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ON THE AF-ALGEBRA OF A HECKE EIGENFORM

IGOR V. NIKOLAEV

The Fields Institute for Mathematical Sciences, 222 College Street, Toronto, Ontario M5T 3J1, Canada (igor.v.nikolaev@gmail.com)

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Abstract An AF-algebra is assigned to each cusp form f of weight 2; we study properties of this operator algebra when f is a Hecke eigenform.

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1. Introduction

1.1. The Modularity Theorem

The modularity theorem asserts that all rational elliptic curves arise from modular forms; this result is tremendously important, since it leads to a spectacular proof of Fermat's Last Theorem. Anyone reading the excellent book [**4**] will find the following interesting object. Denote by $f_N \in S_2(\Gamma_0(N))$ a Hecke eigenform and by f_N^σ all its conjugates and consider a lattice Λ_{f_N} generated by the complex periods of holomorphic forms $\omega_N^\sigma = f_N^\sigma \,\mathrm{d}z$ on the Riemann surface $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ (see § 2 for definitions of these concepts). If $|\sigma|$ is the number of conjugates, the abelian variety $\mathcal{A}_{f_N} := \mathbb{C}^{|\sigma|}/\Lambda_{f_N}$ is said to be *associated* to the eigenform f_N; it has the following remarkable property: *there exists a homomorphism of $\mathcal{A}_{f_N}$ onto a rational elliptic curve.*

Let $\phi_N = \mathrm{Re}(\omega_N)$ be the real part of ω_N; it is a closed form on the surface $X_0(N)$. (Alternatively, one can take for ϕ_N the imaginary part of ω_N.) Clearly, ω_N defines a unique form, ϕ_N; the converse follows from the Hubbard–Masur Theorem [**6**]. Since ω_N and ϕ_N define each other, what object will replace the associated variety $\mathcal{A}_{f_N}$ in the case of ϕ_N? Roughly speaking, it is shown in this paper that such a replacement is given by an operator algebra $\mathfrak{A}_{f_N}$ coming from the real periods of the form ϕ_N; we study the basic properties of such an algebra (Theorem 1.1).

1.2. The AF-algebra $\mathfrak{A}_{f_N}$

Let $f \in S_2(\Gamma_0(N))$ be a cusp form and let $\omega = f\,\mathrm{d}z$ be the corresponding holomorphic differential on $X_0(N)$. We shall denote by $\phi = \mathrm{Re}(\omega)$ a closed form on $X_0(N)$

and consider its periods $\lambda_i = \int_{\gamma_i} \phi$ against a basis γ_i in the (relative) homology group $H_1(X_0(N), Z(\phi); \mathbb{Z})$, where $Z(\phi)$ is the set of zeros of ϕ. Assume $\lambda_i > 0$ and consider the vector $\theta = (\theta_1, \dots, \theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$. The Jacobi–Perron continued fraction of θ [**2**] is given by the formula

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} = \lim_{i\to\infty} B_i \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^{\mathrm{T}}$ is a vector of non-negative integers, I is the unit matrix and $\mathbb{I} = (0, \dots, 0, 1)^{\mathrm{T}}$. By $\mathfrak{A}_f$ we shall understand the approximately finite C^*-algebra (AF-algebra) given by its Bratteli diagram with partial multiplicity matrices B_i. Recall that an AF-algebra is called *stationary* if $B_i = B = \text{const.}$ [**5**]. When two non-similar matrices B and B' have the same characteristic polynomial, the corresponding stationary AF-algebras will be called *companion AF-algebras*. Denote by $\mathfrak{A}_{f_N}$ an AF-algebra such that $f_N \in S_2(\varGamma_0(N))$ is a Hecke eigenform. Our main result can be stated as follows.

Theorem 1.1. *The AF-algebra $\mathfrak{A}_{f_N}$ is stationary unless f_N is a rational eigenform, in which case $\mathfrak{A}_{f_N} \cong \mathbb{C}$; moreover, $\mathfrak{A}_{f_N}$ and $\mathfrak{A}_{f_N^\sigma}$ are companion AF-algebras.*

The paper is organized as follows. The minimal preliminary results are expounded in § 2, where we review the Hecke eigenforms, the AF-algebras and the Jacobi–Perron continued fractions. Theorem 1.1 is proved in § 3.

2. Preliminaries

2.1. The Hecke eigenforms

Let $N > 1$ be a natural number and consider a (finite index) subgroup of the modular group given by the formula

$$\varGamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}.$$

Let $\mathbb{H} = \{z = x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and let $\varGamma_0(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\varGamma_0(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and the compact Riemann surface $X_0(N) = \mathbb{H}^*/\varGamma_0(N)$ is called a *modular curve*. The meromorphic functions $f(z)$ on $\mathbb{H}$ that vanish at the cusps and such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \varGamma_0(N)$$

are called *cusp forms* of weight 2; the (complex linear) space of such forms will be denoted by $S_2(\varGamma_0(N))$. The formula $f(z) \mapsto \omega = f(z)\,\mathrm{d}z$ defines an isomorphism $S_2(\varGamma_0(N)) \cong \varOmega_{\mathrm{hol}}(X_0(N))$, where $\varOmega_{\mathrm{hol}}(X_0(N))$ is the space of holomorphic differentials on the Riemann surface $X_0(N)$. Note that

$$\dim_{\mathbb{C}}(S_2(\varGamma_0(N))) = \dim_{\mathbb{C}}(\varOmega_{\mathrm{hol}}(X_0(N))) = g,$$

where $g = g(N)$ is the genus of the surface $X_0(N)$. A Hecke operator T_n acts on $S_2(\Gamma_0(N))$ by the formula

$$T_n f = \sum_{m\in\mathbb{Z}} \gamma(m) q^m,$$

where $\gamma(m) = \sum_{a|\,\mathrm{GCD}(m,n)} a c_{mn/a^2}$ and $f(z) = \sum_{m\in\mathbb{Z}} c(m)q^m$ is the Fourier series of the cusp form f at $q = \mathrm{e}^{2\pi \mathrm{i} z}$. Furthermore, T_n is a self-adjoint linear operator on the vector space $S_2(\Gamma_0(N))$ endowed with the Petersson inner product; the algebra $\mathbb{T}_N := \mathbb{Z}[T_1, T_2, \dots]$ is a commutative algebra. Any cusp form $f_N \in S_2(\Gamma_0(N))$ that is an eigenvector for one (and hence all) of T_n is referred to as a *Hecke eigenform*; such an eigenform is called *rational* whenever its Fourier coefficients $c(m) \in \mathbb{Z}$. The Fourier coefficients $c(m)$ of f_N are algebraic integers, and we denote by $K_{f_N} = \mathbb{Q}(c(m))$ an extension of the field $\mathbb{Q}$ by the Fourier coefficients of f_N. Then K_{f_N} is a real algebraic number field of degree $1 \leqslant \deg(K_{f_N}/\mathbb{Q}) \leqslant g$, where g is the genus of the surface $X_0(N)$ [**4**, Proposition 6.6.4]. Any embedding $\sigma\colon K_{f_N} \to \mathbb{C}$ conjugates f_N by acting on its coefficients; we write the corresponding Hecke eigenform as

$$f_N^{\sigma}(z) := \sum_{m\in\mathbb{Z}} \sigma(c(m)) q^m.$$

2.2. The AF-algebras

A *C^*-algebra* is an algebra A over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $\|ab\| \leqslant \|a\|\,\|b\|$ and $\|a^*a\| = \|a^2\|$ for all $a, b \in A$. Any commutative C^*-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space X; otherwise, A represents a non-commutative topological space. The C^*-algebras A and A' are said to be *stably isomorphic* (Morita equivalent) if $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators; roughly speaking, the stable isomorphism means that A and A' are homeomorphic as non-commutative topological spaces.

An *AF-algebra* is defined to be the norm closure of an ascending sequence of finite-dimensional C^*-algebras M_n, where M_n is the C^*-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \dots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \dots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \dots$, where M_i are the finite-dimensional C^*-algebras and φ_i are the homomorphisms between such algebras. The homomorphisms φ_i can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \dots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \dots \oplus M_{i'_k}$ be the semi-simple C^*-algebras and $\varphi_i\colon M_i \to M_{i'}$ be the homomorphism. One has two sets of vertices $V_{i_1}, \dots, V_{i_k}$ and $V_{i'_1}, \dots, V_{i'_k}$ joined by b_{rs} edges whenever the summand M_{i_r} contains b_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i. As i varies, one obtains an infinite graph called the *Bratteli diagram* of the AF-algebra. The matrix $B = (b_{rs})$ is known as a *partial multiplicity matrix*; an infinite sequence of B_i defines a unique AF-algebra.

For a unital C^*-algebra A, let $V(A)$ be the union (over n) of projections in the $n \times n$ matrix C^*-algebra with entries in A; projections $p, q \in V(A)$ are *equivalent* if there exists a partial isometry u such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection

p is denoted by $[p]$; the equivalence classes of orthogonal projections can be made into a semigroup by setting $[p]+[q]=[p+q]$. The Grothendieck completion of this semigroup to an abelian group is called the K_0-group of the algebra A. The functor $A\to K_0(A)$ maps the category of unital C^*-algebras into the category of abelian groups, so that projections in the algebra A correspond to a positive cone $K_0^+\subset K_0(A)$ and the unit element $1\in A$ corresponds to an order unit $u\in K_0(A)$. The ordered abelian group (K_0,K_0^+,u) with an order unit is called a *dimension group*; we denote an order-isomorphism class of the latter by (G,G^+).

2.3. The Jacobi–Perron fractions

Let $a_1,a_2\in\mathbb{N}$ such that $a_2\leqslant a_1$. Recall that the greatest common divisor of a_1, a_2, $\mathrm{GCD}(a_1,a_2)$ can be determined from the Euclidean algorithm

$$\begin{aligned}
a_1 &= a_2b_1+r_3,\\
a_2 &= r_3b_2+r_4,\\
r_3 &= r_4b_3+r_5,\\
&\ \ \vdots\\
r_{k-3} &= r_{k-2}b_{k-1}+r_{k-1},\\
r_{k-2} &= r_{k-1}b_k,
\end{aligned}$$

where $b_i\in\mathbb{N}$ and $\mathrm{GCD}(a_1,a_2)=r_{k-1}$. The Euclidean algorithm can be written as the regular continued fraction

$$\theta=\frac{a_1}{a_2}=b_1+\cfrac{1}{b_2+\cfrac{1}{+\cdots+\cfrac{1}{b_k}}}=[b_1,\dots b_k].$$

If a_1, a_2 are non-commensurable in the sense that $\theta\in\mathbb{R}-\mathbb{Q}$, then the Euclidean algorithm never stops, and $\theta=[b_1,b_2,\dots]$. Note that the regular continued fraction can be written in matrix form

$$\begin{pmatrix}1\\ \theta\end{pmatrix}=\lim_{k\to\infty}\begin{pmatrix}0&1\\1&b_1\end{pmatrix}\cdots\begin{pmatrix}0&1\\1&b_k\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

The Jacobi–Perron algorithm and connected (multidimensional) continued fraction generalizes the Euclidean algorithm to the case $\mathrm{GCD}(a_1,\dots,a_n)$ when $n\geqslant 2$. Namely, let $\lambda=(\lambda_1,\dots,\lambda_n)$, $\lambda_i\in\mathbb{R}-\mathbb{Q}$ and $\theta_{i-1}=\lambda_i/\lambda_1$, where $1\leqslant i\leqslant n$. The continued fraction

$$\begin{pmatrix}1\\ \theta_1\\ \vdots\\ \theta_{n-1}\end{pmatrix}=\lim_{k\to\infty}\begin{pmatrix}0&0&\cdots&0&1\\1&0&\cdots&0&b_1^{(1)}\\ \vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&b_{n-1}^{(1)}\end{pmatrix}\cdots\begin{pmatrix}0&0&\cdots&0&1\\1&0&\cdots&0&b_1^{(k)}\\ \vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&b_{n-1}^{(k)}\end{pmatrix}\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix},$$

where $b_i^{(j)} \in \mathbb{N} \cup \{0\}$, is called the *Jacobi–Perron algorithm (JPA)*. Unlike the regular continued fraction algorithm, the JPA may diverge for certain vectors $\lambda \in \mathbb{R}^n$. However, for points of a generic subset of $\mathbb{R}^n$, the JPA converges [**1**]; in particular, the JPA for periodic fractions is always convergent.

3. Proof of theorem 1.1

A standard dictionary [**5**] between AF-algebras and their dimension groups is adopted. Instead of dealing with $\mathfrak{A}_f$, we work with its dimension group $G_{\mathfrak{A}_f} = (G, G^+)$, where $G \cong \mathbb{Z}^n$ is the lattice and $G^+ = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid \theta_1 x_1 + \cdots + \theta_{n-1} x_{n-1} + x_n \geqslant 0\}$ is a positive cone. Recall that $G_{\mathfrak{A}_f}$ is abelian group with an order that defines the AF-algebra $\mathfrak{A}_f$ up to a stable isomorphism. We arrange the proof in a series of lemmas. First, let us show that $\mathfrak{A}_f$ is a correctly defined AF-algebra.

Lemma 3.1. *The $\mathfrak{A}_f$ does not depend, up to a stable isomorphism, on a basis in $H_1(X_0(N), Z(\phi); \mathbb{Z})$.*

Proof. Denote by $\mathfrak{m} := \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ a $\mathbb{Z}$-module in the real line $\mathbb{R}$. Let $\{\gamma_i'\}$ be a new basis in $H_1(X_0(N), Z(\phi); \mathbb{Z})$, such that $\gamma_i' = \sum_{j=1}^n a_{ij}\gamma_j$ for matrix $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$. Using the integration rules, one gets

$$\lambda_i' = \int_{\gamma_i'} \phi = \int_{\sum_{j=1}^n a_{ij}\gamma_j} \phi = \sum_{j=1}^n \int_{\gamma_j} \phi = \sum_{j=1}^n a_{ij}\lambda_j.$$

Thus, $\mathfrak{m}' = \mathfrak{m}$ and a change of basis in the homology group $H_1(X_0(N), Z(\phi); \mathbb{Z})$ amounts to a change of basis in the module $\mathfrak{m}$. It is an easy exercise to show that there exists a linear transformation of $\mathbb{Z}^n$ sending the positive cone G^+ of $G_{\mathfrak{A}_f}$ to the positive cone $(G^+)'$ of $G_{\mathfrak{A}_f'}$. In other words, $\mathfrak{A}_f'$ and $\mathfrak{A}_f$ are stably isomorphic. $\square$

Lemma 3.2. *The (scaled) periods λ_i belong to the field K_{f_N}.*

Proof. Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_{2g}$ be a $\mathbb{Z}$-module generated by λ_i; we seek the effect of the Hecke operators T_m on $\mathfrak{m}$. By the definition of a Hecke eigenform, $T_m f_N = c(m) f_N$ for all $T_m \in \mathbb{T}_N$. In view of the isomorphism $S_2(\Gamma_0(N)) \cong \Omega_{\mathrm{hol}}(X_0(N))$, one gets $T_m \omega_N = c(m)\omega_N$, where $\omega_N = f_N \,\mathrm{d}z$. Then $\mathrm{Re}(T_m\omega_N) = T_m(\mathrm{Re}(\omega_N)) = \mathrm{Re}(c(m)\omega_N) = c(m)\,\mathrm{Re}(\omega_N)$. Therefore, $T_m \phi_N = c(m)\phi_N$, where $\phi_N = \mathrm{Re}(\omega_N)$. The action of T_m on $\mathbb{Z}$-module $\mathfrak{m}$ can be written as

$$T_m(\mathfrak{m}) = \int_{H_1} T_m \phi_N = \int_{H_1} c(m)\phi_N = c(m)\mathfrak{m},$$

where $H_1 := H_1(X_0(N), Z(\phi_N); \mathbb{Z})$. Thus, the Hecke operator T_m acts on the module $\mathfrak{m}$ as multiplication by an algebraic integer $c(m) \in K_{f_N}$.

The action of T_m on $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ can be written as $T_m\lambda = c(m)\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_n)$; thus, T_m is a linear operator (on the space $\mathbb{R}^n$), whose eigenvector λ corresponds to the eigenvalue $c(m)$. It is an easy exercise in linear algebra to show that λ can be scaled so that all λ_i lie in the same field as $c(m)$; Lemma 3.2 follows. $\square$

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Case 1. Let f_N be a non-rational eigenform; then $n = \deg(K_{f_N}/\mathbb{Q}) \geqslant 2$. Note that $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ is a full (i.e. the maximal rank) $\mathbb{Z}$-module in the number field K_{f_N}. Indeed, $\operatorname{rank}(\mathfrak{m})$ cannot exceed n, since $\mathfrak{m} \subset K_{f_N}$ and K_{f_N} is a vector space (over $\mathbb{Q}$) of dimension n. On the other hand, $(\lambda_1, \dots, \lambda_n)$ is a basis of the field K_{f_N} and, as such, $\operatorname{rank}(\mathfrak{m})$ cannot be less than n; thus, $\operatorname{rank}(\mathfrak{m}) = n$.

Lemma 3.3. *The vector $(\lambda_1, \dots, \lambda_n)$ has a periodic (Jacobi–Perron) continued fraction.*

Proof. Since $\mathfrak{m} \subset K_{f_N}$ is a full $\mathbb{Z}$-module, its endomorphism ring $\operatorname{End}(\mathfrak{m}) = \{\alpha \in K_{f_N} : \alpha\mathfrak{m} \subseteq \mathfrak{m}\}$ is an order (a subring of the ring of integers) of the number field K_{f_N}; let u be a unit of the order [**3**, p. 112]. The action of u on $\mathfrak{m}$ can be written in a matrix form $A\lambda = u\lambda$, where λ is a basis in $\mathfrak{m}$ and $A \in \mathrm{GL}_n(\mathbb{Z})$; with no loss of generality, one can assume the matrix A to be non-negative in a proper basis of $\mathfrak{m}$.

According to Proposition 3 of [**1**], the matrix A can be uniquely factorized as

$$A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix},$$

where vectors $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^{\mathrm{T}}$ have non-negative integer entries. By Satz XII of [**7**], the periodic continued fraction

$$\begin{pmatrix} 1 \\ \theta' \end{pmatrix} = \mathrm{Per}\,\overline{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} \tag{3.1}$$

converges to a vector $\lambda' = (\lambda'_1, \dots, \lambda'_n)$, which satisfies the equation $A\lambda' = u\lambda'$. Since $A\lambda = u\lambda$, the vectors λ and λ' are collinear, but collinear vectors have the same continued fractions [**2**]. $\square$

The first case of Theorem 1.1 follows from Lemma 3.3, since $\mathfrak{A}_{f_N}$ is a stationary AF-algebra, whose period is given by the matrix A.

Case 2. Let f_N be a rational eigenform; in this case $n = 1$ and $K_{f_N} = \mathbb{Q}$. The Bratteli diagram of $\mathfrak{A}_{f_N}$ is finite and one dimensional; therefore, $\mathfrak{A}_{f_N} \cong M_1(\mathbb{C}) = \mathbb{C}$. This argument finishes the proof of the first part of Theorem 1.1.

To prove the second part by contradiction, let $A \neq A'$ be similar matrices. To find S such that $A' = S^{-1}AS$, notice that $\mathfrak{m}^\sigma = \lambda_1^\sigma\mathbb{Z} + \cdots + \lambda_n^\sigma\mathbb{Z}$. Since $\mathfrak{m}^\sigma = \mathfrak{m}$, $\lambda_j^\sigma = \sum s_{ij}\lambda_i$, where $S = (s_{ij})$; but $\sigma^k = \mathrm{Id}$ for some integer k and thus $S^k = I$. Therefore, $(A')^k = (S^{-1}AS)^k = A^k$ and $A' = A$, which contradicts our assumption. On the other hand, $\lambda_j^\sigma \in K_{f_N}$ implies that the characteristic polynomials $\mathrm{ch}(A) = \mathrm{ch}(A')$; therefore, $\mathfrak{A}_{f_N}$ and $\mathfrak{A}_{f_N^\sigma}$ are companion AF-algebras.

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