ALMOST MULTIPLICATION RINGS

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Introduction. It is well known that an ideal A in a Dedekind domain has a prime radical if and only if A is a power of a prime ideal. The purpose of this paper is to determine necessary and sufficient conditions in order that a commutative ring with unit element have this property and to study the ideal theory in such rings. Domains with unit element having the above property possess many of the characteristics of Dedekind domains (however, they need not be Noetherian) and will be referred to in this paper as "almost Dedekind domains"—these domains are considered in Section 1. We call a commutative ring with unit element an "almost multiplication ring" provided it has the above property. It is shown in Section 2 that every multiplication ring is an "almost multiplication ring" and that "almost multiplication rings" have several of the important properties of multiplication rings. A summary of the necessary and sufficient conditions obtained in this paper is given by Theorem 1.0 and Theorem 2.0 of Sections 1 and 2 respectively.

Preliminaries. In this paper *ring* will mean a commutative ring with $1 \neq 0$, and *domain* will mean a ring in which the zero ideal is prime. We shall call a ring R a *multiplication ring* if whenever A and B are ideals of R with $A \subset B$, there is an ideal C of R such that $A = BC$ (**6**, p. 2). By a *special primary ring* ("primärer zerlegbarer Ring," (**4**, p. 84)) we mean a ring R with exactly one prime ideal $P \neq R$, such that $P^n = (0)$ for some positive integer n and the only ideals of R are $R, P, P^2, \ldots, P^n = (0)$. *Discrete valuation ring* will mean a Dedekind domain with at most one proper (different from (0) and (1)) prime ideal. A domain J will be called *strongly integrally closed* if J contains each element x of its quotient field for which the polynomial domain $J[x]$ is contained in a finite J-module. The quotient ring R_P (**11**, p. 221) of the ring R with respect to the prime P of R will be called *proper* if P is proper. The symbols A^e, A^c, $A^{(n)}$, and $\mathrm{rad}(A)$ will denote the extension, contraction (**11**, p. 218), nth symbolic power (**11**, p. 232), and radical (**11**, p. 147) of the ideal A respectively. The symbol "$\subset$" will allow equality while "<" will indicate proper containment. Throughout this paper, J will denote a domain with quotient field K, and R will denote a ring with total quotient ring T.

1. Almost Dedekind domains. The necessary and sufficient conditions obtained in this section are included in the following theorem, the proof of which follows from Theorems 1.1, 1.4, 1.5, 1.6, and Corollary 1.2.

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THEOREM 1.0. *In a domain J, these are equivalent:*

(i) *for each proper prime P of J, J_P is a discrete valuation ring;*

(ii) *whenever A is an ideal of J such that* $\text{rad}(A) = P$, *a prime of J, A is then a power P^n of P;*

(iii) *each proper primary ideal of J is a power of a maximal ideal of J;*

(iv) *whenever A, B, and C are ideals of J with $AB = AC$ and A is non-zero, then $B = C$;*

(v) (a) *finitely generated non-zero ideals of J are invertible,*

(b) *proper primes of J are maximal, and*

(c) *J contains no proper idempotent prime ideal;*

(vi) (a) *and* (c) *of* (v) *together with*

(b′) *for each proper prime P of J, J_P is strongly integrally closed.*

DEFINITION 1.1. *A domain with respect to which each proper J_P (P a prime of J) is a discrete valuation ring will be called an almost-Dedekind domain* (AD-domain).

LEMMA 1.1. *If, in J, each ideal with prime radical is a power of its radical, proper primes of J are maximal.*

Proof. It clearly suffices to show that a minimal prime of a non-zero principal ideal is maximal. Let (a) be a non-zero principal ideal of J and P a minimal prime of (a). Since P is minimal for (a),

$$\text{rad}(aJ_P) \cap J = PJ_P \cap J = P.$$

Therefore $aJ_P \cap J = P^n$ for some positive integer n. But then

$$aJ_P = (aJ_P \cap J)J_P = P^nJ_P = (PJ_P)^n$$

and hence PJ_P is invertible since its nth power is principal. This implies that $PJ_P \neq (PJ_P)^2$, which implies that $P \neq P^2$. Now $P^2 \subset P^{(2)} \subset P$ so that $\text{rad}(P^{(2)}) = P$ and either $P^{(2)} = P$ or $P^{(2)} = P^2$. If $P^{(2)} = P$, $P^2J_P = PJ_P$, which cannot happen. Therefore, $P^{(2)} = P^2$ and P^2 is primary. Now let p be an element of $P - P^2$, and m an element of $J - P$. Since P^2 is primary, pm is not in P^2 and hence $P^2 + (pm) = P$. Then let $p = q + rpm$, where q is in P^2, r in J. We have $p(1 - rm)$ is in P^2 and p is not in P^2, primary, which implies that $1 - rm$ is in P. But this implies that 1 is in $P + (m)$ and since m was arbitrary outside P, P is maximal.

THEOREM 1.1. *A domain J is an* AD-*domain if and only if each ideal of J, with prime radical, is a prime power.*

Proof. If each proper J_P is a discrete valuation ring, proper primes of J are clearly maximal. Thus $\text{rad}(A) = P$, a proper prime of J, implies that A is primary, so that $AJ_P \cap J = A$, (**11**, p. 223); but since J_P is a discrete valuation ring, $AJ_P = (PJ_P)^n$ for some positive integer n. Now since A is primary and contained in P, $A = AJ_P \cap J$, and hence $A = P^n$. On the other hand, if each ideal with prime radical is a prime power, then according to Lemma 1.1, each proper prime P of J is maximal, so that J_P has exactly one proper prime ideal. Then since (by the proof of Lemma 1.1) PJ_P is invertible, J_P is a Dedekind domain (**8**, p. 234), thus a discrete valuation ring.

REMARK 1.1. *An* AD-*domain is strongly integrally closed in its quotient field.*

Proof. It is easy to show that any domain J is the intersection of its quotient rings J_P for proper primes P of J. Since each J_P of an AD-domain is strongly integrally closed, it follows that J is strongly integrally closed.

REMARK 1.2. *In an* AD-*domain* J, *the powers of any proper ideal intersect in* (0).

Proof. If A is a proper ideal of J, A is contained in P for some proper prime P of J. Thus $A \subset AJ_P \subset PJ_P$ and

$$\bigcap_{n=1}^{\infty} A^n \subset \bigcap_{n=1}^{\infty} (PJ_P)^n = (0)$$

since J_P is a Dedekind domain.

REMARK 1.3. *A Noetherian* AD-*domain is a Dedekind domain.*

Proof. We have already shown that an AD-domain is integrally closed and has no non-maximal proper prime ideals (**10**, pp. 85, 86).

We state without proof a theorem of Krull (**3**, p. 554).

THEOREM 1.2. *In* J, *these are equivalent:*

(i) J *is a Prüfer domain* (i.e., finitely generated proper ideals of J are invertible);

(ii) *for each proper prime* P *of* J, J_P *is a Prüfer domain;*

(iii) *for each proper prime* P *of* J, J_P *is a valuation ring.*

COROLLARY 1.1. *An* AD-*domain is a Prüfer domain.*

THEOREM 1.3. *If* J *is an* AD-*domain and* $J \subset R \subset K$, *then* R *is an* AD-*domain.*

Proof. Let Q be a proper prime of R and $P = Q \cap J$. Then P is a proper prime of J and J_P is a discrete valuation ring with $J_P \subset R_Q \subset K$ so that R_Q must also be a discrete valuation ring.

It should be mentioned here that Theorem 1.3 is also true with "AD-domain" replaced by either (a) "Prüfer domain" or (b) "Prüfer domain in which proper primes are maximal," since any ring between a valuation ring and its quotient field is a valuation ring.

THEOREM 1.4. *A domain* J *is an* AD-*domain if and only if each proper primary ideal of* J *is a power of a maximal ideal.*

Proof. It has already been shown that in an AD-domain, proper primes are maximal and each primary ideal is a power of its radical. On the other hand, each proper prime of J is a proper primary, hence is maximal. Thus each ideal with prime radical has maximal radical and is primary so that, by hypothesis, it is a maximal (in particular prime) power. So J is an AD-domain by Theorem 1.1.

For completeness we state here without proof a result communicated by Robert Gilmer (see the paper "*The Cancellation Law for Ideals in a Commutative Ring*" on pp. 281–7 of this issue of the *Canadian Journal of Mathematics*).

THEOREM 1.5. *A domain J is an* AD-*domain if and only if for A, B, C non-zero ideals of J such that $AB = AC$, $B = C$.*

THEOREM 1.6. *A domain J is an* AD-*domain if and only if*
(a) *J is a Prüfer domain,*
(b) *proper primes of J are maximal, and*
(c) *J contains no proper indempotent prime.*

Proof. We have already shown that an AD-domain has properties (a), (b), and (c). On the other hand, if J is a Prüfer domain, each J_P is a valuation ring. If J has no non-maximal proper primes, J_P has rank 1; but if J contains no non-zero proper idempotent prime, neither does J_P, since the prime powers of J are primary. Now a rank 1 valuation ring is a Dedekind domain if and only if its maximal ideal is not idempotent (**12**, p. 45; **10**, p. 240).

Since only rank 1 valuation rings are strongly integrally closed (**11**, p. 255; **12**, p. 45), we have

COROLLARY 1.2. *A domain J is an* AD-*domain if and only if* (a), (c) *of Theorem* 1.6 *hold and*
(b′) *each proper J_P is strongly integrally closed.*

COROLLARY 1.3. *The union of a tower of* AD-*domains is an* AD-*domain if and only if it has no proper idempotent primes.*

Proof. It is easily shown that the union J of a tower of AD-domains is a Prüfer domain and each proper J_P is strongly integrally closed.

THEOREM 1.7. *If J is an* AD-*domain, F a finite algebraic extension of* [illegible] *and J^* the integral closure* (*in the polynomial sense*) *of J in F, then J^* is an* AD-*domain.*

Proof. Let Q be a proper prime of J^*. Then $P = Q \cap J$ is a proper prime of J; hence J_P is a discrete valuation ring. Now we know that the integral closure L of J_P in F is a Dedekind domain (**11**, p. 281). It can easily be shown that $L \subset J_Q^* \subset F$ and that F is the quotient field of L. Hence J_Q^* is a Dedekind domain (between a Dedekind domain and its quotient field (**2**, p. 31)). It should be noted that the strong integral closure of J in F is between J^* and F so that it too is, by Theorem 1.3, an AD-domain.

COROLLARY 1.4. *The ring of integral elements of an algebraic number field forms an* AD-*domain if and only if this ring has no proper idempotent primes.*

Proof. This ring can be written as a union of a tower of rings each of which is the integral closure of the rational integers in a finite algebraic extension of the rational numbers. Hence Corollary 1.4 follows from Theorem 1.7 and Corollary 1.3.

Example. Nakano (**7**, p. 426) gives the following example of an algebraic number field K, the integral elements of which form an AD-domain which is not a Dedekind domain. Let K be the field obtained by the adjunction, to the field of rational numbers, of the pth roots of unity for every rational prime p. Let J be the integral elements of K. Nakano showed that J has no idempotent proper primes, so that J is an AD-domain by Corollary 1.4. He also showed that J has no finitely generated proper primes, so that J is not a Dedekind domain.

THEOREM 1.8. *The integral closure (in the polynomial sense) J of an* AD-*domain J' in an algebraic extension of the quotient field of J' is an* AD-*domain if and only if J has no proper idempotent ideals.*

Proof. Let J^* be the union of a maximal tower of AD-domains in J (such a tower exists by the Hausdorff maximality principle and the existence of one AD-domain in J). By Corollary 1.3, J^* is an AD-domain if and only if it has no idempotent proper primes. Now suppose J has no idempotent proper ideals. Then if P^* is any proper prime of J^*, P^*J is a proper ideal of J so that P^* is not idempotent since P^*J is not. So J^* is an AD-domain and if $J^* < J$, there exists an element x in $J - J^*$ and x is integral over J^*. Then the domain J^{**}, which is the integral closure of J^* in $K^*(x)$ (K^* the quotient field of J^*), is an AD-domain by Theorem 1.10, contradicting the maximality of the tower which formed J^*. Therefore $J^* = J$ and the theorem is proved.

2. Almost multiplication rings.

DEFINITION 2.1. *For a prime ideal P of R, let $M(P)$ be $R - P$ and $N(P)$ be the set of elements x of R such that* 0 *is an element of $xM(P)$.* Discrete valuation ring and special primary ring ("primärer zerlegbarer Ring," **4**, p. 84) will be denoted by dvr and spr respectively.

We include the following theorem as a summary of the necessary and sufficient conditions obtained in this section.

THEOREM 2.0. *In a ring R, these are equivalent:*

(i) *for each proper prime P of R, R_P is a* ZPI *ring,*

(ii) *whenever* A *is an ideal of* R *such that* $\text{rad}(A) = P$, *a prime of* R, A *is then a power* P^n *of* P,

(iii) *for each proper prime* P *of* R, R_P *is a multiplication ring,*

(iv) *for each proper primary ideal* Q *of* R, *there is a maximal ideal* M *of* R *such that* Q *is either* $N(M)$ *or a power of* M.

(v) *whenever* A *is an ideal of* R *such that* $\text{rad}(A) = P_1 \ldots P_n$ (*a product of distinct primes*), *then*

$$A = P_1^{e_1} P_2^{e_2} \ldots P_n^{e_n}.$$

DEFINITION 2.2. *R will be called an almost multiplication ring* (AM-*ring*), *if for each proper prime* P *of* R, R_P *is a* ZPI *ring* (**5**, p. 117), *i.e., each ideal of* R_P *is factorable into a product of prime powers.* (In Theorem 2.7 we show that every proper R_P is a ZPI ring if and only if every proper R_P is a multiplication ring. It is well known that every ZPI ring is a multiplication ring and Lemma 2.4 shows that every multiplication ring is an AM-ring).

It has been shown by Asano (**1**, p. 83) that a ZPI ring with a unique maximal ideal is either a dvr or an spr. Therefore, R is an AM-ring if and only if each proper R_P is either a dvr or an spr.

For the proofs in this section it will be convenient to state here a theorem from Zariski and Samuel (**11**, p. 228), namely

THEOREM 2.1. *Let* P *be a prime ideal of* R. *The mapping* $A \to A^e$ *establishes a* 1–1 *correspondence between the set of prime* (*primary*) *ideals of* R *contained in* P, *and the set of all prime (primary) ideals of* R_P.

LEMMA 2.1. *If* R *is an* AM-*ring and* P *a proper prime of* R *such that* $N(P)$ *is not prime, then* $\text{rad}(N(P)) = P$.

Proof. Since $N(P)$ is not prime, R_P is not a domain and hence R_P is an spr. Therefore, there exists a positive integer n such that $(P^e)^n = (0)$ and thus for p in P, $((p)^e)^n = (0)$, i.e., p^n is in $N(P)$. This implies that P is contained in $\text{rad}(N(P))$, but the other containment always holds, so that $\text{rad}(N(P)) = P$.

THEOREM 2.2. *If* R *is an* AM-*ring,* P *a proper prime of* R, *and* $N(P)$ *is not prime, then* P *is minimal and maximal,* R_P *is an* spr *and* $\text{rad}(A) = P$ *implies that* A *is a power of* P.

Proof. As in Lemma 2.1, R_P is an spr and hence contains only one proper prime ideal, P^e; and by Theorem 2.1, there are therefore no prime ideals of R properly contained in P, i.e., P is minimal. Now suppose that P' is a maximal ideal of R containing P. If $R_{P'}$ is an spr, P' is minimal and $P = P'$ is maximal. On the other hand, if $R_{P'}$ is a dvr, $N(P')$ is prime in R. Again, using Theorem 2.1, P' and $N(P)'$ are the only primes of R contained in P' so that either $P = P'$ is maximal or $P = N(P') \subset N(P) \subset P$, which implies that $P = N(P)$ and contradicts $N(P)$ not being prime. Therefore P is maximal and $\text{rad}(A) = P$ implies that A is primary. So by Theorem 2.1, since $A^e = (P^e)^n = (P^n)^e$, $A = P^n$.

THEOREM 2.3. *If R is an* AM-*ring, P a proper prime of R, $N(P)$ is prime, and $P \neq N(P)$, then P is maximal, $N(P)$ is the only prime of R properly contained in P, R_P is a* dvr,

$$\bigcap_{n=1}^{\infty} P^n = N(P),$$

and $\operatorname{rad}(A) = P$ *implies that* $A = P^n$ *for some positive integer* n.

Proof. Since $P \neq N(P)$, $N(P) < P$. Since R_P is a domain, it is a dvr and hence by Theorem 2.1, $N(P)$ and P are the only primes of R contained in P. Let P' be a maximal ideal of R containing P. If $N(P')$ were not prime, P' would be minimal by Theorem 2.2 so that P' would be P and $N(P)$ would not be prime. Therefore $N(P')$ is prime and is the only prime properly contained in P', which implies that either $P = N(P')$ or $P = P'$. If $P = N(P')$, $P = N(P)$; therefore $P = P'$ is maximal. Then each P^n, for a positive integer n, is primary and $(P^n)^e = (P^e)^n$ so that $N(P)$ is contained in each P^n and

$$N(P) \subset \bigcap_{n=1}^{\infty} P^n.$$

But since R_P is a dvr,

$$\bigcap_{n=1}^{\infty} (P^e)^n = (0)$$

so that

$$\bigcap_{n=1}^{\infty} P^n \subset N(P),$$

i.e.,

$$N(P) = \bigcap_{n=1}^{\infty} P^n.$$

Now since P is maximal, $\operatorname{rad}(A) = P$ implies that A is primary, which implies by Theorem 2.1 that $A = P^n$ for some positive integer n.

LEMMA 2.2. *Let A be any ideal of R. Then A is identical with the intersection of all A^{ec} with respect to the prime ideals P such that $A \subset P < R$.*

Proof. It is clear that A is contained in this intersection. Now suppose that x is an element of this intersection. If $A:(x) = R$, x is in A. But if $A:(x) \neq R$, $A:(x) \subset M$ for some maximal ideal M of R. Then $A \subset M < R$ and x is in A^{ec} with respect to M, i.e., there is an element y of $R - M$ such that xy is in A, which contradicts $A:(x) \subset M$.

THEOREM 2.4. *If R is an* AM-*ring and P is a prime of R such that $P = N(P)$, then P is the only ideal with P as radical.*

Proof. Let A be an ideal of R with $\operatorname{rad}(A) = P$. If P' is any proper prime of R such that $P \subset P'$, then it follows as in the proof of Theorem 2.3 that

$N(P') = N(P) = P$. Since $\text{rad}(A) = P$, then $P' \supset A$ implies $P' \supset P$. It follows that A^{ec}, with respect to each proper prime $P' \supset A$, is equal to P and therefore, by Lemma 2.2, $A = P$.

LEMMA 2.3. *If in R each ideal with prime radical is a prime power, this property also holds in each R_P.*

Proof. Let A be an ideal of R_P and $\text{rad}(A) = P^*$, a prime. Then, since $\text{rad}(A^c) = (\text{rad}(A))^c$ and since $(P^*)^c$ is prime, $\text{rad}(A^c) = (P^*)^c$, prime so that $A^c = ((P^*)^c)^n$ and

$$A = A^{ce} = (((P^*)^c)^n)^e = ((P^*)^{ce})^n = (P^*)^n$$

and the lemma is proved.

THEOREM 2.5. *If in R each ideal with prime radical is a prime power, then each proper R_P is a* ZPI *ring.*

Proof. Let P be a proper prime of R. If P is minimal in R, P^e is the only prime of R_P properly contained in R_P and, by Lemma 2.3, each ideal properly contained in R_P is a power of P^e. In this case, R_P is either a field (if $P = N(P)$), or an spr $((0)^e = (P^e)^n)$. If P is not minimal, let P' be a minimal prime contained in P. It can be shown easily that the residue class ring of R modulo P' is an AD-domain so that proper primes of R/P' are maximal and P is maximal in R; and since P was any non-minimal prime, non-minimal primes of R are maximal and there are no primes properly between P' and P in R.

We shall now show that in R,

$$\bigcap_{n=1}^{\infty} P^n = P'$$

is the only prime of R contained in P. Since in R/P' the intersection of the powers of P/P' is (0), we see that

$$\bigcap_{n=1}^{\infty} P^n \subset P'.$$

Now suppose that there is a positive integer n such that $P' \subset P^n$ and $P' \not\subset P^{n+1}$. Then since $\text{rad}(P' + P^{n+1}) = P$, $P' + P^{n+1} = P^n$, i.e., $(P/P')^{n+1} = (P/P')^n$, which cannot happen since R/P' is an AD-domain. Therefore

$$P' \subset \bigcap_{n=1}^{\infty} P^n$$

so that

$$P' = \bigcap_{n=1}^{\infty} P^n.$$

But if P^* is any minimal prime of R contained in P, the same argument shows that

$$P^* = \bigcap_{n=1}^{\infty} P^n$$

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and P' is unique. Therefore, in R_P, P^e and $(P')^e$ are the only primes, and every ideal has prime radical and is thus a prime power. This proves that R_P is a ZPI ring.

From Theorems 2.2 through 2.5 we see that the following theorem is true.

THEOREM 2.6. *A ring R is an* AM-*ring if and only if each ideal of R with prime radical is a prime power.*

LEMMA 2.4. *A multiplication ring is an* AM-*ring.*

Proof. Mori **(6)** has shown that in a multiplication ring primary ideals are prime powers and each ideal is the intersection of its isolated primary components. Therefore, any ideal with prime radical is primary, since it has only one isolated primary component; so any ideal with prime radical is a prime power and the lemma follows from Theorem 2.6.

THEOREM 2.7. *A ring R is an* AM-*ring if and only if each proper R_P is a multiplication ring.*

Proof. The necessity follows from the fact that every ZPI ring is a multiplication ring. Conversely, if each R_P is a multiplication ring, each R_P is an AM-ring by Lemma 2.4; but being its own quotient ring with respect to its maximal ideal, R_P is a ZPI ring.

THEOREM 2.8. *A ring R is an* AM-*ring if and only if for each proper primary ideal Q of R there exists a maximal ideal M of R such that Q is either $N(M)$ or a power of M.*

Proof. The necessity is clear from Theorems 2.2 and 2.3. On the other hand, if each proper primary of R is either $N(M)$ or a power of M, then each R_P is either a field (in case $P = N(M)$), a dvr (in case P is maximal but not minimal), or an spr (in case P is maximal and minimal).

We define, for an ideal A of R, the ideal $\ker(A)$, the *kernel* of A **(4**, p. 119**)**, to be the intersection of the isolated primary components of A, i.e., the intersection of all A^{ec} with respect to the minimal primes of A.

THEOREM 2.9. *In an* AM-*ring, each ideal is identical with its kernel.*

Proof. Using Lemma 2.2 we need only to show that A^{ec} with respect to a proper prime is the same as A^{ec} with respect to some minimal prime of A. Suppose M is a prime containing A, but not a minimal prime of A. Then if $A \subset P < M$, P a prime of R, P is minimal containing A and $N(\mathrm{P}) = N(M) = P$ by Theorems 2.2 and 2.3. But then A^{ec} with respect to M is the same as A^{ec} with respect to P, namely $N(P)$.

COROLLARY 2.1. *If R is an* AM-*ring and P a prime of R, A an ideal of R properly containing P, then $PA = P$.*

Proof. Since $\text{rad}(PA) = P$, P is the only minimal prime of PA, and by Theorem 2.9, $(PA)^{ec} = PA$ ($(PA)^{ec}$ with respect to P). Since $P < A$, $A^e = R_P$ and

$$(PA)^{ec} = ((PA)^e)^c = (P^e A^e)^c = P^{ec} = P;$$

hence $PA = P$.

THEOREM 2.10. *A ring R is an* AM*-ring if and only if whenever A is an ideal of R such that* $\text{rad}(A) = P_1 \cdot P_2 \cdot \ldots \cdot P_n$ *(the product of n distinct primes), then*

$$A = P_1^{e_1} \cdot P_2^{e_2} \cdot \ldots \cdot P_n^{e_n}$$

for some collection $e_1, \ldots, e_n$ of n positive integers.

Proof. The sufficiency is clear from Theorem 2.5. Conversely, suppose R is an AM-ring and A is an ideal of R with $\text{rad}(A) = P_1 \cdot P_2 \cdot \ldots \cdot P_n$. Taking note of Corollary 2.1 we can assume that the P_i are relatively prime and minimal for A. Then by Theorem 2.9, A is the intersection of its isolated primary components. Using Lemma 2.3, we see that the isolated primary components of A are powers of the minimal primes containing A, namely $P_1, P_2, \ldots, P_n$. Then since $P_1, P_2, \ldots, P_n$ are relatively prime, these isolated primary components are relatively prime and A is their product. We can now reinsert any deleted non-minimal primes without changing the product and the theorem is proved.

LEMMA 2.5. *Let R be a ring and A, B ideals of R. Then $A = B$ if and only if $A^e = B^e$ in every proper R_P.*

Proof. If $A^e = B^e$, $A^{ec} = B^{ec}$ and the lemma follows from Lemma 2.2.

THEOREM 2.11. *If R is an* AM*-ring, and A, B, and C are ideals of R with A regular (i.e., A contains at least one non zero-divisor), then $AB = AC$ only if $B = C$.*

Proof. For each proper prime P of R, $A^eB^e = A^eC^e$ and A^e is regular, and since R_P is a dvr or an spr, this implies that $B^e = C^e$. Thus the theorem follows from Lemma 2.5.

In concluding this section, we summarize from Theorems 2.2 through 2.5 the classification of the proper primes of an AM-ring.

THEOREM 2.12. *In an* AM*-ring, a proper prime P is either*

(1) *maximal and minimal, in which case $N(P) = P^n$ and R_P is an* spr,

(2) *maximal and not minimal, in which case $N(P)$ is the only prime below P,*

$$N(P) = \bigcap_{n=1}^{\infty} P^n,$$

and R_P is a dvr, *or*

(3) *minimal and not maximal, in which case*

$$P = N(P) = N(M) = \bigcap_{n=1}^{\infty} M^n$$

for M any maximal ideal containing P, P is the only P-primary ideal of R and R_P *is a field, the quotient field of* R_M.

(4) *If P is a minimal prime which is not maximal, then P is the only ideal with radical equal to P—in particular,* $P = P^2$. *If P and P* are any two prime ideals, then* $P \subset P^*$ *or* $P^* \subset P$ *or* $P + P^* = R$.

References

1. K. Asano, *Über kommutative Ringe, in denen jedes Ideal als Produkt von Primidealen darstellbar ist,* J. Math. Soc. Japan, *1* (1951), 82–90.
2. I. S. Cohen, *Commutative rings with restricted minimum condition*, Duke Math. J., *17* (1950), 27–42.
3. W. Krull, *Beitrag zur Arithmetik kommutativer Integritätsbereiche*, Math. Z., *41* (1936), 545–77.
4. ———*Idealtheorie* (New York, 1948).
5. S. Mori, *Allgemeine Z.P.I.-Ringe*, J. Sc. Hiroshima Univ. (A), *10* (1940), 117–36.
6. ———*Über allgemeine Multiplikationsringe* I, J. Sci. Hiroshima Univ. (A), *4* (1934), 1–26.
7. N. Nakano, *Idealtheorie in einem speziellen unendlichen algebraischen Zahlkörper*, J. Sci. Hiroshima Univ. (A), *16* (1953), 425–39.
8. ———*Über die Umkehrbarkeit der Ideale in Integritätsbereichen*, Proc. Imp. Acad. Tokyo, *19* (1943), 230–4.
9. H. Prüfer, *Untersuchungen über Teilbarkeitseigenschaften in Körpern*, J. R. Angew. Math., *148* (1931), 1–36.
10. B. L. Van der Waerden, *Modern algebra I* (New York, 1950).
11. O. Zariski and P. Samuel, *Commutative algebra I* (New York, 1958).
12. ———*Commutative algebra II* (New York, 1960).

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