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ENTIRE MEAN PERIODIC FUNCTIONS

P. G. LAIRD

Introduction. Let H denote the set of all entire functions of a single complex variable equipped with the topology of convergence uniform on all compact subsets of C, the set of complex numbers. Then an entire function f is mean periodic if the subspace spanned by f and its complex translates is not dense in H. It was shown by Schwartz [**13**, p. 922] in 1947, to whom this definition is due, that any such function is the limit in H of a certain sequence of exponential polynomials. Here, by an exponential polynomial is meant a finite linear combination of terms $u_n e_a : z \longrightarrow z^n e^{az}$ where n is any non-negative integer and a is any complex number.

In Section 2, a new transform is introduced that enables another proof of this property to be given. This transform has properties resembling those of the transform introduced by Kahane [**5**; **6**] in 1952 and used by him to give a simple proof of the main property of continuous mean periodic functions of a single real variable. Our transform utilizes a convolution product of entire functions that is described, along with other preliminaries, in Section 1.

Further properties of this convolution product and of entire mean periodic functions are given in Section 3. As well, results concerning special cases of systems of convolution equations considered by Malgrange [**11**] are alluded to. These new properties resemble those of continuous mean periodic functions of a single real variable discussed elsewhere by the author (Laird, [**7**; **10**]).

Section 4 is concerned briefly with properties of mean periodic functions of several complex variables that differ from results given in the first three sections.

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1. Preliminaries. Throughout this article, we shall assume, following Schwartz [**13**], certain facts although some change in notation is made. Let H' denote the dual space of the complex Fréchet space H. Any element of H'

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may be identified with at least one measure of compact support in the plane. If L and M are two non-zero elements of H', it is possible to define their convolution $L * M$ that is a unique non-zero element of H'. Moreover, H' with the operations of addition and convolution is an integral domain with identity δ where $\delta(f) = f(0)$ for all $f \in H$.

The Fourier-Laplace transform of an element $L \in H'$ is defined as $\hat{L}(z) = L(e_{-z})$. This transform is an isomorphism between $H'(+, *)$ and the integral domain comprising the set of entire functions of exponential type taken with the operations of addition and pointwise multiplication.

If f is any entire function, we set $f^{\vee} : \xi \to f(-\xi)$ and $T_z f : \xi \to f(z - \xi)$ for any complex number z. Also, let W_f denote the closed subspace of H spanned by f and its complex translates. Then f is mean periodic in H if $W_f \neq H$. Schwartz [**13**] showed that his definition is equivalent to each of the following:

A. there exists a non-zero element of L of H' for which $L * f(z) = L(T_z f^{\vee}) = 0$ for all complex z,

B. f is the limit in H of a sequence of exponential polynomials belonging to a proper closed translation invariant subspace of H,

C. there is a sequence $\{a_n\}$ of complex numbers, not all zero, with

$$\limsup \sqrt[n]{|a_n|} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} a_n D^n f(z)/n! = 0 \quad \text{for all complex } z,$$

and

D. the closed subspace spanned by f and its derivatives is distinct from H.

We may note that condition C corresponds with $L * f = 0, \hat{L}(z) = \sum_{n=0}^{\infty} a_n z^n/n!$ and results by use of the Laplace-Borel transform of $\hat{L}(z)$. The only lengthy proof of any of these statements is that of the necessity of condition B. Before giving a new proof of this, we consider the convolution product of two entire functions. Throughout, the term 'locally uniform convergence' will mean convergence uniform on all compact subsets of C.

THEOREM 1. *Let f, g be two entire functions. Then*

$$f \otimes g : z \to \int_0^z f(z - \xi) g(\xi) d\xi$$

is an entire function. Moreover,

(a) *if $\{f_n\}$, $\{g_n\}$ are sequences of entire functions that converge, locally uniformly, to f, g, then*

$$f_n \otimes g_n \to f \otimes g \text{ locally uniformly as } n \to \infty,$$

and

(b) *the equation $w - w \otimes g = f$ always has a unique entire function as a solution.*

Proof. It is clear that $f \otimes g$ is a function of z since for a fixed value of z, $f(z - \xi)g(\xi)$ is an entire function of ξ. That $f \otimes g$ is an entire function follows from the fact that any entire function is the locally uniform limit of a sequence of polynomials. With $u_m : z \to z^m$,

$$(1) \quad \frac{u_m}{m!} \otimes \frac{u_n}{n!} = \frac{u_{m+n+1}}{(m+n+1)!}$$

and so $\{f_n \otimes g_n\}$ is a sequence of polynomials when $\{f_n\}$, $\{g_n\}$ are sequences of polynomials. Moreover, if $f_n \to f$, $g_n \to g$ locally uniformly, an application of elementary estimates to

$$f \otimes g - f_n \otimes g_n = (f - f_n) \otimes g + f_n \otimes (g - g_n)$$

on any compact subset of C shows that $f_n \otimes g_n \to f \otimes g$ locally uniformly on C as $n \to \infty$. Hence $f \otimes g$ is an entire function. As well, statement (a) is established.

For part (b), let $g^{\otimes 1} = g$ be entire and $g^{\otimes (n+1)} = g^{\otimes n} \otimes g$ for $n = 1, 2, 3, \ldots$. If r is any positive number and if $c = \sup \{|g(z)| : z \leqq r\}$, then

$$|g^{\otimes (n+1)}(z)| \leqq c|cz|^n/n! \leqq c(cr)^n/n! \quad \text{for } |z| \leqq r.$$

Thus, $w_n = f + f \otimes g + \ldots f \otimes g^{\otimes n}$ is a Cauchy sequence in H whose limit w satisfies $w - w \otimes g = f$. If h is the difference between any two entire solutions to this equation, then $h = h \otimes g$. So $h = (h \otimes g) \otimes g = h \otimes g^{\otimes n}$ for $n = 1, 2, 3 \ldots$. Hence $h = 0$.

Remark. The above properties are analogous to the well known properties of continuous functions defined on a half line (see, for example, Erdélyi [**4**]). Also, if $D^n f$ denotes the n-th derivative of f and if f, g are entire, then

$$(2) \quad D^n(f \otimes g) = f(0)D^{n-1}g + Df(0) \cdot D^{n-2}g + \ldots D^{n-1}f(0) \cdot g + (D^n f) \otimes g.$$

2. A transform. Throughout this section, f will denote an entire mean periodic function and L will denote a non-zero element of H' satisfying $L * f = 0$.

Definition. The transform of f is

$$K(f)(z) = \frac{L(f \otimes e_z)^{\vee}}{\hat{L}(z)}.$$

We may compare this definition with the transform $J(F)$ introduced by Kahane [**5**; **6**] for a continuous mean periodic function F of a single real variable. If μ is a non-zero measure with a compact real support such that $\mu * F = 0$, and if $F^-(t) = F(t)$ for negative t and is otherwise zero, then

$$J(F)(z) = \int e^{-izt}(\mu * F^-(t))dt \Big/ \int e^{-izt} d\mu(t).$$

As it can be shown that $\int e^{-izt}(\mu * F^-)(t)dt = -\mu(F \otimes e_{iz})^{\vee}$, $J(F)(z)$ corresponds to $-K(F)(iz)$. As well, we may note that Delsarte [**1**; **2**] used the

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expression $\mu(F \otimes e_z)^\vee$ to obtain the coefficients c_k of a continuous mean periodic function F whose formal expansion is $\sum c_k e^{a_k t}$. For such a function, it can also be shown that c_k is equal to the residue of the pole of the transform $K(F)(z)$ at $z = a_k$.

It is necessary to show that $K(f)$ is independent of the choice of non-zero L satisfying $L * f = 0$. This is an immediate consequence of the following lemma for if $L, M \in H'$ and if $L * f = 0$, $M * f = 0$, then

$$\hat{L}(z) \cdot M(f \otimes e_z)^\vee = \hat{M}(z) \cdot L(f \otimes e_z)^\vee.$$

LEMMA 2. *Let g be any entire function and let L, M be any elements of H'. Then*

(3) $(L * M)(g \otimes e_z)^\vee = \hat{L}(z) \cdot M(g \otimes e_z)^\vee + L((M * g) \otimes e_z)^\vee$, *and*

(4) $(L * M)(g \otimes e_z)^\vee = \hat{M}(z) \cdot L(g \otimes e_z)^\vee + M((L * g) \otimes e_z)^\vee$.

Proof. We have

$$(L * M)(g \otimes e_z)^\vee = L_\xi[M_\eta\{g \otimes e_z(-\xi-\eta)\}] = L_\xi G(\xi, z) \quad \text{where}$$

$$\begin{aligned} G(\xi, z) &= M_\eta\left\{\int_0^{-\xi-\eta} e^{z(-\xi-\eta-\theta)} g(\theta) d\theta\right\} \\ &= e^{-z\xi} M(g \otimes e_z)^\vee + \int_0^{-\xi} e^{-z\xi-z\psi} M_\eta\{g(\psi-\eta)\} d\psi \\ &= e^{-z\xi} M(g \otimes e_z)^\vee + ((M * g) \otimes e_z)^\vee(\xi). \end{aligned}$$

Thus, (3) is established. Formulae (4) follows from (3) and the fact that $*$ is commutative.

From the next lemma, we see that the transform $K(f)$ is a meromorphic function that is the quotient of two entire functions of exponential type.

LEMMA 3. *Let g be any entire function and let M be any element of H'. Then $h(z) = M(g \otimes e_z)^\vee$ is an entire function of exponential type.*

Proof. By use of the linearity and continuity of M and $\otimes$, one can show that h is complex-differentiable at each point of the complex plane. So h is an entire function. Since M may be extended to a measure having a compact support in the plane, there exists positive constants c, T such that

$$|M(w)| \leqq c. \sup\{|w(\eta)| : |\eta| \leqq T\} \quad \text{for all } w \in H.$$

Let $\alpha = \sup\{|g(z)| : |z| \leqq T\}$. Then if $|\eta| \leqq T$,

$$|(g \otimes e_z)(\eta)| \leqq \alpha T \exp(2|\eta z|) \quad \text{so} \quad |h(z)| \leqq c\alpha T \exp(2T|z|)$$
for all complex z.

THEOREM 4. *A necessary and sufficient condition that $u_n e_a \in W_f$ is that $K(f)(z)$ has a pole at $z = a$ of order exceeding n.*

Proof. Suppose that $K(f)(z)$ has a pole at $z = a$ of order exceeding n, and, that L is any element of H' satisfying $L * W_f = \{0\}$. Then $L * f = 0$, $\hat{L}(z)$

has a zero at $z = a$ of order exceeding n and, from

$$(5)\quad L * u_n e_a = \sum_{k=0}^{n} {}^nC_k D^k \hat{L}(a) \cdot u_{n-k} e_a,$$

$L * u_n e_a = 0$. Hence $u_n e_a \in W_f$.

Conversely, suppose that $L * f = 0$ and $u_n e_a \in W_f$. From (5), $D^k\hat{L}(a) = 0$ for $k = 0, 1, \ldots n$. If $L(f \otimes e_a)^\vee$ is non-zero, the proof is complete; if not, define M by $M(g) = -L(g \otimes e_{-a})$ for each $g \in H$. By the linearity and continuity of L and $\otimes$, $M \in H'$. As well, $\hat{M}(z) = \hat{L}(z)/(z - a)$ and so $L = M*(D\delta - a\delta)$. So $D(M * f) = aM * f$ showing that $M * f = ce_a$. Since $c = M * f(0) = L(f \otimes e_a)^\vee = 0$, $M * f = 0$.

If $M(f \otimes e_a)^\vee$ is non-zero, the proof is completed at this stage. Otherwise, the order of the zero of $M(f \otimes e_z)^\vee$ is one less than that of $L(f \otimes e_z)^\vee$ at $z = a$. Continuation of this process yields after a finite number of steps the existence of an $N \in H'$ with $N * f = 0$ and $N(f \otimes e_a)^\vee \neq 0$. Hence, when $u_n e_a \in W_f$, $K(f)(z)$ has a pole of order exceeding n at $z = a$.

As a consequence of the above theorem, the following definition is consistent with the definition of Schwartz [**13**] of the spectrum of an entire mean periodic function.

Definition. The spectral set, S_f, of an entire mean periodic function f is the set of poles of $K(f)(z)$. The spectrum, Λ_f, is the set of pairs (a_k, p_k) where $a_k \in S_f$ and p_k is the order of the pole of $K(f)(z)$ at $z = a_k$.

THEOREM 5. *If $K(f)$ is entire, then $f = 0$.*

Proof. As usual, let L be a non-zero element of H' with $L * f = 0$. Since $K(f)$ is the quotient of two entire functions of exponential type, $K(f)$ is also of exponential type when it is entire (see, for example, Kahane [**6**, p. 135]). Then $K(f)(z) = \hat{M}(z)$ for some $M \in H'$ and so $L(f \otimes e_z)^\vee = \hat{N}(z)$ where $N = L * M$. Letting $h = L * (f \otimes e_z)$,

$$Dh = L * D(f \otimes e_z) = L * (f + zf \otimes e_z) = zh.$$

Thus $h = h(0)e_z$ and as $h(0) = \hat{N}(z)$, $L * (f \otimes e_z) = N * e_z$. By the linearity and continuity of L, N, $*$, $\otimes$, it follows that $L * (f \otimes u_n e_z) = N * u_n e_z$ for $n = 1, 2, \ldots$. So $L * (f \otimes g) = N * g$ where g is any polynomial and so also when g is any entire function.

From $L * (f \otimes g) = L * M * g$, we obtain $L * (f^{\otimes n} \otimes g) = L * M^{\otimes n} * g$ and then $L(f^{\otimes n} \otimes e_z)^\vee = \hat{L}(z)(w(z))^n$ for $n = 1, 2, \ldots$ where $w(z) = \hat{M}(z) = K(f)(z)$. As $f^{\otimes n} \to 0$ locally uniformly, $\hat{L}(z) \cdot (w(z))^n \to 0$ as $n \to \infty$ showing that if $\hat{L}(z) \neq 0$, $|w(z)| < 1$. Since the zeros of $\hat{L}$ are isolated, $|w(z)| \leqq 1$ for all complex z. So, by Louiville's Theorem, $w(z)$ is a constant, say c.

Thus $w(z) = \hat{M}(z) = c\hat{\delta}(z)$ and then $L * (f \otimes g - cg) = 0$ for all $g \in H$. If $c \neq 0$, and w is any entire function then by Theorem 1, g may be chosen so that $f \otimes g - cg = w$. Hence $L * w = 0$ for all entire functions w showing $L = 0$. As this is a contradiction, $c = 0$ and then $L * (f \otimes g) = 0$ for all $g \in H$.

If we now suppose that $f \neq 0$, then $f(0) \neq 0$ or there is a positive integer n for which $D^n f(0) \neq 0$ but $D^k f(0) = 0$ for $k = 0, 1, \ldots, n - 1$. In either case, $L * (f \otimes g) = 0$ yields after differentiation $n + 1$ times

$$L * [D^n f(0) g + (D^{n+1} f) \otimes g] = 0$$

for all $g \in H$. Again, by Theorem 1, it follows that $L * w = 0$ where w is any entire function and so $L = 0$. As this is a contradiction, we see $f = 0$ when $K(f)$ is entire.

COROLLARY. *Let $L * f = 0$ where $f \neq 0$ and L is a non-zero element of H'. Then there exists a non-zero $N \in H'$ such that $L * (f \otimes g) = N * g$ and $\hat{N}(z) = L(f \otimes e_z)^\vee$ for all $g \in H$.*

THEOREM 6. *An entire mean periodic function, f, is the limit of a sequence of exponential polynomials $\{f_n\} \subset W_f$.*

Proof. Let W denote the closed translation invariant subspace spanned by the exponential polynomials belonging to W_f. To show that $f \in W$, it suffices to show that if M is any non-zero element of H' with $M * W = \{0\}$, then $M * f = 0$.

Choose and fix a non-zero element L of H' for which $L * f = 0$. Put $h = M * f$ so that $L * h = 0$. By use of Lemma 2,

$$\begin{aligned} L(h \otimes e_z)^\vee &= (L * M)(f \otimes e_z)^\vee - \hat{L}(z) \cdot M(f \otimes e_z)^\vee \\ &= \hat{M}(z) \cdot L(f \otimes e_z)^\vee - \hat{L}(z) \cdot M(f \otimes e_z)^\vee \end{aligned}$$

and so

$$K(h)(z) = \hat{M}(z) \cdot K(f)(z) - M(f \otimes e_z)^\vee.$$

Since $M * W = \{0\}$, we see by Theorem 4 that $\hat{M}(z) \cdot K(f)(z)$ is entire. Thus $K(h)(z)$ is entire so by Theorem 5, $h = 0$. Hence $f \in W$.

3. New properties. With addition and the convolution product defined in § 1, it is apparent that H is a commutative ring and an algebra over C. Moreover H has no non-zero divisors of zero. To see this, let $f, g \in H$, $f \otimes g = 0$ and $g \neq 0$. Then, $f(0)g + (Df) \otimes g = D(f \otimes g) = 0$ and as $g = 0$ if $f(0) \neq 0$ by Theorem 1, one has $f(0) = 0$. Inductively, $D^n f(0) = 0$ for $n = 0, 1, 2, \ldots$ and since f is entire, $f = 0$.

That H can be shown to have no non-zero divisors of zero in a manner that avoids the use of Titchmarch's convolution Theorem [**14**] was known to Rubel [**12**] who also recognized that this ring would form the basis of an operational calculus, similar to that of Mikusinki's convolution quotients of continuous functions on a half line (see, for example, Erdélyi [**4**]).

We now show that H has a single descending chain of ideals, where an ideal I of H is a subspace of H that contains $f \otimes g$ whenever $f \in I$ and $g \in H$. A similar property holds for a ring of exponential polynomials (Laird, [**9**]). Here, the *degree* of an entire function f is zero if $f(0) \neq 0$ and n if $D^k f(0) = 0$ for $k = 0, ., \ldots n - 1$ with $D^n f(0) \neq 0$. (If $f = 0$, then f has infinite degree.)

THEOREM 7. (a) *Let f, g be entire functions and $g \neq 0$. Then a necessary and sufficient condition for the integral equation $w \otimes g = f$ to have an entire function as a solution is that the degree of f exceeds the degree of g.*

(b) *Let I be any non-trivial ideal of H. Then $I = X_n$ where $X_n = \{g \in H$: degree $g \geqq n\}$ for some positive integer n.*

Proof. The necessity of condition (a) is due to the fact that if w and g are entire functions and $g \neq 0$ then the degree of $w \otimes g$ exceeds the degree of g.

Conversely, let g have degree m and suppose that the degree of f exceeds m. If $m = 0$, then $g(0) \neq 0$, $f(0) = 0$ and by Theorem 1, there exists an entire function w satisfying $g(0)w + w \otimes Dg = Df$. On integration of this equation, we obtain

$$g(0)w \otimes e + w \otimes (e \otimes Dg) = e \otimes Df = f - f(0)e = f \quad \text{or}$$
$$w \otimes g = f \quad \text{where} \quad e : z \to 1.$$

If m is positive, let $a = D^m g(0)$ so $a \neq 0$. Then, by Theorem 1, an entire function w can be found with

$$aw + w \otimes D^{m+1}g = D^{m+1}f.$$

On repeated integration, with

$$D^k f(0) = 0 = D^k g(0) \quad \text{for} \quad k = 0, 1 \ldots m - 1 \quad \text{and} \quad D^m f(0) = 0,$$

we obtain $w \otimes g = f$.

For (b), let I be any non-trivial ideal of H. Also let m be the smallest degree of all functions in I so $I \subset X_m$. Choose and fix g to be an element of I with degree m. Then $g(z) = \sum_{j=m}^{\infty} g_j z^j/j!$ and $g_m \neq 0$.

Let h be any element of X_m with $h(z) = \sum_{j=m}^{\infty} h_j z^j/j!$. Then $h - h_m g/g_m$ has degree exceeding m so that it is equal to $w \otimes g$ for some $w \in H$ by (a). Thus h belongs to the ideal I showing that $X_m \subset I$. Hence $X_m = I$.

If I contains an element f with $f(0) \neq 0$, then $m = 0$ and $I = H$. Otherwise $I = X_m$ for some positive integer m.

The next theorem concerns the transform $T(f)$ of an entire function f defined by

$$T(f)(\xi) = \sum_{j=0}^{\infty} f_j \xi^{j+1} \quad \text{when} \quad f(z) = \sum_{j=0}^{\infty} f_j z^j/j!$$

THEOREM 8. *Let f be an entire mean periodic function. Then*

$$K(f)(z) = T(f)(1/z).$$

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Proof. Choose a non-zero element L of H' satisfying $L * f = 0$. Let

$$\hat{L}(z) = \sum_{n=0}^{\infty} a_n z^n/n! \quad \text{so that} \quad L * f(z) = \sum_{n=0}^{\infty} a_n D^n f(z)/n!$$

With $L * D^m f = 0$ and $f = \sum_{n=0}^{\infty} f_n z^n/n!$,

$$(6) \quad \sum_{n=0}^{\infty} a_n f_{n+m}/n! = 0 \quad \text{for } m = 0, 1, 2, \dots .$$

Now

$$\begin{aligned} L(f \otimes e_z)^{\vee} &= L * (f \otimes e_z)(0) \\ &= \sum_{n=0}^{\infty} a_n D^n (f \otimes e_z)(0)/n! \\ &= 0 + \sum_{n=1}^{\infty} a_n \left(\sum_{j=0}^{n=1} f_j z^{n-j-1} \right) /n! \quad \text{(using (2))} \\ &= \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} f_{n-k-1} a_n/n!. \end{aligned}$$

Also

$$\begin{aligned} T(f)(1/z)\hat{L}(z) &= \sum_{j=0}^{\infty} f_j z^{-j-1} \sum_{n=0}^{\infty} a_n z^n/n! \\ &= L(f \otimes e_z)^{\vee} + \sum_{n=0}^{\infty} z^{-m-1} \sum_{n=0}^{\infty} a_n f_{n+m}/n! \\ &= L(f \otimes e_z)^{\vee} \end{aligned}$$

from (6). The result then follows from the definition of $K(f)$.

This also shows that $K(f)(z)$ is none other than Borel's form of the Laplace transform of f.

The transform T has other properties. It is clear that it is a linear map and from (1), $T(u_m \otimes u_n) = T(u_m) \cdot T(u_n)$. Hence $T(f \otimes g) = T(f) \cdot T(g)$ for all $f, g \in H$ and so T is an isomorphism between $H(+, \otimes)$ and a subalgebra of the ring of formal power series in one indeterminate.

For the remainder of this section, MH will denote the set of all entire mean periodic functions.

THEOREM 9. *MH is a subalgebra of H and*

(a) *the transform K is a monomorphism from MH into the field of meromorphic functions, and*

(b) *if $f \in MH$ and if h is any exponential polynomial, then $fh \in MH$.*

Proof. Let $f, g \in MH$ and let L, M be non-zero elements of H' with $L * f = 0$, $M * g = 0$. Also let $a, b \in C$. As $L * M \neq 0$ and $L * M * (af + bg) = 0$, $af + bg \in MH$. From the Corollary to Theorem 5, $L * (f \otimes g) = N * g$ for some $N \in H'$ whence $L * M * (f \otimes g) = 0$. Hence MH is a subalgebra of H.

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That K is a homomorphism from MH to the field of meromorphic functions can be shown by two applications of Lemma 2 that give

$$(L * M)((af + bg) \otimes e_z)^\vee = a\hat{M}(z) \cdot \hat{L}(f \otimes e_z) + b\hat{L}(z) \cdot M(g \otimes e_z)^\vee$$

and

$$(L * M)((f \otimes g) \otimes e_z)^\vee = L(f \otimes e_z)^\vee \cdot M(g \otimes e_z)^\vee.$$

Alternatively, this property of K is a consequence of the preceding theorem and remarks.

For part (b), since MH is a subspace of H, it suffices to show that $u_n e_a f \in MH$ when $f \in MH$. In turn, this reduces to showing that uf and $e_a f \in MH$.

Let L_1 and L_a be elements of H' defined by $L_1(w) = L(uw)$ and $L_a(w) = L(e_a w)$ for each $w \in H$. Then

$$L * uf(z) = L((z - u)T_z f)^\vee = zL * f(z) - L_1 * f(z),$$

so

$$L * L * uf = -L * L_1 * f = 0,$$

and

$$(L_a * e_a f)(z) = L(e_a T_z(e_a f)^\vee) = e^{az} L * f(z) = 0.$$

As $L * L$ and L_a are non-zero, uf and $e_a f \in MH$.

COROLLARY. *If h is any exponential polynomial, then $K(h)(z)$ is a rational function of z.*

Proof. The proof is a consequence of $K(e_a)(z) = 1/(z - a)$ and $e_a^{\otimes(n+1)} = u_n e_a / n!$.

Remark. It would appear, following the remarks of Schwartz [**13**, p. 927], that the pointwise product of two entire mean periodic functions need not be mean periodic. A specific example is given by the two mean periodic functions exp (exp $i\alpha z$), exp (exp $i\beta z$) where α, β are real and incommensurable. The product of these functions is

$$h(z) = \sum \sum \frac{1}{m!n!} \exp\,(i\alpha m z + i\beta n z) \quad m, n = 0, 1, 2, \ldots.$$

As $\{\alpha m + \beta n : m, n = 0, 1, 2, \ldots\}$ is a set of real numbers with infinite density, h is not mean periodic.

THEOREM 10. *Let f, g be entire mean periodic functions. Then*

(a) *if $g \neq 0$ and if w is an entire function satisfying $w \otimes g = f$, then w is mean periodic, and*

(b) *the solution to $w - w \otimes g = f$ is mean periodic.*

Also, if J is any non-trivial ideal of MH, then $J = Y_n$ where $Y_n = \{f \in MH$: degree of $f \geqq n\}$ *for some positive integer n.*

Proof. Let L, M be non-zero elements of H' with $L * f = 0$, $M * g = 0$. If $w \in H$ and if $g \neq 0$, by the Corollary to Theorem 5, there exists a non-zero element N of H' for which $M * (w \otimes g) = N * w$. Hence $L * N * w = 0$ and as $L * N \neq 0$, w is mean periodic.

The equation $w - w \otimes g = f$ always has an entire solution and is equivalent to $w \otimes (e - g \otimes e) = f \otimes e$. From Theorem 1, $e - g \otimes e \neq 0$. Also $e - g \otimes e$ and $e \otimes f$ are mean periodic and so w is mean periodic.

A necessary and sufficient condition for $w \otimes g = f$ to have an entire solution has already been given in Theorem 7. This, along with the details of the remainder of Theorem 7 holding for MH in place of H shows that the ring $MH(+, \otimes)$ has a single descending chain of ideals.

The final results of this section are concerned with systems of equations. Here, reference to a mean periodic vector will mean a vector whose components are entire mean periodic functions.

THEOREM 11 *For the system of equations*

$$w'(z) = Aw(z) + f(z) \quad \textit{with } w(a) = b,$$

where A is a constant $n \times n$ matrix and f is a n-vector whose components are entire functions, a necessary and sufficient condition that w be mean periodic is that f is mean periodic.

THEOREM 12. *Let $A(z)$ be an $n \times n$ matrix whose elements are entire functions with a complex period c. Let f be an n-vector whose j-th component is of the form $g_j h_j$ where each g_j is an exponential polynomial and h_j is an entire function with complex period d_j. If each d_j is a real, positive and rational multiple of c, then all entire solutions to the system of equations*

$$w'(z) = A(z)w(z) + f(z)$$

are mean periodic in H.

THEOREM 13. *Let f be a mean periodic vector and $G = \{g_{jk}\}$ be an $n \times n$ matrix whose elements are entire mean periodic functions. Then the system of equations*

$$w_j - \sum_{k=1}^{n} g_{jk} \otimes w_k = f_j \quad (j = 1, 2, \ldots n)$$

has a unique mean periodic solution.

THEOREM 14. *Consider the system of equations*

$$\sum_{j=0}^{m} [A_j w'(z - a_j) + B_j w(z - b_j)] = f(z)$$

where

(i) *for $j = 0, 1, \ldots m$, A_j, B_j are $n \times n$ matrices of complex numbers with $A_0 = I$, the unit matrix,*

(ii) $a_0 = 0$, $a_1, a_2, \ldots a_m$ *are non-zero complex numbers lying in a sector* $\{z = re^{i\theta} : |\theta + \alpha| < \pi/2\}$ *for some fixed* α *and* $b_0, b_1, \ldots b_m$ *are any complex numbers, and,*

(iii) *the components of the n-vector f are entire functions.*

Then (a) *the homogeneous equation has at least one non-zero entire mean periodic solution,*

(b) *if f is mean periodic, then all solutions that are entire are also mean periodic, and,*

(c) *if any entire solution w is mean periodic, then f is mean periodic.*

The proofs of the above four theorems are given in a thesis of the author or are akin to the proofs of similar theorems for continuous mean periodic functions of a single real variable (Laird [**8**, Chapter 5] or [**7**; **10**]). They are therefore omitted. However, it may be noted that the major conclusion of Theorem 14, part (b), is a special case of Proposition 8 given by Malgrange [**11**, p. 318].

4. Several variables. It is natural to enquire as to how many of the previous results are valid for entire mean periodic functions of several variables. These functions have been considered by Ehrenpreis [**3**] although our tentative observations hold also for the indefinitely differentiable functions of several real variables discussed by Malgrange [**11**].

We let E denote the space of entire functions of n complex variables and equipped with the topology of convergence uniform on all compact subsets of Euclidean complex n-space. Then $f \in E$ is mean periodic if the subspace spanned by f and its translates is not dense in E; this being equivalent to the existence of a non-zero element S of the dual space of E satisfying a convolution equation $S * f = 0$. Since this dual space is an integral domain, the set of all such mean periodic functions is a subspace of E.

When f, g are any entire functions of n complex variables, their convolution may be defined as

$$f \otimes g(z) = \int_0^{z_1} \int_0^{z_2} \ldots \int_0^{z_n} f(z - \xi) g(\xi) d\xi$$

where $z = (z_1, z_2, \ldots z_n)$ and $\xi = (\xi_1, \xi_2, \ldots \xi_m)$. The method of proof used in Theorem 1 may be adapted to show that $f \otimes g$ is entire.

It is well known that a linear partial differential equation with constant co-efficients may have solutions that are not exponential polynomials (i.e., finite linear combinations of terms $z_1^{q_1} z_2^{q_2} \ldots z_n^{q_n} \exp(a_1 z_1 + a_2 z_2 + \ldots a_n z_n)$). However, it is possible to give a simple characterization of an exponential polynomial of n variables by

LEMMA 15. *Let* $g \in E$. *Then* g *is an exponential polynomial if, and only if, there exists n non-zero linear partial differential operators* $P_1, P_2, \ldots P_n$, *where*

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each P_i has constant coefficients and only involves the partial derivatives with respect to z_i and $P_ig = 0$ for $i = 1, 2, \ldots . n$.

Proof. If $g \in E$ and $P_1g = 0$, then g is a finite sum of terms $A(z_2, z_3, \ldots z_n)z_1^q \times \exp(az_1)$ where q is any non-negative integer and a is any complex number. With $P_2g = 0$, $P_2A = 0$ and so each A is a finite sum of terms $B(z_3, z_4, \ldots z_n)z_2^q \times \exp(az_2)$. Continuing in this manner, we find that with $P_ig = 0$ for $i = 1, 2, \ldots n$, so g is an exponential polynomial.

The converse portion of the proof is obvious.

Using this characterization, we have

THEOREM 16. *Let $f, g, h \in E$ where f is mean periodic and g, h are exponential polynomials. Then $f \otimes g$ is mean periodic and $g \otimes h$ is an exponential polynomial.*

Proof. Let $P_ig = 0$ for $i = 1, 2, \ldots n$ where each P_i is as in Lemma 15. Then, with $D_1 = \partial/\partial z_1$,

$$D_1(f \otimes g)(z) = \int_0^{z_2} \ldots \int_0^{z_n} f(z_1, \xi_2, \ldots \xi_n)g(0, z_2 - \xi_2, \ldots z_n - \xi_n) \times d\xi_2 \ldots d\xi_n + f \otimes D_1g(z),$$

$$D_1^k(f \otimes g)(z) = \int_0^{z_2} \ldots \int_0^{z_n} \sum_{j=0}^{k} D_1^jf(z_1, \xi_2, \ldots \xi_n) \times D_1^{k-j-1}g(0, z_2 - \xi_2, \ldots z_n - \xi_n)d\xi_2 \ldots d\xi_n + f \otimes D_1^kg(z)$$

and as P_1 is a linear operator in D_1 with constant coefficients,

$$P_1(f \otimes g)(z) = \int_0^{z_2} \ldots \int_0^{z_n} \sum_{j,l} A_{jl}D_1^jf(z_1, \xi_2, \ldots \xi_n) \times D_1^lg(0, z_2 - \xi_2, \ldots z_n - \xi_n)d\xi_2 \ldots d\xi_n + 0 \tag{7}$$

where A_{jl} are constants. Next,

$$D_2P_1(f \otimes g)(z) = \int_0^{z_3} \ldots \int_0^{z_n} \sum A_{jl}D_1^jf(z_1, z_2, \xi_3, \ldots \xi_n) \times D_1^lg(0, 0, z_3 - \xi_3, \ldots z_n - \xi_n)d\xi_3 \ldots d\xi_n + \int_0^{z_2} \ldots \int_0^{z_n} \times \sum A_{jl}D_1^jf(z_1, \xi_2, \ldots \xi_n)D_2D_1^lg(0, z_2 - \xi_2, \ldots z_n - \xi_n)d\xi_2 \ldots d\xi_n$$

whence, by use of $P_2g = 0$,

$$P_2P_1(f \otimes g)(z) = \int_0^{z_3} \ldots \int_0^{z_n} \sum B_{jj'll'}D_1^jD_2^{j'}f(z_1, z_2, \xi_3, \ldots \xi_n) \times D_1^lD_2^{l'}g(0, 0, z_3 - \xi_3, \ldots z_n - \xi_n)d\xi_3 \ldots d\xi_n$$

where $B_{jj'll'}$ are constants. Continuing in this manner and using $P_ig = 0$

for $i = 1, 2, \ldots n$, we finally obtain

$$P_n \ldots P_2P_1(f \otimes g)(z) = \sum C_{j_1 j_2 \ldots j_n} D_1^{j_1} D_2^{j_2} \ldots D_n^{j_n} f(z)$$

where $C_{j_1 j_2 \ldots j_n}$ are constants. Rephrasing this relation, we have

$$(P_n \ldots P_2P_1)\delta * (f \otimes g) = T * f$$

where $(P_n \ldots P_2P_1)\delta$ and T are elements of the dual space of E. If f is mean periodic with $S * f = 0$ where S is a non-zero element of the dual space of E, so also is $T_1 = S * (P_n \ldots P_2P_1)\delta$. With $T_1 * (f \otimes g) = S * T * f = 0$, we see that $f \otimes g$ is mean periodic.

Now let h be another exponential polynomial with n non-zero linear partial differential operators, Q_i, each with constant coefficients only involving partial derivatives with respect to z_i and satisfying $Q_ih = 0$ for $i = 1, 2, \ldots, n$. From (7), replacing f by h, we find

$$Q_1P_1(h \otimes g)(z) = \int_0^{z_2} \ldots \int_0^{z_n} \sum_{j,l} A_{jl} Q_1 D_1^{j} h(z_1, \xi_2, \ldots \xi_n) \times D_1^{l} g(0, z_2 - \xi_2, \ldots z_n - \xi_n) d\xi_2 \ldots d\xi_n$$

so that $Q_1P_1(h \otimes g) = 0$. In a like manner, it may be verified that $Q_iP_i(h \otimes g) = 0$ for $i = 2, 3, \ldots n$ as well as $i = 1$. As each Q_iP_i is non-zero, Lemma 15 entails that $h \otimes g$ is an exponential polynomial.

We note that the proof of Theorem 9, part (b) could be extended to show that the pointwise product of an exponential polynomial with a mean periodic function in E is mean periodic.

Since the remaining remarks are negative in nature, they shall be confined to entire functions of two complex variables, z, w.

If $f(z, w) = 2ze^{z^2}$ and $g(z, w) = 2we^{w^2}$, then f, g are mean periodic in E as $D_2\delta * f = 0$, $D_1\delta * g = 0$ where $D_1 = \partial/\partial z$, $D_2 = \partial/\partial w$. However, their pointwise product, fg, is not mean periodic since the subspace spanned by fg and its partial derivatives includes the pointwise product of any polynomial with fg and so is dense in E.

Hence $f \otimes g(z, w) = (e^{z^2} - 1)(e^{w^2} - 1)$ is not mean periodic. A consequence of this is that the Corollary to Theorem 5 and the first part of Theorem 9 does not hold for functions of several variables.

No transform appears to be yet available for mean periodic functions of several variables. If $e(z, w) = 1$ and if $S = D_1\delta$ with $S * e = 0$, a modification of the transform introduced in § 2 to

$$K(f)(z, w) = S(f \otimes \exp(z, w))^{\vee}/\hat{S}(z, w) \quad \text{with} \quad \exp(z, w)(\xi, \eta) = \exp(z\xi + w\eta)$$

is inadequate, simply because,

$$S(e \otimes \exp(z, w))^{\vee} = (D_1(e^z - 1)(e^w - 1))_{z=0\, z=w} = 0.$$

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The University of Wollongong,
Wollongong, N.S.W. 2500,
Australia