A Note on Double Limits.

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Let u_{mn}, v_{mn} be functions of m and n, v_{mn} being real and positive for all positive values of m and n.[1] Suppose that either v_{mn} increases steadily to infinity with n, or that u_{mn} and v_{mn} both tend to zero (the latter steadily) as $n \to \infty$, for any fixed value of m. Denote $\dfrac{u_{m,\,n+1} - u_{mn}}{v_{m,\,n+1} - v_{mn}}$ by w_{mn}, and assume that $\lim\limits_{n \to \infty} w_{mn}$ exists for every value of m, being denoted by l_m. Then from Stolz' extension of a result proved by Cauchy, and an allied theorem,[2] we have $\lim\limits_{n \to \infty} \dfrac{u_{mn}}{v_{mn}} = l_m$, for all values of m. It follows from Pringsheim's Theorem that if the double limit of $\dfrac{u_{mn}}{v_{mn}}$ exists, being l, then $l_m \to l$ as $m \to \infty$.

As a particular case, if u_{mn} is a real function, if the double sequence $\dfrac{u_{mn}}{v_{mn}}$ is monotonic[3] (either increasing or decreasing) in both m and n, and it is known that $l_m \to l$ as $m \to \infty$, then the double limit of $\dfrac{u_{mn}}{v_{mn}}$ will also be l.

For example, let $S_{\mu\nu}$ be any monotonic (increasing or decreasing) double sequence in both μ and ν, converging to S, and take $\sum\limits_{\mu=1}^{m} \sum\limits_{\nu=1}^{n} S_{\mu\nu}/mn$ as $\dfrac{u_{mn}}{v_{mn}}$. Then $\dfrac{u_{mn}}{v_{mn}}$ is also monotonic (in the same sense as $S_{\mu\nu}$) in both m and n. We then have $w_{mn} = \sum\limits_{r=1}^{m} S_{r,\,n+1}/m$. If $S_{m\infty}$

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[1] Or for all values of m and n greater than fixed values, say m_1 and n_1.

[2] See Bromwich *Infinite Series*, pp. 377–378, for both of these.

[3] Hereafter when the word "monotonic" is used, the functions concerned are to be regarded as real.

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denote $\lim_{n\to\infty} S_{mn}$ (which can be seen to exist for all values of m)

$\lim_{n\to\infty} w_{mn} = \dfrac{\sum_1^m S_{r\infty}}{m}$. Since a_n and $\dfrac{\sum_1^n a_r}{n}$ converge to the same limit

if a_n converges, it follows that

$$\lim_{m\to\infty}\ \lim_{n\to\infty} w_{mn} = \lim_{m\to\infty} \frac{\sum_1^m S_{r\infty}}{m} = \lim_{m\to\infty} S_{m\infty} = \lim_{m\to\infty}\ \lim_{n\to\infty} S_{mn} = S.$$

Thus the double limit of $\dfrac{\sum_1^m \sum_1^n S_{\mu\nu}}{mn}$ is S.

Again if the double limits of $\dfrac{u_{mn}}{v_{mn}}$ and w_{mn} are both known to exist, being U and W respectively, we have

$U = \lim_{m\to\infty}\ \lim_{n\to\infty} \dfrac{u_{mn}}{v_{mn}} = \lim_{m\to\infty}\ \lim_{n\to\infty} w_{mn} = W$, so U and W will be equal.

In addition if $\lim_{m\to\infty}\ \lim_{n\to\infty} \dfrac{u_{mn}}{v_{mn}} = l$, and the double sequence w_{mn} is monotonic (either way) in both m and n, the double limit of w_{mn} will be l, this being true also in the more general case where the double limit of w_{mn} is known to exist.

If $l m \to \infty$, and $\dfrac{u_{mn}}{v_{mn}}$ is monotonic increasing in m, the double limit is $+\infty$. For, given any number R, however large we can find M so that $l_m > R$ if $m \geqslant M$. Then N can be found so that if $n > N$, $\dfrac{u_{Mn}}{v_{Mn}} > l_M - \dfrac{R}{2} > \dfrac{R}{2}$, and now if $m > M$, $n > N$,

$$\frac{u_{mn}}{v_{mn}} > \frac{u_{Mn}}{v_{Mn}} > \frac{R}{2}.$$

This proves the result.

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