BULL. AUSTRAL. MATH. SOC. MOS 4160

VOL. 2 (1970), 233-236.

A Stone-Weierstrass theorem for random functions

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It is shown in this note that if Q is an algebra of uniformly bounded mean-square continuous real-valued random functions indexed in a compact set T, containing all bounded random variables and separating points of T (i.e., given t_1 and t_2 in T, there is a random function X_t in Q such that $\left|X_{t_1} - X_{t_2}\right| = 1$), then given any mean square continuous random function, there is a sequence in Q converging in mean square to the given random function uniformly on T.

The purpose of this note is to present a Stone-Weierstrass type theorem for random functions which might find possible future applications in probability theory or analysis. Tzannes in [2] showed that a mean square continuous (m.s.c.) second order random function (r.f.) can be approximated uniformly in mean square by a sequence of random polynomials (i.e., polynomials with random variables as co-efficients). So it is natural to consider the same problem in the more general situation which we describe in the following paragraph.

Let T be a compact set in some topological space. Let us restrict our attention to real-valued random functions on some probability space indexed in the parameter set T. A r.f. X_t is said to be m.s.c. on T if for every t in T, $E\left[X_t^2\right] = \int X_t^2 < \infty$ and

Received 22 December 1969.

and $n(X_t - X_s) = \left\{\int \left[X_t - X_s\right]^2\right\}^{\frac{1}{2}}$ tends to 0 as s tends to t. A r.f. X_t is uniformly bounded if there is a constant M such that for every t, $|X_t| < M$. A family Q of random functions is called an algebra if

(i) X_t and Y_t in Q implies that $X_t . Y_t$ (pointwise multiplication) is also in Q and

(ii) X_t and Y_t in Q implies that $X_t + Y_t$ is also in Q.

The uniformly bounded m.s.c. random functions can be easily seen to form an algebra. Q is said to separate points in T if given t_1 and t_2 in T, there exists a r.f. X_t in Q such that $|X_{t_1} - X_{t_2}| = 1$. If $T = [0, 1]$, the algebra of random polynomials separate points of T. This is the desired Stone-Weierstrass setting in which we consider the problem mentioned in the first paragraph. We have, as can be expected, the following theorem.

THEOREM. *Let Q be an algebra of uniformly bounded* m.s.c. *random functions containing all bounded random variables. Let Q also separate points of T. Then given a* m.s.c. r.f., *there exists a sequence in Q which converges in mean square to the given* r.f. *uniformly on T.*

Proof. The proof follows closely the classical pattern.

Following the classical proof (see page 131, [1]), one can easily check that if X_t is in Q, then $|X_t|$ is in $\bar{Q}$, the closure of Q in the uniform mean square limit sense.

Next, given t_1 and t_2 and any two random variables X_1 and X_2 in Q, we can find Z_t in $\bar{Q}$ such that $Z_{t_1} = X_1$ and $Z_{t_2} = X_2$; for we can take $Z_t = X_1 + |X_t - X_{t_1}|.(X_2 - X_1)$ where X_t is in Q such that $|X_{t_1} - X_{t_2}| = 1$.

Now let W_t be any m.s.c. non-negative r.f.. We wish to show that W_t is in $\bar{Q}$. With no loss of generality, we can assume that W_t is uniformly bounded. For, given $\varepsilon > 0$, using the mean square continuity

https://doi.org/10.1017/S0004972700041848 Published online by Cambridge University Press

of W_t and the r.f. $W_{tm} = \inf\{m, W_t\}$, where m is a constant, and the compactness of T, we can find a m such that $n(W_t - W_{tm}) < \varepsilon$ for every t in T. [Note that here n denotes the L_2-norm.]

So we assume that W_t is uniformly bounded by a constant m. Let I_D be the characteristic function of the measurable set D and so it is a random variable in Q. Let t_0 be in T. Then for every t' in T, we can find a neighbourhood $N_{t'}$ of t' and $Y_t^{t'}$ in $\bar{Q}$ such that $Y_{t_0}^{t'} = W_{t_0}$ and $n\left(Y_t^{t'} . I_D\right) < n\left(W_t . I_D\right) + \varepsilon/3m$ for every t in $N_{t'}$ and every measurable set D. Then using the compactness of T and noting that $\inf\{X_t, Y_t\}$ is in $\bar{Q}$ for X_t and Y_t in $\bar{Q}$, we can find a $Y_t^{t_0}$ in $\bar{Q}$ such that

$$Y_{t_0}^{t_0} = W_{t_0} \quad \text{and} \quad n\left(Y_t^{t_0} . I_D\right) < n\left(W_t . I_D\right) + \varepsilon/3m$$

for every t in T and every measurable set D. Now we can find a neighbourhood N_{t_0} of t_0 such that for every t in N_{t_0} and every measurable set D,

$$n\left(Y_t^{t_0} . I_D\right) > n\left(W_t . I_D\right) - \varepsilon/3m .$$

Doing this for every t_0 in T, then we can find a Y_t in $\bar{Q}$ such that $\left|n\left(Y_t . I_D\right) - n\left(W_t . I_D\right)\right| < \varepsilon/3m$ for every t in T and every measurable set D. Then $\left|E\left(Y_t^2 . I_D\right) - E\left(W_t^2 . I_D\right)\right| < \varepsilon$. Now let $A_t = \left[W_t \geqq Y_t\right]$. Then

$$n\left(Y_t - W_t\right) \leqq n\left[I_{A_t} . \left(Y_t - W_t\right)\right] + n\left[I_{A_t^c} . \left(Y_t - W_t\right)\right] ,$$

each of which is less than $\sqrt{\varepsilon}$; for

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$$E\left(I_{A_t}\cdot\left(Y_t-W_t\right)^2\right) = E\left(W_t^2.I_{A_t}\right) + E\left(Y_t^2.I_{A_t}\right) - 2E\left(Y_t.W_t.I_{A_t}\right)$$
$$\leqq E\left(W_t^2.I_{A_t}\right) - E\left(Y_t^2.I_{A_t}\right) < \varepsilon$$

and similarly the other one.

Finally, let W_t be any m.s.c. r.f. . Then since T is compact and W_t is m.s.c. , given $\varepsilon > 0$, we can find $\beta > 0$ such that $P(B) < \beta$ (where P is the measure in the probability space) implies that $n\left(W_t.I_B\right) < \varepsilon$ for every t in T . Noting that $E\left(W_t^2\right)$ is a bounded function of t , we can find a number $k > 0$ such that for every t in T , there is a B_t , a measurable set such that $P\left(B_t\right) < \beta$ and on B_t^c , $\left|W_t\right|$ is less than k . Then we write $U_t = \sup\left\{-k, \inf\left(W_t, k\right)\right\}$ so that U_t is clearly a m.s.c. r.f. bounded by k for all t . We note that on B_t^c , $U_t = W_t$ and therefore, since $\left|U_t\right| \leqq \left|W_t\right|$, it is easy to see that $n\left(W_t-U_t\right) = n\left(I_{B_t}\cdot\left(W_t-U_t\right)\right) < 2\varepsilon$. Now $k - U_t$ is a non-negative m.s.c. r.f. and so we can find Y_t in $\bar{Q}$ such that $n\left(U_t - \left(k-Y_t\right)\right) < \varepsilon$ and this proves that there is a $Z_t = k - Y_t$ in $\bar{Q}$ such that $n\left(W_t-Z_t\right) < 3\varepsilon$ for every t in T . This completes the proof of the theorem.

References

[1] J. Dieudonné, *Foundations of modern analysis* (Academic Press, New York, London, 1960).

[2] Nicolaos S. Tzannes, "Polynomial expansions of random functions", *IEEE Trans. Information Theory* IT-13 (1967), 314.

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