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EXTENDING JORDAN IDEALS AND JORDAN HOMOMORPHISMS OF SYMMETRIC ELEMENTS IN A RING WITH INVOLUTION

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Introduction. In this work, we show how the ideas in [**3**, pp. 6–12] can be used to give conditions under which Jordan ideals in the set of symmetric elements in an associative ring R with involution extend to associative ideals of R in a natural way. We also give conditions under which a Jordan homomorphism of the set of symmetric elements will extend to an associative homomorphism of R. Such work has been done on matrix rings with involution in [**5**; **6**]. An abstract definition of a Jordan ring may be found in [**3**] as well as other background information.

Let R be an associative ring with involution $r \to r^*$; that is, a mapping $r \to r^*$ such that

$$(r_1 + r_2)^* = r_1^* + r_2^*,$$
$$(r_1 r_2)^* = r_2^* r_1^*,$$
$$(r^*)^* = r.$$

We will denote by S the set of *-symmetric elements of R, namely $S = \{s \in R | s^* = s\}$. Likewise, let $K = \{k \in R | k^* = -k\}$, the set of *-skew symmetric elements of R. If I is an ideal of R then we will call I a **-ideal* if I is invariant under the involution on R, i.e. if $i^* \in I$ for every $i \in I$.

If juxtaposition denotes the multiplicative binary operation on R, then $\cdot$, defined by $s_1 \cdot s_2 = s_1 s_2 + s_2 s_1$, $s_i \in S$, makes the additive group S into a *Jordan ring*. Similarly, K forms a *Lie ring* under $[k_1, k_2] = k_1 k_2 - k_2 k_1$, $k_i \in K$.

Throughout this paper our assumptions on R are:

(1) $2r = 0$ implies $r = 0$, $r \in R$;
(2) $A = \{2a | a \in A\}$ for every *-ideal A of R and every Jordan ideal A of S.

For example, R may be any algebra over a field of characteristic not two or R may be any finite ring satisfying (1). We note that condition (2) says that the mapping $r \to 2r$ of R is an onto mapping for every *-ideal of R and every Jordan ideal of S. Our use of conditions (1) and (2) will be to allow divisibility by 2. The notation $\frac{1}{2}a$ will mean that element $r \in R$ such that $2r = a$.

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If $r \in R$, then $r = \frac{1}{2}(r + r^*) + \frac{1}{2}(r - r^*)$ and so every element in R can be written as the sum of an element in S and one in K. Since $S \cap K = \{0\}$, this representation is unique. We will keep this property of R in mind by writing $R = S + K$.

Extending Jordan ideals of S**.** Let I be a *-ideal of R. Then * induces an involution on the ring I. So $I = U + L$ where U is the set of symmetric elements of I and L is the set of skew symmetric elements of I. An easy check shows that U is a Jordan ideal of S and L is a Lie ideal of K. We now seek conditions under which a Jordan ideal U of S is the set of symmetric elements of a *-ideal I of R. If such is the case for a particular ideal U of S then we will say that *U extends to a *-ideal of R*.

Let E be the subring of the rationals generated by $\frac{1}{2}$. Using E we may, if R does not have a unit element, imbed R in a ring $\bar{R}$ such that $1 \in \bar{R}$. Such a ring is $\bar{R} = \{(e, r) | e \in E, r \in R\}$ under the usual operations. It is easy to check that $\bar{R}$ satisfies conditions (1) and (2). $\bar{R}$ is a ring with involution $'$ defined by $(m, r)' = (m, r^*)$. We note that $\bar{R} = \bar{S} + \bar{K}$ where

$$\bar{S} = \{(m, s) | m \in E, s \in S\} \quad \text{and} \quad \bar{K} = \{(0, k) | k \in K\}.$$

If U is a Jordan ideal of S we can correspond U with $\bar{U} = \{(0, u) | u \in U\}$, a Jordan ideal of $\bar{S}$. It is easy to see that U extends in R if and only if $\bar{U}$ extends in $\bar{R}$. For easy reference we write this as the first lemma.

LEMMA 1. *If $1 \notin R = S + K$, let $\bar{R}$ be the ring with 1 in which R is imbedded in the usual way. Then $\bar{R} = \bar{S} + \bar{K}$ is a ring with involution $'$ and if U is an ideal of S then U extends to a *-ideal of R if and only if its corresponding ideal $\bar{U}$ of $\bar{S}$ extends to a $'$-ideal in $\bar{R}$.*

LEMMA 2. *Let $R = S + K$ be a ring with involution *. A Jordan ideal U of S extends to a *-ideal of R if and only if $aub + b^*ua^* \in U$ for every $u \in U$, $b \in R$.*

Proof. We may assume that $1 \in R$; for, if we identify U with $\bar{U} = \{(0, u) | u \in U\}$ in $\bar{R}$, then $\bar{U}$ satisfies the conditions of Lemma 2 (using assumption (2) on R), and by Lemma 1, $\bar{U}$ extends in $\bar{R}$ if and only if U extends in R.

Let L be the Lie ideal of K generated by $\{aub - b^*ua^* | a, b \in R\}$. K consists simply of all finite sums of its generators. We let $I = U + L$ and proceed to show that I is a *-ideal of R. It is clear that the set I is invariant under the involution. For every $h \in L$ we know that

$$h = \sum_i a_i u_i b_i - b_i^* u_i a_i^*,$$

a finite sum, where $a_i, b_i \in R$, $u_i \in U$. So if $s \in S$, we have

$$\begin{aligned} sh = {} & \tfrac{1}{2} \textstyle\sum_i (sa_i) u_i b_i - b_i^* u_i (sa_i)^* + \tfrac{1}{2} \sum_i a_i u_i (b_i s) - (b_i s)^* u_i a_i^* \\ & + \tfrac{1}{2} \textstyle\sum_i (sa_i) u_i b_i + b_i^* u_i (sa_i)^* + \tfrac{1}{2} \sum_i - a_i u_i (b_i s) - (b_i s)^* u_i a_i^*. \end{aligned}$$

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This means that $sh \in I$ for every $s \in S$ and $h \in L$. For $k \in K$ we have

$$kh = \tfrac{1}{2}\sum_i (ka_i)u_i b_i + b_i^* u_i (ka_i)^* + \tfrac{1}{2}\sum_i a_i u_i (b_i k) + (b_i k)^* u_i a_i^* \\ + \tfrac{1}{2}\sum_i (ka_i)u_i b_i - b_i^* u_i (ka_i)^* + \tfrac{1}{2}\sum_i (b_i k)^* u_i a_i^* - a_i u_i (b_i k).$$

This shows that $kh \in I$ for every $k \in K$, $h \in L$. For $s \in S$, $u \in U$, $k \in K$ we have

$$su = \tfrac{1}{2}(su + us) + \tfrac{1}{2}(su - us),$$
$$ku = \tfrac{1}{2}(ku + uk^*) + \tfrac{1}{2}(ku - uk^*),$$

which show that su and ku belong to L. Since $R = S + K$, all of the above calculations show that I is a left ideal of R. Since I is invariant under the involution, I is also a right ideal and hence an ideal of R.

We let $\{s_1s_2 \ldots s_n\} \equiv s_1s_2 \ldots s_n + s_n \ldots s_2s_1$, where each $s_i \in S$. Clearly, $\{s_1s_2 \ldots s_n\} \in S$. Following Cohn [**1**], we will call $\{s_1s_2s_3s_4\}$ a *tetrad* in s_1, s_2, s_3, s_4.

If U is a Jordan ideal of S then, clearly, $\{us\} = us + su \in S$ for every $u \in U$, $s \in S$. We show now that $\{us_1s_2\} \in U$. For $2sus = [s(su + us) + (su + us)s] - [s^2u + us^2]$ belongs to U and thus $\{s_1us_2\} = (s_1 + s_2)u(s_1 + s_2) - s_1us_1 - s_2us_2 \in U$. So since $\{us_1s_2\} = \{(us_1 + s_1u)s_2\} - \{s_1us_2\}$, we have $\{us_1s_2\} \in U$. We will give examples later to show that the tetrad $\{us_2s_3s_4\}$ need not be in U. This leads us to the main theorem of this section.

THEOREM 1. *Let $R = S + K$ be an associative ring with involution * satisfying properties* (1)–(2) *and assume that the set of symmetric elements S generates R associatively. Then a Jordan ideal U of S extends to a *-ideal I of R if and only if $\{us_2s_3s_4\} \in U$ for every $s_2, s_3, s_4 \in S$, $u \in U$.*

Proof. The necessity of $\{us_2s_3s_4\}$ being in U is clear. For the converse, we note first that since S generates R, Lemma 2 tells us that it is enough to show that $\{s_2s_3 \ldots s_ius_{i+1} \ldots s_n\} \in U$ for $n = 2, 3, \ldots$. We proceed to do this by induction on n. Clearly, $\{us\} = \{su\} \in U$ which is the case $n = 2$. Now we assume that we have shown that for every $s_i \in S$, $u \in U$, we have $\{s_2s_3 \ldots s_ius_{i+1} \ldots s_{n-1}\} \in U$ regardless of the position of u. Then we have $\{us_2s_3 \ldots s_n\} = \{(us_2 + s_2u)s_3 \ldots s_n\} - \{s_2us_3 \ldots s_n\}$. Since $us_2 + s_2u \in U$ as well as $\{(us_2 + s_2u)s_3 \ldots s_n\} \in U$ (by induction hypothesis), we conclude that $\{us_2s_3 \ldots s_n\} \in U$ if and only if $\{s_2us_3 \ldots s_n\} \in U$. Continuing, we get $\{us_2s_3 \ldots s_n\} \in U$ if and only if $\{s_2 \ldots s_ius_{i+1} \ldots s_n\} \in U$. So to finish the proof of the theorem, it is enough to show that $\{us_2s_3 \ldots s_n\} \in U$ for every $u \in U$, $s_i \in S$.

For this goal we need the following general identities found in [**1**]:

(4) $$\{(s_1s_2 + s_2s_1)s_3 \ldots s_n\} = \{s_1s_2s_3 \ldots s_n\} + \{s_2s_1s_3 \ldots s_n\};$$

(5) $$\{s_1s_2s_3 \ldots s_{n-1}\} \cdot s_n = \{s_1s_2s_3 \ldots s_n\} + \{s_ns_1s_2 \ldots s_{n-1}\};$$

(6) $$\{s_1s_2s_3s_4\} \cdot \{s_5 \ldots s_n\} = \{s_n \ldots s_5s_4s_3s_2s_1\} + \{s_4s_3s_2s_1s_n \ldots s_5\} \\ + \{s_1s_2s_3s_4s_n \ldots s_5\} + \{s_n \ldots s_5s_1s_2s_3s_4\}.$$

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Finally, relative to the ideal U of S we have, using our induction hypothesis,

$$\{us_2s_3 \ldots s_n\} \equiv (-1)^\sigma\{t_1t_2 \ldots t_n\} \text{ modulo } U, \tag{7}$$

where the t_i are some permutation of $u, s_2, s_3, \ldots, s_n$ and $\sigma = 0$ or 1 depending on whether the permutation is even or odd, respectively.

Case 1. Suppose that n is odd. Let $s_1 = u$ in (5) and get (using the induction hypothesis)

$$\{us_2s_3 \ldots s_n\} \equiv -\{s_nus_2 \ldots s_{n-1}\} \text{ modulo } U. \tag{8}$$

Permuting $u, s_2, s_3, \ldots, s_n$ to $s_n, u, s_2, \ldots, s_{n-1}$ is an even permutation, since n is odd. So by (7) we have

$$\{us_2s_3 \ldots s_n\} \equiv \{s_nus_2 \ldots s_{n-1}\} \text{ modulo } U. \tag{9}$$

Addition of equations (8) and (9) gives $2\{us_2s_3 \ldots s_n\} \in U$ and thus $\{us_2s_3 \ldots s_n\} \in U$.

Case 2. Suppose that n is even. Let $s_1 = u$ in (6) and get

$$\{s_n \ldots s_5s_4s_3s_2u\} + \{s_4s_3s_2us_n \ldots s_5\} \equiv -\{s_n \ldots s_5us_2s_3s_4\} - \{us_2s_3s_4s_n \ldots s_5\} \text{ modulo } U, \tag{10}$$

where we have used the assumption that $\{us_2s_3s_4\} \in U$. Since

$$s_n, \ldots, s_5, s_4, s_3, s_2, u \quad \text{and} \quad s_4, s_3, s_2, u, s_n, \ldots, s_5$$

differ by an even permutation, as do

$$s_n, \ldots, s_5, u, s_2, s_3, s_4 \quad \text{and} \quad u, s_2, s_3, s_4, s_n, \ldots, s_5,$$

we have from (7) and (10)

$$\{us_2s_3 \ldots s_n\} \equiv -\{s_n \ldots s_5us_2s_3s_4\} \text{ modulo } U. \tag{11}$$

If $u, s_2, s_3, \ldots, s_n$ and $s_n, \ldots, s_5, u, s_2, s_3, s_4$ differ by an even permutation, which will be the case if 4 divides n, then (7) and (11) imply that $\{us_2s_3 \ldots s_n\} \in U$. If 4 does not divide n, then $u, s_2, s_3, \ldots, s_n$ and $s_n, s_{n-1}, \ldots, s_1$ differ by an odd permutation and so (7) says

$$\{us_2s_3 \ldots s_n\} \equiv -\{s_n \ldots s_3s_2u\} \text{ modulo } U. \tag{12}$$

On the other hand, we always have

$$\{us_2s_3 \ldots s_n\} = \{s_n \ldots s_3s_2u\}. \tag{13}$$

Comparing (12) and (13) gives $\{us_2s_3 \ldots s_n\} \in U$, completing the proof of Theorem 1.

Let $[S, S]$ denote the additive subgroup of K generated by

$$\{s_is_j - s_js_i | s_i, s_j \in S\}$$

Using this notation we have the following corollary.

COROLLARY 1. *If $R = S + K$ such that $[S, S] = K$, then every Jordan ideal U of S extends to a *-ideal of R.*

Proof. We are assuming that S generates R in a special way. For $s_1, s_2 \in S$, $u \in U$ we have

$$(s_1s_2 - s_2s_1)u - u(s_1s_2 - s_2s_1) = [(s_2u + us_2)s_1 + s_1(s_2u + us_2)] \\ -[s_2(s_1u + us_1) + (s_1u + us_1)s_2],$$

and hence $(s_1s_2 - s_2s_1)u - u(s_1s_2 - s_2s_1) \in U$. Since $[S, S] = K$, every element of K is a sum of elements of the form $s_1s_2 - s_2s_1$. Thus, $[K, U] \subset U$. Since U is a Jordan ideal, we have $S \cdot U \subset U$. This shows that $ru + ur^* \in U$ for every $r \in R$. Hence, $u(s_1s_2s_3) + (s_1s_2s_3)^*u = \{us_1s_2s_3\} \in U$. Now we apply Theorem 1.

COROLLARY 2. *Let $R = S + K$ such that S generates R. If U is a Jordan ideal of S having the property that $U^{\cdot 2} = U$, then U extends to a *-ideal of $R = S + K$.*

Proof. For every $u \in U$, $k \in K$ we have $u^2k - ku^2 \in U$ since $u^2k - ku^2 = (uk - ku)u + u(uk - ku)$. Also, $u^2s + su^2 \in U$. This means that $ru^2 + u^2r^* \in U$ for every $r \in R$, $u \in U$. Linearization gives $r(u_1u_2 + u_2u_1) + (u_1u_2 + u_2u_1)r^* \in U$. Since $U^{\cdot 2} = U$, we have $ru + ur^* \in U$, so Theorem 1 applies.

COROLLARY 3. *If $R = S + K$ is generated by two symmetric elements, then every Jordan ideal U of S extends to an invariant associative ideal of R.*

Proof. Choose $u \in U$, $s_1, s_2, s_3 \in S$. If $\{us_1s_2s_3\} \in U$, then the same is true of any tetrad obtained from a permutation of u, s_1, s_2, s_3 and conversely, as seen in the proof of Theorem 1. Suppose that $s_3 = x_1x_2 + x_2x_1$ where $x_1, x_2 \in S$. Then

$$\{us_1s_2(x_1x_2 + x_2x_1)\} = \{\{us_1s_2x_1\}x_2\} + \{\{us_1s_2x_2\}x_1\} - \{x_1us_1s_2x_2\} \\ -\{x_1s_2s_1ux_2\} + \{\{us_1s_2x_1\}x_2\} + \{\{us_1s_2x_2\}x_1\} - \{x_1\{us_1s_2\}x_2\}.$$

This shows, since $\{x_1\{us_1s_2\}x_2\} \in U$, that $\{us_1s_2s_3\} \in U$ if both $\{us_1s_2x_1\}$ and $\{us_1s_2x_2\}$ are in U.

Now let v and w be two symmetric generators of R. It is known [**1**, pp. 305–306] that v and w generate S solely by the Jordan product. Thus, by the above argument, $\{us_1s_2s_3\} \in U$ if $\{ut_1t_2t_3\} \in U$ for $t_i = v$ or w, $i = 1, 2, 3$. Since a duplication of either v or w must occur, it is easy to check that $\{ut_1t_2t_3\} \in U$.

Corollary 3 fails for more than two symmetric generators. For, let $R = F[x_1, x_2, x_3]$, the free algebra over a field F generated by three independent elements x_1, x_2, x_3. Let * be the involution on R which reverses the order of the generators; for example, $(x_1x_2 + x_3x_2x_1)^* = x_2x_1 + x_1x_2x_3$. Let U be the Jordan ideal of S in R generated by $x_1x_2 + x_2x_1$. Then it has been shown [**1**, pp. 307–308] that $\{(x_1x_2 + x_2x_1)x_1x_2x_3\} \notin U$. So U does not extend to a *-ideal of R.

For an easy example of a Jordan ideal which does not extend, let R be an algebra over F generated by x_1, x_2, x_3, x_4 such that $x_ix_j + x_jx_i = 0$ if $i \neq j$. Let the involution in R be the one that reverses the order of the generators, as before. Let U be the Jordan ideal of S generated by x_1, x_2, x_3, x_4. It is clear, since $x_ix_j + x_jx_i = 0$ if $i \neq j$, that $\{x_1x_2x_3x_4\} \notin U$, so U does not extend.

THEOREM 2. *Let $R = S + K$ be a ring with involution *. Let U be the maximal nilpotent ideal of S. Then U extends to the maximal nilpotent ideal I of R.*

Proof. A Zorn's lemma argument applied to the set of all nilpotent ideals of S proves the existence of a maximal nilpotent ideal U. Since the sum of two nilpotent Jordan ideals is another nilpotent Jordan ideal, U must be unique. Similarly, we can show the existence of a unique nilpotent ideal I of R which must necessarily be a *-ideal of R. Hence, $I = U_1 + L$ where U_1 is a nilpotent ideal of S. We must have $U_1 \subseteq U$ due to the maximality of U. To show that $U \subseteq U_1$, we adapt an argument by Herstein [**2**, p. 633]. Consider R/I, the associative ring having an involution induced by *. R/I has no non-zero nilpotent ideals. For every $\bar{r} \in \bar{R} = U/I$, $\bar{u} \in \bar{U}$ we have $(\bar{u}^2)\bar{r} + (\bar{r}^*)(\bar{u}^2) \in \bar{U}$, the image of U, as seen in the proof of Corollary 2. So if n is the exponent of nilpotency of $\bar{U}$, then $[(\bar{u}^2)\bar{r} + (\bar{r}^*)(\bar{u}^2)]^n = \bar{0}$. Let $\bar{u}$ have exponent m. If $m > 2$, then there is an even integer t such that $\bar{u}^t \neq \bar{0}$ but $(\bar{u}^t)^2 = \bar{0}$. We have $[(\bar{u}^{t/2})^2(\bar{r}) + (\bar{r}^*)(\bar{u}^{t/2})^2] \in U$ and hence $[(\bar{u}^{t/2})^2(\bar{r}) + (\bar{r}^*)(\bar{u}^{t/2})^2]^n = \bar{0}$. So $\bar{r}[(\bar{u}^{t/2})^2\bar{r} + (\bar{r}^*)(\bar{u}^{t/2})^2]^n(\bar{u})^t = \bar{0}$, which means that $\bar{r}[(\bar{u}^t)\bar{r}]^n(\bar{u}^t) = \bar{0}$. Therefore, $[\bar{r}(\bar{u}^t)]^{n+1} = \bar{0}$ and the left ideal of $\bar{R}$ generated by $\bar{u}^t$, $\bar{R}\bar{u}^t$, is nilpotent. It is well-known that the sum of all nilpotent left ideals of $\bar{R}$ is a nilpotent two-sided ideal, which is a contradiction, unless $\bar{u}^2 = 0$. We may therefore assume that $\bar{u}^2 = 0$ for every $\bar{u} \in \bar{U}$. Since $\bar{u}_1\bar{u}_2 + \bar{u}_2\bar{u}_1 = (\bar{u}_1 + \bar{u}_2)^2 - \bar{u}_1{}^2 - \bar{u}_2{}^2 = \bar{0}$, we have $\bar{U}^2 = \{\bar{0}\}$. If $\bar{u} \in \bar{U}$, $\bar{s} \in S$ then $\bar{u}\bar{s}\bar{u} = \bar{0}$ since $\bar{u}(\bar{s}\bar{u} + \bar{u}\bar{s}) + (\bar{s}\bar{u} + \bar{u}\bar{s})\bar{u} = \bar{0}$. Hence, if $\bar{r} = \bar{s} + \bar{k}$ then $\bar{u}\bar{r}\bar{u}\bar{r}\bar{u} = \bar{u}(\bar{s} + \bar{k})\bar{u}(\bar{s} + \bar{k})\bar{u} = \bar{u}(\bar{k}\bar{u}\bar{k})\bar{u} = \bar{0}$. So $\bar{R}\bar{u}$ is a left ideal of $\bar{R}$ in which every element cubes to $\bar{0}$. Again, this leads to a nilpotent associative ideal of $\bar{R}$. This shows that $\bar{U} = \{\bar{0}\}$ and $U \subseteq U_1$. So U extends to I.

COROLLARY 1. *If $R = S + K$ is an associative ring with involution * such that S is nilpotent, then R is nilpotent.*

Proof. By Theorem 2, S extends to the maximal nilpotent *-ideal I of R. If $R \neq I$, consider R/I. R/I contains no nilpotent ideals, since I is maximal. On the other hand, R/I has an involution induced by * and the only symmetric element is $\bar{0}$. This means that R/I contains only skew elements which must square to $\bar{0}$; i.e., R/I is nil. Moreover, $\bar{k}_1\bar{k}_2 + \bar{k}_2\bar{k}_1 = \bar{0}$ and thus $\bar{k}_1\bar{k}_2\bar{k}_1 = \bar{0}$ for every $\bar{k}_1, \bar{k}_2 \in \bar{R} = R/I$. Since $\{\bar{k}_1\bar{k}_2\bar{k}_3\} = (\bar{k}_1 + \bar{k}_3)\bar{k}_2(\bar{k}_1 + \bar{k}_3) - \bar{k}_1\bar{k}_2\bar{k}_1 - \bar{k}_3\bar{k}_2\bar{k}_3$, we have

$$\{\bar{k}_1\bar{k}_2\bar{k}_3\} = \bar{0}. \tag{14}$$

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Also, $\bar{k}_1\bar{k}_2\bar{k}_3 - \bar{k}_3\bar{k}_2\bar{k}_1$ is symmetric and so

$$\bar{k}_1\bar{k}_2\bar{k}_3 - \bar{k}_3\bar{k}_2\bar{k}_1 = 0. \tag{15}$$

Adding (14) and (15) shows that $\bar{k}_1\bar{k}_2\bar{k}_3 = \bar{0}$ for every $\bar{k}_1, \bar{k}_2, \bar{k}_3 \in \bar{R}$. Hence, $\bar{R}$ is nilpotent, which is a contradiction. So $\bar{R} = \{\bar{0}\}$ and R is nilpotent.

COROLLARY 2. *Let $R = S + K$ have a nil Jacobson radical N. Then the maximal nil ideal U of S extends to N.*

Proof. N is a *-ideal of R, so $N = U_1 + L$ and the maximality of U implies that $U_1 \subseteq U$. We let $\bar{R} = R/I$ and let $\bar{U}$ be the image of U in $\bar{R}$. If $\bar{U} \neq \{0\}$, the proof of Theorem 2 shows that either $\bar{U}$ is nilpotent or else there exists a $\bar{u} \in \bar{U}$ and an even integer t such that $\bar{u}^t \neq \bar{0}$ and the left ideal $\bar{R}\bar{u}^t$ is nil. In either case, we are led to a contradiction of the fact that $\bar{R}$ has zero Jacobson radical. Hence, $\bar{U} = \{\bar{0}\}$ and $U \subseteq U_1$. So $U = U_1$ and U extends to N.

Extending Jordan homomorphisms of S. Let Φ be a Jordan homomorphism of S. In other words, Φ is a mapping of S such that

$$\Phi(s_1 + s_2) = \Phi(s_1) + \Phi(s_2),$$
$$\Phi(s_1s_2 + s_2s_1) = \Phi(s_1)\Phi(s_2) + \Phi(s_2)\Phi(s_1).$$

Let R' be an associative ring generated by $\{\Phi(s)|s \in S\}$. We seek conditions on R' and Φ which will insure an extension of Φ to an associative homomorphism of $R = S + K$ onto R'. We note that if the elements of S generate R associatively, and if Φ extends to an associative homomorphism of R onto R', then this extension is unique.

THEOREM 3. *Let $R = S + K$ be a ring with involution such that the elements of S generate R. Then any Jordan homomorphism Φ of S into an associative ring R' generated by $\{\Phi(s) = s'\}$ can be extended to a unique associative homomorphism of R onto R' if:*

(i) $\{s_1s_2s_3s_4\}' = \{s_1's_2's_3's_4'\}$, *the tetrad identity; and*

(ii) *R' contains no nilpotent central elements.*

Proof. If $r \in R$, then since S generates R, we have $r = \sum_i s_{1i}s_{2i} \ldots s_{ni}$. If Φ extends, we must have $\Phi(r) = \sum_i s_{1i}'s_{2i}' \ldots s_{ni}'$. It suffices to prove that this extension is well-defined; in other words, we will show that $\sum_j\prod_i s_{ij} = 0$ implies that $\sum_j\prod_i s_{ij}' = 0'$ if conditions (i) and (ii) are satisfied.

Suppose that $s_1s_2 \ldots s_n$ is the longest term in a given expression $\sum_j\prod_i s_{ij} = 0$ and assume that $n > 4$. Because $s_1s_2 \ldots s_n = (s_1s_2 + s_2s_1)s_3 \ldots s_n - s_2s_1s_3 \ldots s_n$, we can change $s_1s_2 \ldots s_n$ into $-s_2s_1s_3 \ldots s_n$ plus a term of smaller length, since $s_1s_2 + s_2s_1 \in S$, without disturbing the value of $\sum_j\prod_i s_{ij}$ under Φ. Similarly, we may then change $-s_2s_1s_3 \ldots s_n$ into $s_2s_3s_1s_4 \ldots s_n$ plus another term of smaller length without changing the value of $\sum_j\prod_i s_{ij}$ under Φ. Continuing in this fashion, we ultimately change $s_1s_2s_3s_4 \ldots s_n$ into $s_4s_3s_2s_1 \ldots s_n$ plus many terms of smaller length. We do this with every term in $\sum_j\prod_i s_{ij}$ of maximal length n. Adding the original expression $(\sum_j \prod_i s_{ij})$ and the resulting expression, we obtain an expression for 0 having terms of the form $\{s_1s_2s_3s_4\}s_5 \ldots s_n$ as well as other terms of length less then n. Since $\{s_1s_2s_3s_4\}' = \{s_1's_2's_3's_4'\}$ the new expression of terms of length less then n will have the same value under Φ as $\sum_j \prod_i s_{ij}$ does. This shows that we may assume that $\sum_j \prod_i s_{ij}$ is an expression of terms of length less than or equal to 3. Since $R = S + K$, we may also assume that $\sum_j \prod_i s_{ij}$ is either skew-symmetric or symmetric.

Suppose that $\sum_j \prod_i s_{ij}$ is symmetric. Then $(s_1s_2 + s_2s_1)' = s_1's_2' + s_2's_1'$ and $(s_1s_2s_3 + s_3s_2s_1)' = (s_1's_2's_3' + s_3's_2's_1')$. (The latter is true since $ss_1s = [s(ss_1 + s_1s) + (ss_1 + s_1s)s] - [s^2s_1 + s_1s^2] \Rightarrow (ss_1s)' = s's_1's'$, and

$$(s_1 + s_3)s_2(s_1 + s_3) = s_1s_2s_1 + s_3s_2s_3 + \{s_1s_2s_3\} \Rightarrow \{s_1s_2s_3\}' = \{s_1's_2's_3'\}.)$$

This shows that Φ is well defined on S.

Suppose that $\sum_j \prod_i s_{ij}$ is skew-symmetric. For any $s \in S$ we have

$$0' = [s(\textstyle\sum_j \prod_i s_{ij}) - (\sum_j \prod_i s_{ij})s]' = s'(\sum_j \prod_i s_{ij}') - (\sum_j \prod_i s_{ij}')s',$$

since $s(\sum_j \prod_i s_{ij}) - (\sum_j \prod_i s_{ij})s \in S$. This shows that $\sum_j \prod_i s_{ij}'$ belongs to the centre of R'. Also, since $(\sum_j \prod_i s_{ij})^2 \in S$, we have $0' = [(\sum_j \prod_i s_{ij})^2]' = (\sum_j \prod_i s_{ij}')^2$, and so $\sum_j \prod_i s_{ij}'$ is nilpotent. But by assumption, R' contains no non-zero nilpotent central elements, so $\sum_j \prod_i s_{ij}' = 0'$ and Φ is well defined on K.

COROLLARY 1. *Suppose that the symmetric elements of $R = S + K$ generate R. Let I be a Jordan homomorphism of S into an associative ring R' generated by $\{\Phi(s) = s'\}$. Furthermore, assume that Φ satisfies the tetrad identity. Then Φ extends uniquely to an associative homomorphism of R onto a homomorphic image of R'.*

Proof. As in the proof of Theorem 3, we first try to extend Φ to a homomorphism of R onto R' by defining $\Phi(\sum_j \prod_i s_{ij}) = \sum_j \prod_i s_{ij}'$. We see in the proof of Theorem 3 that the tetrad identity implies that if $\sum_j \prod_i s_{ij}$ is symmetric, then $\sum_j \prod_i s_{ij} = 0$ means that $\sum_j \prod_i s_{ij}' = 0'$. On the other hand, if $\sum_j \prod_i s_{ij}$ is skew symmetric, then $\sum_j \prod_i s_{ij} = 0$ means that $\sum_j \prod_i s_{ij}'$ is a central nilpotent element a' of R'. Let H' be the ideal of R' generated by the set of all such $a' \in R'$. Then R'/H' is generated by the equivalence classes $s' + H'$, and considering Φ as a Jordan homomorphism of S into R'/H', we have that Φ extends to a Jordan homomorphism of R onto R'/H'.

For an example to illustrate Theorem 3 and its Corollary 1, we use one that is given in [**4**, p. 483]. Let $R = F[x, y]$ be the polynomial algebra over the field F in two commuting indeterminants. R is a ring with involution using the identity involution, so $S = R$. Let $R' = F[X, Y, Z]$ be the algebra over F generated by X and Y subject to the relations $Z = XY - YX$, $Z^2 = 0$,

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$XZ = ZX$, $YZ = ZY$. It is shown in [**4**] that the linear mapping Φ that sends $x^k y^l$ into $\frac{1}{2}(X^k Y^l + Y^l X^k)$ is a Jordan homomorphism of R into R' which is not an associative homomorphism. Note that Z is a central nilpotent element of R'. Let H' be the ideal of R' generated by Z. Then R'/H' is isomorphic with R, and Φ becomes as associative isomorphism of R onto R'/H'.

For another example of a Jordan homomorphism which does not extend, let R be the algebra over the field F generated by s_1, s_2, s_2, s_4 subject to the relations $s_i s_j + s_j s_i = 0$ if $i \neq j$ and $s_i^2 = \alpha_i 1 \neq 0$, $\alpha_i \in F$. R is a Clifford algebra, 16 dimensional over F, with a basis consisting of the 16 elements of the form $s_1^{\beta_1} s_2^{\beta_2} s_3^{\beta_3} s_4^{\beta_4}$ where each β_i equals 0 or 1. The involution in R reverses the order of the s_i's. Hence, S has a basis consisting of 1, $s_1, s_2, s_3, s_4, \{s_1 s_2 s_3 s_4\}$. Let $R' = R$ and define Φ on the basis elements of S by $\Phi(1) = 1$, $\Phi(s_i) = s_i$, $i = 1, 2, 3, 4$ and $\Phi(\{s_1 s_2 s_3 s_4\}) = -\{s_1 s_2 s_3 s_4\}$. We extend Φ linearly to all of S and check that Φ is a Jordan automorphism of S. It is clear that Φ cannot extend to an automorphism of R since

$$\Phi(\{s_1 s_2 s_3 s_4\}) = -\{s_1 s_2 s_3 s_4\} \neq \{s_1 s_2 s_3 s_4\},$$

violating the tetrad identity. We note that associatively we have $\Phi(s_i s_j) = \Phi(s_i)\Phi(s_j)$ and $\Phi(s_1 s_j s_k) = \Phi(s_i)\Phi(s_j)\Phi(s_k)$ which means that Φ may be extended uniquely to the subspace of R spanned by at most three of the generators s_1, s_2, s_3, s_4. We may extend this example by letting R be the associative algebra over F generated by $s_1, s_2, s_3, \ldots, s_n$ where $n \equiv 1$ modulo 4, subject to the following conditions:

(i) $s_i s_j + s_j s_i = 0$, if $i \neq j$;

(ii) $s_{2i}^2 = -1$, $s^2_{2i+1} = 1$;

(iii) if s_i, s_j, s_k, s_l are four distinct generators such that $i < j < k < l$ then $s_i s_j s_k s_l$ equals the product of all the other generators in order; that is, if $m < q$ then s_m precedes s_q. For example, if $n = 9$ then

$$s_1 s_2 s_3 s_4 = s_5 s_6 s_7 s_8 s_9, \quad s_1 s_3 s_4 s_6 = s_2 s_5 s_7 s_8 s_9, \text{ etc.}$$

We let R' be the Clifford algebra over F generated by $s_1', \ldots, s_n'$ where n is the same as above. So we have $s_i' s_j' + s_j' s_i' = 0'$ for $i \neq j$ and let $(s_{2i}')^2 = -1$, $(s'_{2i+1})^2 = 1$. The involutions in R and R' reverse the orders of the generators. Since S is generated by the Jordan products of $s_1, s_2, \ldots, s_n$ and all their tetrads (see [**1**]), it suffices to define a Jordan homomorphism $\Phi: S \to R'$ by $\Phi(s_i) = s_i'$ for $i = 1, 2, \ldots, n$ and if

$$\{s_i s_j s_k s_l\} = \prod_{i=1}^{n-4} s_{m_i}$$

where $i < j < k < l$ and $m_1 < m_2 < \ldots < m_{n-4}$, then

$$\Phi(\{s_i s_j s_k s_l\}) = \prod_{i=1}^{n-4} s_{m_i}'.$$

A check will show that Φ is a Jordan homomorphism of S into R' which does not extend. Once more the tetrad identity is violated.

Finally, we give another corollary of Theorem 3 similar to Corollary 3 of Theorem 1, and since the proofs are similar we omit the proof here.

COROLLARY 2. *If* $R = S + K$ *is generated by three symmetric elements, then any Jordan homomorphism* Φ *of* S *into* R' *satisfies the tetrad identity. Hence,* Φ *extends to a homomorphism of* R *onto perhaps a homomorphic image of* R'.

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