg_n^{\Delta}\|_{L_p[x_{nn},x_{1n}]}. \tag{51}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

STEP 2. Estimate $\|J_2 g_n^\Delta\|$.

(We begin with J_2 as it is easier to handle). We may write

$$\Delta = \frac{1}{4} + r \tag{52}$$

where

$$-\frac{1}{p} < r < 1 - \frac{1}{p} \quad \text{and} \quad r < \frac{1}{2}\left(1 + \frac{1}{p}\right). \tag{53}$$

Using our bound (26) for p_n, we then have

$$\begin{aligned}
\|J_2 g_n^\Delta\|_{L_p[x_{nn},x_{1n}]} &\le C\left\| H\left[\sum_{k=1}^n y_{kn} \frac{\chi_{kn}}{|I_{kn}|}\right] g_n^r \right\|_{L_p[x_{nn},x_{1n}]} \\
&\le C\left\| \left[\sum_{k=1}^n y_{kn} \frac{\chi_{kn}}{|I_{kn}|}\right] g_n^r \right\|_{L_p[x_{nn},x_{1n}]} \\
&= C\left[\sum_{k=1}^n \left\{\frac{|y_{kn}|}{|I_{kn}|}\right\}^p \int_{I_{kn}} g_n^{rp}\right]^{1/p}.
\end{aligned}$$

where, in the second last line, we used Lemma 3.1. Now by (21),

$$\int_{I_{kn}} g_n^{rp} \sim g_n(x_{kn})^{rp} |I_{kn}|$$

and by (22),

$$|y_{kn}| \sim |Pw|(x_{kn}) |I_{kn}| g_n(x_{kn})^{1/4}. \tag{54}$$

Then using (52) followed by (20), we have

$$\begin{aligned}
\|J_2 g_n^\Delta\|_{L_p[x_{nn},x_{1n}]} &\le C\left[\sum_{k=1}^n |I_{kn}|\, |Pwg_n^\Delta|^p(x_{kn})\right]^{1/p} \\
&\le C\left[\sum_{k=1}^n \lambda_{kn} w^{-2}(x_{kn}) |Pwg_n^\Delta|^p(x_{kn})\right]^{1/p}.
\end{aligned} \tag{55}$$

STEP 3. Estimate $\|J_1 g_n^\Delta\|$.

By Lemma 3.3,

$$|J_1(x)| = \left|\sum_{k=1}^n y_{kn}\tau_{kn}(x)\right| \le C\sum_{k=1}^n |y_{kn}| f_{kn}(x)$$

so

$$\|J_1 g_n^\Delta\|_{L_p[x_{nn},x_{1n}]} \le C\left[\sum_{j=2}^n \int_{I_{jn}} \left[\sum_{k=1}^n |y_{kn}| f_{kn}(x)\right]^p g_n^{\Delta p}(x)\, dx\right]^{1/p}.$$

Using the spacing (20), (21) and the definition (46) of f_{kn}, we see that

$$f_{kn}(x) \sim \frac{|I_{kn}|}{(x_{jn} - x_{kn})^2} g_n(x_{jn})^{-1/4}, \quad x \in I_{jn}$$

uniformly in n and in j, k with $j \ne k$. We deduce that

$$\|J_1 g_n^\Delta\|_{L_p[x_{nn},x_{1n}]} \le C[S_1 + S_2], \tag{56}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

where (recall (52) and (54))

$$S_1 := \left[\sum_{j=2}^{n} |I_{jn}| \left[\sum_{\substack{k=1\\k\neq j}}^{n} |y_{kn}| \frac{|I_{kn}|}{(x_{jn}-x_{kn})^2} g_n(x_{jn})^r\right]^p\right]^{1/p}$$

and by (46) and (21),

$$S_2 := \left[\sum_{j=2}^{n} |y_{jn}|^p |I_{jn}|^{1-p} g_n(x_{jn})^{rp}\right]^{1/p}.$$

Exactly as in the last part of Step 2, we see that (54) gives

$$S_2 \leq C\Big[\sum_{j=1}^{n} \lambda_{jn} w^{-2}(x_{jn}) |Pwg_n^{\Delta}|^p(x_{jn})\Big]^{1/p}. \tag{57}$$

To deal with S_1, we use Lemma 3.2 with a discrete measure space. Using (54), we see that

$$S_1 \leq C\Big[\sum_{j=1}^{n}\Big[\sum_{k=1}^{n} b_{jk}\{|I_{kn}|^{1/p}|PWg_n^{\Delta}|(x_{kn})\}\Big]^p\Big]^{1/p}$$

where

$$b_{kk} = b_{1k} = 0, \quad 1 \leq k \leq n$$

and for $j \neq k$,

$$b_{jk} := |I_{kn}|^{2-\frac{1}{p}} |I_{jn}|^{1/p} (x_{kn}-x_{jn})^{-2} \left(\frac{g_n(x_{jn})}{g_n(x_{kn})}\right)^r.$$

Defining the $n \times n$ matrix $B := (b_{jk})_{j,k=1}^n$, we see that if ℓ_p^n denotes $\mathbb{R}^n$ with the usual ℓ_p norm, then

$$S_1 \leq C\|B\|_{\ell_p^n \to \ell_p^n}\Big[\sum_{k=1}^{n} |I_{kn}|\,|PWg_n^{\Delta}|^p(x_{kn})\Big]^{1/p}.$$

If we can show that for some C_1 independent of n, that

$$\|B\|_{\ell_p^n \to \ell_p^n} \leq C_1, \quad n \geq 1, \tag{58}$$

then, we obtain, taking account of (56) and our estimate (57) on S_2 that

$$\|J_1 g_n^{\Delta}\|_{L_p[x_{nn},x_{1n}]} \leq C\Big[\sum_{k=1}^{n} \lambda_{kn} w^{-2}(x_{kn}) |Pwg_n^{\Delta}|^p(x_{kn})\Big]^{1/p}.$$

Together with (50) and our estimate (55) for J_2, we then obtain the desired inequality (7).

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

STEP 4. The proof of (58).

Let us set $\Omega = \{1, 2, 3, \ldots, n\}$ in Lemma 3.2, and let us set there $\mu(\{j\}) = 1$, $1 \le j \le n$, and

$$S(j,k) := b_{jk}; \quad R(j,k) := \left(\frac{|I_{kn}|}{|I_{jn}|}\right)^{\frac{1}{pq}} \left(\frac{g_n(x_{kn})}{g_n(x_{jn})}\right)^{\frac{\alpha}{pq}}$$

where α will be chosen later. We see that Lemma 3.2 gives (58) if we can show that

$$\sup_j \sum_{k=1}^{n} S(j,k)R(j,k)^q \le C;$$
$$\sup_k \sum_{j=1}^{n} S(j,k)R(j,k)^{-p} \le C,$$

that is, if we recall the choice of $\{b_{jk}\}$, S, R and that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sup_j \sum_{\substack{k=1\\k\neq j}}^{n} |I_{kn}|^2 (x_{kn} - x_{jn})^{-2} \left(\frac{g_n(x_{kn})}{g_n(x_{jn})}\right)^{-r+\alpha/p} \le C;$$
$$\sup_k \sum_{\substack{j=1\\j\neq k}}^{n} |I_{kn}|\,|I_{jn}| (x_{kn} - x_{jn})^{-2} \left(\frac{g_n(x_{kn})}{g_n(x_{jn})}\right)^{-r-\alpha/q} \le C.$$

Now recall (20) and (21). Then we see that we can reformulate these sums as integrals, and it suffices to show that for $\sigma = 0, 1$,

$$\sup_{x\in[-a_n,a_n]} \frac{\phi_n(x)^{1-\sigma}}{n} \int_{\{t\in[-a_n,a_n]:|x-t|\ge\frac{C}{n}\phi_n(x)\}} \frac{\phi_n^{\sigma}(t)}{(x-t)^2} \left(\frac{g_n(t)}{g_n(x)}\right)^{\beta_\sigma} dt \le C_1, \tag{59}$$

where

$$\beta_0 := r + \frac{\alpha}{q}; \quad \beta_1 := -r + \frac{\alpha}{p}. \tag{60}$$

(Note that our range of integration and range of x may exclude small intervals around x_{1n} or x_{nn}, but this is fine in view of (20), (21); the term $\sigma = 0$ corresponds to the second sum and $\sigma = 1$ corresponds to the first sum). We need only estimate the integral for $x \in [0, a_n]$ and thus need only show for $\sigma = 0, 1$,

$$\sup_{x\in[0,a_n]} \frac{\phi_n(x)^{1-\sigma}}{n} \int_{\{t\in[0,a_n]:|x-t|\ge\frac{C}{n}\phi_n(x)\}} \frac{\phi_n^{\sigma}(t)}{(x-t)^2} \left(\frac{g_n(t)}{g_n(x)}\right)^{\beta_\sigma} dt \le C_1. \tag{61}$$

Now for $x, t \in [0, a_n]$,

$$x - t = a_n[g_n(t) - g_n(x)]$$

and recall that $\phi_n(t)$ is given by (18) and $g_n(t)$ is given by (5). Thus making the substitutions $u = g_n(t)$, $v = g_n(x)$ it suffices to show for $\sigma = 0, 1$,

$$\sup_{v\in[\delta_n,1+\delta_n]} \frac{\psi_n(v)^{1-\sigma}}{n} \int_{\{u\in[\delta_n,1+\delta_n]:|u-v|\ge\frac{C}{n}\psi_n(v)\}} \frac{\psi_n^{\sigma}(u)}{(u-v)^2} \left(\frac{u}{v}\right)^{\beta_\sigma} du \le C_1, \tag{62}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

where

$$\psi_n(s) := \max\left\{\sqrt{s}, \frac{1}{T(a_n)\sqrt{s}}\right\}. \tag{63}$$

Let us make the further substitution $u = vy$. We see that it suffices to show that for our range of v, and $\sigma = 0, 1$,

$$I := \frac{\psi_n(v)^{1-\sigma}}{nv} \int_{\{y\in[\delta_n/v, 2/v]:|y-1|\geq \frac{C}{nv}\psi_n(v)\}} \frac{\psi_n^\sigma(vy)y^{\beta_\sigma}}{(y-1)^2}\, dy \leq C_1. \tag{64}$$

Note that

$$\begin{aligned}\frac{\psi_n(v)}{nv} &= \max\left\{\frac{1}{n\sqrt{v}}, \frac{1}{nT(a_n)v^{3/2}}\right\} \leq \max\left\{\frac{1}{n\sqrt{\delta_n}}, \frac{1}{nT(a_n)\delta_n^{3/2}}\right\} \\ &= \max\left\{\left(\frac{T(a_n)}{n^2}\right)^{1/3}, 1\right\} \leq 1,\end{aligned}$$

for large n, by (31). Thus if C in the limit of integration in (64) is small enough, we may estimate

$$\begin{aligned} I &\leq \frac{\psi_n(v)^{1-\sigma}}{nv}\left[\int_{\delta_n/v}^{1/2} + \int_{\{y\in[1/2,2]:|y-1|\geq\frac{C}{nv}\psi_n(v)\}} + \int_2^{2/v}\right]\frac{\psi_n^\sigma(vy)y^{\beta_\sigma}}{(y-1)^2}\, dy \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{65}$$

Firstly in I_2, we have $\psi_n(vy) \sim \psi_n(v)$, so

$$I_2 \leq C\frac{\psi_n(v)}{nv}\int_{\{y\in[1/2,2]:|y-1|\geq\frac{C}{nv}\psi_n(v)\}} \frac{dy}{(y-1)^2} \leq C. \tag{66}$$

We also see that

$$I_1 \leq C\frac{\psi_n(v)^{1-\sigma}}{nv}\int_{\delta_n/v}^{1/2} \psi_n^\sigma(vy)y^{\beta_\sigma}\, dy \tag{67}$$

and

$$I_3 \leq C\frac{\psi_n(v)^{1-\sigma}}{nv}\int_2^{2/v} \psi_n^\sigma(vy)y^{\beta_\sigma-2}\, dy. \tag{68}$$

We now distinguish two ranges of x:

RANGE I. $v \in [\delta_n, \frac{1}{T(a_n)}]$
Here

$$\psi_n(v) \sim \frac{1}{T(a_n)\sqrt{v}}$$

and

$$y \in \left[0, \frac{1}{2}\right] \Rightarrow \psi_n(vy) \sim \frac{1}{T(a_n)\sqrt{vy}}$$

so

$$I_1 \leq \frac{C}{nT(a_n)v^{3/2}}\int_0^{1/2} y^{\beta_\sigma-\sigma/2}\, dy \leq \frac{C}{nT(a_n)\delta_n^{3/2}} = C,$$

provided

$$\beta_\sigma - \frac{\sigma}{2} > -1, \quad \sigma = 0, 1. \tag{69}$$

Next, in estimating I_3, we must take account of the fact that

$$y \le \frac{1}{T(a_n)v} \Rightarrow \psi_n(vy) \sim \frac{1}{T(a_n)\sqrt{vy}};$$
$$y > \frac{1}{T(a_n)v} \Rightarrow \psi_n(vy) \sim \sqrt{vy}.$$

Then

$$\begin{aligned} I_3 \le{}& \frac{C}{nT(a_n)v^{3/2}} \int_2^{\max\{2, \frac{1}{T(a_n)v}\}} y^{-\sigma/2+\beta_\sigma-2}\, dy \\ &+ \frac{C}{nT(a_n)v^{3/2}} [T(a_n)v]^\sigma \int_{\max\{2, \frac{1}{T(a_n)v}\}}^{2/v} y^{\sigma/2+\beta_\sigma-2}\, dy \le C \end{aligned}$$

as $T(a_n)v \le 1$ and provided

$$\beta_\sigma + \sigma/2 < 1, \quad \sigma = 0, 1. \tag{70}$$

RANGE II. $v \in [\frac{1}{T(a_n)}, 1 + \delta_n]$
Here

$$\psi_n(v) \sim \sqrt{v}$$

and

$$y \in [2, \infty) \Rightarrow \psi_n(vy) \sim \sqrt{vy}$$

so

$$I_3 \le \frac{C}{n\sqrt{v}} \int_2^{2/v} y^{\sigma/2+\beta_\sigma-2}\, dy \le C\frac{T(a_n)^{1/2}}{n} = o(1)$$

by (31) and provided (70) holds. Next, we estimate

$$\begin{aligned} I_1 &\le \frac{C(\sqrt{v})^{1-\sigma}}{nv} \left[\int_{\frac{\delta_n}{v}}^{\max\{\frac{\delta_n}{v}, \frac{1}{T(a_n)v}\}} \left(\frac{1}{T(a_n)\sqrt{vy}} \right)^\sigma y^{\beta_\sigma}\, dy + \int_{\max\{\frac{\delta_n}{v}, \frac{1}{T(a_n)v}\}}^{\frac{1}{2}} \left(\sqrt{vy}\right)^\sigma y^{\beta_\sigma}\, dy \right] \\ &\le \frac{C}{n\sqrt{v}} \left(T(a_n)v\right)^{-\sigma} \int_0^1 y^{-\sigma/2+\beta_\sigma}\, dy + \frac{C}{n\sqrt{v}} \int_0^1 y^{\sigma/2+\beta_\sigma}\, dy \le C \end{aligned}$$

provided (69) holds. Recall too that $T(a_n)v \ge 1$ and (31), which implies that $v \ge Cn^{-2+\varepsilon}$ in this present range.

In summary, we have shown that for $v \in [\delta_n, 1 + \delta_n]$ and $\sigma \in \{0, 1\}$,

$$I \le C(I_1 + I_2 + I_3) \le C$$

and hence have completed the proof of (58), provided we can choose the parameter α in β_σ to satisfy

$$1 - \frac{\sigma}{2} > \beta_\sigma > \frac{\sigma}{2} - 1, \quad \sigma = 0, 1. \tag{71}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

STEP 5. The proof of (71) for a suitable choice of α.

We see from (60) that we need

$$-1 < r + \frac{\alpha}{q} < 1; \quad -\frac{1}{2} < -r + \frac{\alpha}{p} < \frac{1}{2}.$$

Rearranging these inequalities leads to

$$-q(1+r) < \alpha < q(1-r); \quad p\Big(r - \frac{1}{2}\Big) < \alpha < p\Big(r + \frac{1}{2}\Big).$$

We see that we can choose such an α provided

$$-q(1+r) < p\Big(r + \frac{1}{2}\Big); \quad p\Big(r - \frac{1}{2}\Big) < q(1-r).$$

Solving for r leads to

$$-\frac{p/2+q}{p+q} < r < \frac{p/2+q}{p+q} \iff -\frac{1}{2}\Big(1 + \frac{1}{p}\Big) < r < \frac{1}{2}\Big(1 + \frac{1}{p}\Big).$$

Finally, we also needed (38) for the application of Lemma 3.1, namely that

$$-\frac{1}{p} < r < 1 - \frac{1}{p}.$$

Comparison of the lower and upper bounds for r shows that we need

$$-\frac{1}{p} < r < \min\Big\{1 - \frac{1}{p}, \frac{1}{2}\Big(1 + \frac{1}{p}\Big)\Big\},$$

which is precisely (53). So we have (71) and hence (58).

Finally, recalling that $\Delta = r + \frac{1}{4}$ gives the condition (6) of Theorem 1.2. ∎

5. Proof of Theorems 1.3 to 1.6.

We begin with the

PROOF OF THE SUFFICIENCY PART OF THEOREM 1.3. Assume (9). Now if (8) holds for a given Δ, then it also holds for any larger Δ, as g_n is bounded in $[-1,1]$, independently of n. Thus we may assume that Δ is so small that (6) holds. Then setting $P := L_n[f]$ in (7), we have

$$\begin{aligned}\|L_n[f]wg_n^\Delta\|_{L_p[-1,1]} &\le C\Big(\sum_{k=1}^{n}\lambda_{kn}w^{-2}(x_{kn})|fwg_n^\Delta|^p(x_{kn})\Big)^{1/p}\\ &\le \|fw\|_{L_\infty[-1,1]}C\Big(\sum_{k=1}^{n}(x_{k-1,n}-x_{k,n})g_n^{\Delta p}(x_{kn})\Big)^{1/p}\\ &\le \|fw\|_{L_\infty[-1,1]}C\Big(\int_{-a_n}^{a_n} g_n^{\Delta p}(x)\,dx\Big)^{1/p}\end{aligned}$$

by (20) and (21). Then we continue this as

$$= \|fw\|_{L_\infty[-1,1]}Ca_n^{1/p}\Big(\int_{-1}^{1}\Big[\big|1-|t|\big| + \delta_n\Big]^{\Delta p}dx\Big)^{1/p} \le C\|fw\|_{L_\infty[-1,1]},$$

as $\Delta p > -1$. ∎

In the proof of the necessity part of all the theorems, we use the following:

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

LEMMA 5.1. *Let* $[a,b]$ *be a closed subinterval of* $(-1,0)$ *and for* $n \geq 1$, *let* $f_n\colon (-1,1) \longrightarrow \mathbb{R}$, *with* $f_n = 0$ *outside* $[a,b]$, *and*

$$f_n(x_{jn}) = w^{-1}(x_{jn})\operatorname{sign}\left(p_n'(x_{jn})\right), \quad x_{jn} \in (a,b). \tag{72}$$

Then there exists n_0 *such that for* $n \geq n_0$ *and* $x \in [0,1]$,

$$|L_n[f_n](x)| \geq C|p_n(x)|. \tag{73}$$

PROOF. Since $[a,b] \subset (-1,0)$, we have for $x \in [0,1]$,

$$\begin{aligned} L_n[f_n](x) &= p_n(x) \sum_{x_{jn}\in(a,b)} \frac{1}{|p_n'w|(x_{jn})(x-x_{jn})} \\ &\sim p_n(x) \sum_{x_{jn}\in(a,b)} \frac{(x_{jn}-x_{j+1,n})}{x+|x_{jn}|} \\ &\sim p_n(x) \sum_{x_{jn}\in(a,b)} (x_{jn}-x_{j+1,n}) \sim p_n(x). \end{aligned}$$

Here we have used (22), and the fact that $-1 < a < b < 0$, so that $g_n \sim 1$ in $[a,b]$ for large n. ∎

PROOF OF THE NECESSITY PART OF THEOREM 1.3. Assume (8). Construct f_n as in Lemma 5.1 so that f_n also satisfies

$$\|f_n w\|_{L_\infty[-1,1]} = 1.$$

(We may also assume that f_n is continuous, but that is irrelevant to the proof). Then for some C_1 independent of n,

$$\begin{aligned} C &= C\|f_n w\|_{L_\infty[-1,1]} \geq \|L_n[f_n]wg_n^\Delta\|_{L_p[-1,1]} \\ &\geq C\|p_n w g_n^\Delta\|_{L_p[0,x_{1n}]} \geq C\|g_n^{\Delta-\frac14}\|_{L_p[0,x_{1n}]} \end{aligned} \tag{74}$$

by first Lemma 5.1 and then Lemma 2.3. Now (19) and an easy calculation (compare (27)) shows that

$$\|g_n^r\|_{L_p[0,x_{1n}]} \sim \begin{cases} 1, & r > -\frac1p \\ (\log n)^{1/p}, & r = -\frac1p \\ \delta_n^{r+\frac1p}, & r < -\frac1p \end{cases}.$$

Since the last two terms on the right-hand side grow to ∞ with n, we deduce from (74) that

$$\Delta - \frac14 > -\frac1p,$$

that is, (9) holds. ∎

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

PROOF OF THEOREM 1.4. Let f satisfy (10) and P be a polynomial. Then from Theorem 1.2 with $\Delta = 0$, and n large enough,

$$\begin{aligned}\|(f - L_n[f])w\|_{L_p[-1,1]} &\le \|(f-P)w\|_{L_p[-1,1]} + \|L_n[P-f]w\|_{L_p[-1,1]} \\ &\le \|(f-P)w\|_{L_p[-1,1]} + C\Big(\sum_{k=1}^n \lambda_{kn} w^{-2}(x_{kn})|(P-f)w|^p(x_{kn})\Big)^{1/p}.\end{aligned}$$

Now by hypothesis, fw is Riemann integrable over each compact subinterval of $(-1,1)$, and for some $\alpha < \frac{1}{p}$,

$$\lim_{|x|\to 1-} (fw)(x)(1-x^2)^\alpha = 0.$$

The same is true of Pw (even with $\alpha = 0$). Next, by Lemma 10.1 in [8, p. 106], there exists

$$H(x) = \sum_{j=0}^\infty h_{2j} x^{2j}, \quad \text{all } h_{2j} \ge 0,$$

with

$$H(x) \sim w^{-2}(x) \quad \text{in } (-1,1).$$

Defining

$$G(x) := H(x)(1-x^2)^{-\alpha p}, \quad x \in (-1,1),$$

we see that G is even, has a Maclaurin series with all non-negative coefficients, and by (10),

$$\lim_{|x|\to 1-} w^{-2}(x)|(P-f)w|(x)^p / G(x) = 0.$$

Next, given any fixed $M > 0$, we have for large enough n, (see (33))

$$a_n + M\delta_n = a_n\left(1 + o\left(\frac{1}{T(a_n)}\right)\right) \le a_{2n} < 1.$$

Hence, given a large enough $M > 0$, we have for large n,

$$\begin{aligned}\sum_{j=1}^n \lambda_{jn} G(x_{jn}) &\sim \sum_{j=1}^n \lambda_{jn} w^{-2}(x_{jn})(1-x_{jn}^2)^{-\alpha p} \\ &\le C\sum_{j=1}^n \lambda_{jn} w^{-2}(x_{jn})(a_n + M\delta_n - x_{jn}^2)^{-\alpha p} \\ &\le C\sum_{j=1}^n \lambda_{jn} w^{-2}(x_{jn}) g_n(x_{jn})^{-\alpha p} \le C\int_{-a_n}^{a_n} g_n(x)^{-\alpha p}\,dx \le C\end{aligned}$$

by an easy calculation, as $\alpha p < 1$. Here we have also used (19), (21). Then Theorem 1.6(b) [5, p. 94] shows that

$$\lim_{n\to\infty} \sum_{k=1}^n \lambda_{kn} w^{-2}(x_{kn})|(P-f)w|^p(x_{kn}) = \int_{-1}^1 |(P-f)w|^p$$

and hence

$$\limsup_{n\to\infty} \|(f - L_n[f])w\|_{L_p[-1,1]} \le C\|(f - P)w\|_{L_p[-1,1]}.$$

Here C is independent of P and n (it came only from the converse quadrature sum estimate). Moreover, our condition (10) on f and density of polynomials in $L_p[-1,1]$, easily imply that this last right-hand side may be made as small as we please. ∎

PROOF OF THE SUFFICIENCY PART (A) OF THEOREM 1.5. Let

$$F(x) := 1 + Q^{2/3}(x)T(x). \tag{75}$$

We first show that

$$g_n(x)F(x) \ge C, \quad x \in (-1,1), n \ge 1. \tag{76}$$

We need only do this for $x \in [0,1)$ and consider three ranges of x.

(I) $x \in [0, a_{n/2}]$

Write $x = a_r$. Then

$$g_n(x) \ge 1 - \frac{a_r}{a_n} \ge 1 - \frac{a_r}{a_{2r}} \sim \frac{1}{T(x)}$$

by (33). Then

$$g_n(x)F(x) \ge C\left[\frac{1}{T(x)} + Q^{2/3}(x)\right] \ge C$$

so (76) follows.

(II) $x \in [a_{n/2}, a_{2n}]$

Here by (29), (30),

$$F(a_n) \sim Q^{2/3}(a_n)T(a_n) \sim \left(nT(a_n)\right)^{2/3} = \delta_n^{-1}. \tag{77}$$

As $g_n \ge \delta_n$, (76) follows.

(III) $x \in [a_{2n}, 1)$

As both F and g_n are increasing over this range of x, (76) follows from the previous range of x.

Next let P be a polynomial and f satisfy the hypotheses of Theorem 1.5(a). We proceed similarly to Theorem 1.4. Note that $\Delta > 0$ follows from (12). We also note that if the conclusion of Theorem 1.5(a) holds for a given Δ, then it holds for any larger Δ, so we may assume that Δ is small enough to satisfy (6). Then using (76),

$$\begin{aligned}
&\|(f - L_n[f])wF^{-\Delta}\|_{L_p[-1,1]} \\
&\qquad \le C[\|(f-P)wg_n^{\Delta}\|_{L_p[-1,1]} + \|L_n[P-f]wg_n^{\Delta}\|_{L_p[-1,1]}] \\
&\qquad \le C\left[\|(f-P)wg_n^{\Delta}\|_{L_p[-1,1]} + \left(\sum_{k=1}^{n} \lambda_{kn} w^{-2}(x_{kn})|(P-f)wg_n^{\Delta}|^p(x_{kn})\right)^{1/p}\right] \\
&\qquad \le C\left[\|(f-P)w\|_{L_p[-1,1]} + \left(\sum_{k=1}^{n} \lambda_{kn} w^{-2}(x_{kn})|(P-f)w|^p(x_{kn})\right)^{1/p}\right]
\end{aligned}$$

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

as g_n^{Δ} is bounded in $[-1,1]$ independently of n. Then proceeding as in the previous proof, we obtain

$$\limsup_{n\to\infty} \|(f - L_n[f])wF^{-\Delta}\|_{L_p[-1,1]} \le C\|(f-P)w\|_{L_p[-1,1]}$$

with C independent of P and the result follows. ∎

PROOF OF THE NECESSITY PART (B) OF THEOREM 1.5. Define f_n as in Lemma 5.1 with $[a,b] = [-\frac{1}{2}, -\frac{1}{4}]$ and the additional restrictions that f_n is continuous in $(-1,1)$ and

$$\|f_n w\|_{L_\infty[-1,1]} = 1.$$

Let F be given by (75). By the conclusion of Lemma 5.1,

$$\begin{aligned}\|L_n[f_n]wF^{-\Delta}\|_{L_p[-1,1]} &\ge C\|p_n wF^{-\Delta}\|_{L_p[a_{n/2},x_{n1}]}\\ &\ge CF(a_n)^{-\Delta}\|g_n^{-1/4}\|_{L_p[a_{n/2},x_{n1}]}\end{aligned}$$

by Lemma 2.3. A straightforward calculation and (77) show that we may continue this as

$$\ge C\delta_n^{\Delta-\frac{1}{4}+\frac{1}{p}} \times \begin{cases}(\log n)^{1/4}, & p=4\\ 1, & p>4\end{cases}. \tag{78}$$

Next by applying the uniform boundedness principle to suitable (and obviously defined) spaces of functions, we deduce from the hypothesis of (b) that for $n \ge 1$ and for every continuous $h\colon(-1,1) \longrightarrow \mathbb{R}$ vanishing outside $[-\frac{1}{2}, \frac{1}{2}]$ that

$$\|L_n[h]wF^{-\Delta}\|_{L_p[-1,1]} \le C\|hw\|_{L_\infty[-1,1]},$$

where C is independent of h and n. Applying this to $h = f_n$ gives

$$C \ge \delta_n^{\Delta-\frac{1}{4}+\frac{1}{p}} \times \begin{cases}(\log n)^{1/4}, & p=4\\ 1, & p>4\end{cases}.$$

Recall that δ_n decays to 0 as $n \longrightarrow \infty$ faster than $n^{-2/3}$. Then for $p = 4$, we deduce that $\Delta > 0$ and for $p > 4$, we deduce that $\Delta \ge \frac{1}{4} - \frac{1}{p}$. ∎

Finally, we turn to

THE PROOF OF THEOREM 1.6. Assume that for continuous $f\colon(-1,1) \longrightarrow \mathbb{R}$ such that f vanishes outside $[-\frac{1}{2}, \frac{1}{2}]$ we have

$$\limsup_{n\to\infty} \|L_n[f]wU\|_{L_p[-1,1]} < \infty. \tag{79}$$

Let f_n be as in the previous proof and F be given by (75). Then as above

$$\begin{aligned}C &\ge \|L_n[f_n]wU\|_{L_p[-1,1]} \ge C\|p_n wU\|_{L_p[a_{n/2},x_{n1}]}\\ &\ge C\|g_n^{-1/4}U\|_{L_p[a_{n/2},x_{n1}]}.\end{aligned}$$

Now given $\lambda > 0$, we can by hypothesis (15), and then (77) continue this for large n as

$$\ge \lambda F(a_{n/2})^{\frac{1}{p}-\frac{1}{4}}\|g_n^{-1/4}\|_{L_p[a_{n/2},x_{n1}]} \ge C_1\lambda\delta_n^{\frac{1}{4}-\frac{1}{p}}\|g_n^{-1/4}\|_{L_p[a_{n/2},x_{n1}]} \ge C_2\lambda,$$

where C_1, C_2 are independent of λ. For large enough λ, we obtain a contradiction. ∎

https://doi.org/10.4153/CJM-1998-062-5 Published online by Cambridge University Press

REFERENCES

1. S. B. Damelin, *The Weighted Lebesgue Constant of Lagrange Interpolation for Non-Szegö Weights on* $[-1,1]$. Acta Math. Hungar., to appear.

2. ———, *Lagrange Interpolation for non-Szegö Weights on* $[-1,1]$. Proceedings of the International Workshop on Approximation Theory and Numerical Analysis, to appear.

3. S. B. Damelin and D. S. Lubinsky, *Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Erdős Weights*. Canad. J. Math. **40**(1996), 710–736.

4. ———, *Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Erdős Weights II*. Canad. J. Math. **40**(1996), 737–757.

5. G. Freud, *Orthogonal Polynomials*. Akademiai Kiado/Pergamon Press, Budapest, 1971.

6. H. König, *Vector Valued Lagrange Interpolation and Mean Convergence of Hermite Series*. Proc. Essen Conference on Functional Analysis, North Holland.

7. H. König and N. J. Nielsen, *Vector Valued* L_p *Convergence of Orthogonal Series and Lagrange Interpolation*. Math. Forum **6**(1994), 183–207.

8. A. L. Levin and D. S. Lubinsky, *Christoffel Functions and Orthogonal Polynomials for Exponential Weights on* $[-1,1]$. Mem. Amer. Math. Soc. **535**(1994).

9. D. S. Lubinsky, *An Extension of the Erdős-Turan Inequality for the Sum of Successive Fundamental Polynomials*. Ann. of Math. **2**(1995), 305–309.

10. D. S. Lubinsky and D. Matjila, *Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Freud Weights*. SIAM J. Math. Anal. **26**(1995), 238–262.

11. H. N. Mhaskar and E. B. Saff, *Where Does The Sup Norm of a Weighted Polynomial Live?*. Constr. Approx. **1**(1985), 71–91.

12. G. Mastroianni, *Boundedness of the Lagrange Operator in Some Functional Spaces. A Survey*. to appear.

13. G. Mastroianni and M. G. Russo, *Weighted Marcinkiewicz Inequalities and Boundedness of the Lagrange Operator*. to appear.

14. B. Muckenhoupt, *Mean Convergence of Jacobi Series*. Proc. Amer. Math. Soc. **23**(1970), 306–310.

15. P. Nevai, *Orthogonal Polynomials*. Mem. Amer. Math. Soc. **213**(1979).

16. ———, *Mean Convergence of Lagrange Interpolation II*. J. Approx. Theory **30**(1980), 263–276.

17. ———, *Mean Convergence of Lagrange Interpolation III*. Trans. Amer. Math. Soc. **282**(1984), 669–698.

18. ———, *Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study*. J. Approx. Theory **48**(1986), 3–167.

19. J. Szabados and P. Vertesi, *Interpolation of Functions*. World Scientific, Singapore, 1990.

20. ———, *A Survey on Mean Convergence of Interpolatory Processes*. J. Comput. Appl. Math. **43**(1992), 3–18.

Mathematics Department
Witwatersrand University
Wits 2050
South Africa
e-mail: 036dsl@cosmos.wits.ac.za