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PURELY INFINITE SIMPLE C^*-CROSSED PRODUCTS II

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ABSTRACT. We study the pure infiniteness of C^*-crossed products by endomorphisms and automorphisms. Let A be a purely infinite simple unital C^*-algebra. At first we show that a crossed product $A \times_\rho N$ by a corner endomorphism ρ is purely infinite if it is simple. From this observation we prove that any simple C^*-crossed product $A \times_\alpha Z$ by an automorphism α is purely infinite. Combining this with the result in [Je] on pure infiniteness of crossed products by finite groups, one sees that if α is an outer action by a countable abelian group G then the simple C^*-algebra $A \times_\alpha G$ is purely infinite.

1. **Introduction.** One of the most important problems concerning the structure of simple unital C^*-algebras is whether there is an example of a simple unital C^*-algebra which does not belong to the following classes: the class of stably finite simple unital C^*-algebras and the class of purely infinite simple unital C^*-algebras. In the context of general C^*-algebras Clarke [Cl] and Blackadar [Bl] independently showed the existence of finite unital C^*-algebras which are not stably finite. However there is no known example of a simple C^*-algebra that does not belong to the classes described above.

A projection p in A is said to be *infinite* if there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* < p$. A C^*-algebra A is said to be *purely infinite* if for each non-zero positive element a in A, $\overline{aAa}$ has an infinite projection. If A is simple, this is equivalent to say that for each non-zero positive element a in $A, xax^* = 1$ for some $x \in A$ by [Cu2] and [Cu3]. Note that if a simple C^*-algebra A has a purely infinite hereditary C^*-subalgebra B, then A itself is purely infinite. In fact, for any hereditary C^*-subalgebra C in A, there exists a unitary u in the multiplier algebra $M(A)$ of A such that $uBu^* \cap C \neq 0$ [Rø1, Lemma 3.4]. Cuntz algebras O_n $(2 \leq n \leq \infty)$ are typical examples of purely infinite simple separable unital C^*-algebras [Cu2].

Kishimoto [Ki2] showed that the reduced C^*-crossed product $A \times_\alpha G$ of a simple C^*-algebra by an outer action α of a discrete group G is always simple. If A is purely infinite simple then $A \times_\alpha G$ is obviously infinite simple since it contains A as a C^*-subalgebra. It was shown [Je] that this crossed product is actually purely infinite if G is finite.

In this note, we investigate the pure infiniteness of crossed products by endomorphisms and automorphisms to show the following:

(i) If ρ is a proper corner endomorphism on a purely infinite simple unital C^*-algebra A then $A \times_\rho \mathbf{N}$ is purely infinite if it is simple.

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(ii) If α is an automorphism on a purely infinite simple unital (or stable) C^*-algebra A then $A \times_\alpha \mathbf{Z}$ is purely infinite if it is simple.

Recall that a unital C^*-algebra A is said to have *the comparability property* if it has at least one normalized trace and if for every pair e, f of projections in A for which $\tau(e) < \tau(f)$ for each normalized trace τ on A it follows that e is equivalent to a subprojection of f.

Rørdam proved in [Rø3] that for any simple unital C^*-algebra A of real rank zero with the comparability property and a proper corner endomorphism ρ on A, $A \times_\rho \mathbf{N}$ is simple and purely infinite. The Proof of (i) follows the ideas in the Proof of [Rø3, Theorem 3.1].

Combining (ii) and the result in [Je] we see that if α is an outer action of a countable discrete abelian group G on a purely infinite simple unital C^*-algebra A, then $A \times_\alpha G$ is also purely infinite.

2. **A crossed product $A \times_\rho \mathbf{N}$ by an endomorphism.** An endomorphism ρ on a unital C^*-algebra A is called *a corner endomorphism* if its image is equal to the corner $\rho(1)A\rho(1)$ of A [Rø3]. We call ρ a *proper* corner endomorphism if $\rho(1) \neq 1$.

We suppose that a C^*-algebra A acts faithfully on a Hilbert space H. An isometry s on H is said to *implement* ρ if $\rho(a) = sas^*$ for any $a \in A$. The crossed product $A \times_\rho \mathbf{N}$ is defined to be the universal C^*-algebra generated by A and the isometry s which implements ρ. From this definition we know that the $*$-algebra generated by the following set

$$\{(s^*)^n a_{-n} + \cdots + s^* a_{-1} + a_0 + a_1 s + \cdots + a_n s^n \mid a_{-n}, \ldots, a_n \in A, n \in \mathbf{N}\}$$

is dense in $A \times_\rho \mathbf{N}$. Moreover the universal property of $A \times_\rho \mathbf{N}$ yields a circle(dual) action $\{\hat{\rho}_\lambda\}_{\lambda \in \mathbf{T}}$ given by

$$\hat{\rho}_\lambda(a) = a, \quad a \in A$$
$$\hat{\rho}_\lambda(s) = \lambda s, \quad \lambda \in \mathbf{T},$$

so we can define the conditional expectation $E: A \times_\rho \mathbf{N} \longrightarrow A$ by

$$E(x) = \int_{\mathbf{T}} \hat{\rho}_\lambda(x)\, dm(\lambda),$$

where m is the normalized Lebesgue measure on the circle $\mathbf{T}$. The Cuntz algebra O_n is a typical example of this crossed product $A \times_\rho \mathbf{N}$ with A a UHF-algebra of type n^∞ if n is finite and A an AF-algebra if n is infinite. If ρ is an automorphism of a unital C^*-algebra A then $A \times_\rho \mathbf{N}$ is a usual C^*-crossed product $A \times_\rho \mathbf{Z}$. We can extend Kishimoto's *strong Connes spectrum* [Ki1, Ki2] for automorphisms to endomorphims, that is,

$$\tilde{\mathbf{T}}(\rho) = \{\lambda \in \mathbf{T} \mid \hat{\rho}_\lambda(I) = I \text{ for all } I \in \mathrm{Prim}(A \times_\rho \mathbf{N})\},$$

where $\mathrm{Prim}(A \times_\rho \mathbf{N})$ denotes the primitive ideal space of $A \times_\rho \mathbf{N}$. It is easy to see that $\tilde{\mathbf{T}}(\rho)$ is a subgroup of $\mathbf{T}$.

THEOREM 2.1. *Let A be a purely infinite simple unital C^*-algebra and ρ a proper corner endomorphism implemented by an isometry s. Then the following are equivalent:*

(i) $\tilde{\mathbf{T}}(\rho^n) = \mathbf{T}$ *for* $n \in \mathbf{N}$,

(ii) For any $n \in \mathbf{N}$ and $a \in A$, we have

$$\begin{aligned}\inf\{\|xa\rho^n(x)\| \mid x \in A,\, x \geq 0, \text{ and } \|x\| = 1\}\\ = \inf\{\|ea\rho^n(e)\| \mid e \text{ is a non zero projection in } A\}\\ = 0,\end{aligned}$$

(iii) $A \times_\rho \mathbf{N}$ *is simple and purely infinite,*

(iv) $A \times_\rho \mathbf{N}$ *is simple.*

(v) $\rho^n \neq \mathrm{Ad}\, v$, *for any isometry $v \in A$ and $n \in \mathbf{N}$.*

That (i) implies (ii) is proved in the following proposition by modifying the Proof of [Ki2, Lemma 1.1] slightly.

PROPOSITION 2.2 ((i) $\Rightarrow$ (ii)). *Let ρ be an endomorphism of a unital C^*-algebra A with $\tilde{\mathbf{T}}(\rho) \neq \{1\}$. Then for any $a \in A$ we have*

$$\inf\{\|xa\rho(x)\| \mid x \in A,\, x \geq 0, \text{ and } \|x\| = 1\} = 0.$$

Moreover, for a C^*-algebra of real rank zero the infimum taken over non zero projections in A is zero.

PROOF. Suppose that there is an element $a \in A$ such that the infimum is $\delta > 0$. Let φ be a pure state of $B = \rho(A)$, and let $(\pi_\varphi, \Omega_\varphi)$ be the GNS-representation by φ. By the Proof of [Ki2, Lemma 1.1] we can find a projection p in A^{**} such that $\bar{\pi}_\varphi(p)$ is the one dimensional projection onto $\mathbf{C}\Omega_\varphi$ and $\bar{\pi}_\varphi\big(\bar{\rho}(p)\big)$ is non-zero and so one dimensional, where $\bar{\pi}_\varphi(\bar{\rho}$, respectively) is an extension map of $\pi_\varphi(\rho$, respectively) onto A^{**}.

Let $\xi \in H_\varphi$ be a unit vector such that

$$\bar{\pi}_\varphi\big(\bar{\rho}(p)\big)\xi = \xi.$$

Define an operator V on H_φ by

$$V\pi_\varphi(x)\Omega_\varphi = \pi_\varphi \circ \rho(x)\xi, \qquad x \in A.$$

Since

$$\begin{aligned}\|\pi_\varphi \circ \rho(x)\xi\|^2 &= \big(\bar{\pi}_\varphi \circ \bar{\rho}(px^*xp)\xi, \xi\big)\\ &= \varphi(x^*x)\\ &= \|\pi_\varphi(x)\Omega_\varphi\|^2\end{aligned}$$

by $px^*xp = \varphi(x^*x)p$, and since π_φ is irreducible, V has an extension to an isometry on H_φ, which is denoted by V again. We get a contradiction from the same argument in the Proof of [Ki2, Lemma 1.1] (V was a unitary there). ■

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LEMMA 2.3. *Let A and ρ satisfy (ii) in the Theorem 2.1 with the same assumptions. If $x \in A \times_\rho \mathbf{N}$ is an element such that $E(x) = 1$ where $E: A \times_\rho \mathbf{N} \longrightarrow A$ is the conditional expectation, then for any $\varepsilon > 0$ there is an isometry $v \in A \times_\rho \mathbf{N}$ such that $\|v^*xv - 1\| < \varepsilon$.*

PROOF. The Proof of [Rø3, Lemma 3.4] works under our assumptions since [Rø3, Lemma 3.2] is true (with $m = 1$) for a purely infinite simple C^*-algebra. ∎

PROOF OF (ii) ⇒ (iii). In a similar way as in [Rø3] we prove that for any positive element $x \in A \times_\rho \mathbf{N}$ there is an element $z \in A \times_\rho \mathbf{N}$ such that $zxz^* = 1$.

Let x be a non-zero positive element in $A \times_\rho \mathbf{N}$. It suffices to find an element $z \in A \times_\rho \mathbf{N}$ so that $E(z^*xz) = 1$. In fact if we can find such an element $z \in A \times_\rho \mathbf{N}$, by Lemma 2.3 there is an isometry $v \in A \times_\rho \mathbf{N}$ such that $\|v^*z^*xzv - 1\| < 1$. Hence since v^*z^*xzv is invertible, there is a positive element $y \in A \times_\rho \mathbf{N}$ such that $yv^*z^*xzvy = 1$. Thus we can see that $A \times_\rho \mathbf{N}$ is simple and purely infinite.

The element $a_0 = E(x)$ is a non-zero positive element in A. Let $0 < \varepsilon < \|a_0\|$ and set $f(t) = \max\{t - \varepsilon, 0\}$ ($t \in \mathbf{R}^+$). Then

$$\{0\} \neq \overline{f(a_0)Af(a_0)} \subset a_0^{\frac{1}{2}} A a_0^{\frac{1}{2}}$$

and $\overline{f(a_0)Af(a_0)}$ contains a non-zero projection p since A is purely infinite. Write $p = a_0^{\frac{1}{2}} y a_0^{\frac{1}{2}}$ with y a positive element in A. Put $e = y^{\frac{1}{2}} a_0 y^{\frac{1}{2}}$. Then e is a non-zero projection in A. Since A is a purely infinite simple unital C^*-algebra, there is a partial isometry $u \in A$ such that $u^*u = ss^*$ and $uu^* \leq e$. Then $s^*u^*eus = 1$. With $z = y^{\frac{1}{2}}us$ we have

$$E(z^*xz) = s^*u^*y^{\frac{1}{2}} a_0 y^{\frac{1}{2}} us = s^*u^*eus = 1.$$

∎

To prove (iv) ⇒ (i), we need the following well known proposition.

PROPOSITION 2.4. *Let α be an automorphism of a simple C^*-algebra A. Then the following are equivalent:*

(i) $A \times_\alpha \mathbf{Z}$ *is simple,*
(ii) $\tilde{\mathbf{T}}(\alpha) = \mathbf{T}$,
(iii) $\tilde{\mathbf{T}}(\alpha^n) \neq \{1\}$ *for* $n \in \mathbf{Z} \setminus \{0\}$.

(It is known (see [Ki2]) that if A is simple then $\tilde{\mathbf{T}}(\alpha) \neq \{1\}$ is equivalent to α being outer).

PROOF. See [Pe, 8.11.12] for (i) ⇔ (ii). (iii) ⇒ (i) was proved by Kishimoto in [Ki2].

(i) ⇒ (iii) (See [OP, Lemma 10.1]). Let v be the unitary in the multiplier algebra of the crossed product by which α is implemented. Suppose that α^n is inner and let $u \in M(A)$ be a unitary with $\alpha^n = \operatorname{Ad} u$. We may assume that $n > 0$. Then for each k, $1 \leq k \leq n-1$, $\alpha^k(u)$ implements α^n, so $\alpha^{n^2} = \operatorname{Ad} w$, $w = u\alpha(u)\cdots\alpha^{n-1}(u)$. Since $\alpha^n(u) = u$, we see that $\alpha(w) = w$. It is easily checked that $v^{n^2}w^*$ is in the center of $M(A \times_\alpha \mathbf{Z})$ which is trivial by Dauns-Hofmann theorem. But $v^n w^*$ is not an element in $M(A)$, that is, $v^n w^* \notin \mathbf{C}1$ and we get a contradiction. ∎

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PROOF OF (iv) $\Rightarrow$ (i). There are an embedding φ of A onto a corner of $\bar{A}$, where $\bar{A}$ is the direct limit of the directed system

$$A \xrightarrow{\rho} A \xrightarrow{\rho} A \xrightarrow{\rho} \cdots \longrightarrow \bar{A}$$

and an automorphism α on $\bar{A}$ so that φ extends to an embedding of $A \times_\rho \mathbf{N}$ onto a corner of $\bar{A} \times_\alpha \mathbf{Z}$, and the diagram commutes for $n \in \mathbf{N}$ (see [Pa] and [Rø3, Proposition 2.1]). In fact $A \times_\rho \mathbf{N}$ is isomorphic to the corner $\varphi(1)(\bar{A} \times_\alpha \mathbf{Z})\varphi(1)$ $(= C^*(\varphi(A), \bar{s}))$ of $\bar{A} \times_\alpha \mathbf{Z}$, which is generated by $\varphi(A)$ and $\bar{s} = u\varphi(1)$, where u is the unitary in $M(\bar{A} \times_\alpha \mathbf{Z})$ implementing α.

$$\begin{array}{ccccc} A & \xrightarrow{\rho^n} & A & \xrightarrow{i} & A \times_{\rho^n} \mathbf{N} \\ \varphi \downarrow & & \varphi \downarrow & & \varphi \downarrow \\ \bar{A} & \xrightarrow{\alpha^n} & \bar{A} & \xrightarrow{i} & \bar{A} \times_{\alpha^n} \mathbf{Z}. \end{array}$$

Since $A \times_\rho \mathbf{N}$ is simple we see that $\bar{A} \times_\alpha \mathbf{Z}$ is simple. Then by Proposition 2.4., $\bar{A} \times_{\alpha^n} \mathbf{Z}$ is simple for all $n \in \mathbf{N}$ since $\bar{A}$ is (purely infinite) simple. It follows that $A \times_{\rho^n} \mathbf{N}$ is simple as a corner of $\bar{A} \times_{\alpha^n} \mathbf{Z}$ and $\tilde{\mathbf{T}}(\rho^n) = \mathbf{T}$. ∎

PROOF OF (ii) $\Rightarrow$ (v). Suppose that $\rho^n = \mathrm{Ad}\, v$ for some isometry $v \in A$ and $n \in \mathbf{N}$. Then for any non-zero projection $e \in A$,

$$\|ev^*\rho^n(e)\| = \|ev^*vev^*\| = \|ev^*\| = 1.$$

∎

PROOF OF (v) $\Rightarrow$ (iv). Suppose that $A \times_\rho \mathbf{N}$ is not simple. Then $\bar{A} \times_\alpha \mathbf{Z}$ is not simple and by Proposition 2.4. $\alpha^n = \mathrm{Ad}\, w$ for some unitary $w \in M(\bar{A})$ and $n \neq 0$. Let $u \in M(\bar{A} \times_\alpha \mathbf{Z})$ be the unitary, so that $\alpha(x) = uxu^*$, $x \in \bar{A}$. Since $A \times_\rho \mathbf{N}$ is isomorphic to the corner $\varphi(1)(\bar{A} \times_\alpha \mathbf{Z})\varphi(1)\big(= C^*\big(\varphi(A), \bar{s}\big)\big)$ of $\bar{A} \times_\alpha \mathbf{Z}$ and

$$\varphi\big(\rho^n(a)\big) = \alpha^n\big(\varphi(a)\big) = w\varphi(a)w^*$$

we see that $w^*\varphi\big(\rho^n(1)\big)w = \varphi(1)$ and it follows that $\varphi(1)w^*\varphi(1)w\varphi(1) = \varphi(1)$, that is, $\varphi(1)w\varphi(1)$ is an isometry in $\varphi(1)\bar{A}\varphi(1) = \varphi(A)$. So there exists an isometry $v \in A$ such that $\varphi(v) = \varphi(1)w\varphi(1)$. From $\varphi\big(\rho^n(a)\big) = \varphi(v)\varphi(a)\varphi(v)^*$, $a \in A$, we conclude that $\rho^n(a) = vav^*$, $a \in A$. ∎

3. **A crossed product $A \times_\alpha \mathbf{Z}$ by an automorphism.** Throughout this section, for a projection $p \in A$, A_p denotes the hereditary C^*-subalgebra pAp of A generated by p.

THEOREM 3.1. *Let A be a purely infinite simple unital C^*-algebra and let $\alpha: A \longrightarrow A$ be an automorphism. Then the crossed product $A \times_\alpha \mathbf{Z}$ is purely infinite if it is simple.*

PROOF. We may assume that there is a non-zero projection $p \in A$ with

$$\begin{aligned} s^*s &= \alpha(p), & ss^* &= e < p, \\ t^*t &= 1 - \alpha(p), & tt^* &= 1 - e \end{aligned}$$

for some partial isometries $s, t \in A$ by choosing nonzero projections $p, p < 1$, and $e \neq 0$, $e < p$, $[e] = [p]$ in $K_0(A)$.

It follows that $s = es\alpha(p)$ and $t = (1-e)t\big(1-\alpha(p)\big)$. It is obvious that $u_1 = s + t$ is a unitary. Define an automorphism $\rho: A \longrightarrow A$ by $\rho(a) = u_1\alpha(a)u_1{}^*$, $a \in A$, and define a function $u: \mathbf{Z} \longrightarrow \mathcal{U}(A)$ (the unitary group of A) by

$$u(0) = 1, \quad u(1) = u_1,$$
$$u(m) = u_1\alpha\big(u(m-1)\big) \text{ for } m \geq 2,$$
$$u(-m) = \alpha^{-m}\big(u(m)^*\big) \text{ for } m \geq 1.$$

Then it is easily checked that $\rho^m = (\mathrm{Ad}\, u_m)\alpha^m$, $m \in \mathbf{Z}$ with $u_m = u(m)$. Hence two systems $(A, \mathbf{Z}, \alpha)$ and $(A, \mathbf{Z}, \rho)$ are exterior equivalent [Pe, 8.11.2].

It suffices to show that $A \times_\rho \mathbf{Z}$ is purely infinite. Note that $\rho(p) = u_1\alpha(p)u_1{}^* = e < p$. The restriction $\rho_0 = \rho|_{A_p}$ is a proper corner endomorphism.

Let v be a unitary in $A \times_\rho \mathbf{Z}$ which implements ρ. Set $s_1 = vp$. Then $s_1 \in A \times_\alpha \mathbf{Z}$, $s_1{}^*s_1 = p$, and $s_1s_1{}^* = vpv^* = \rho(p) = e < p$. Moreover, $\rho_0(a) = s_1as_1{}^*$ for $a \in A_p$. It is easily checked that $(A \times_\rho \mathbf{Z})_p = C^*(A_p, s_1)$, the C^*-subalgebra of $A \times_\rho \mathbf{Z}$ generated by A_p and s_1. Since $A \times_\rho \mathbf{Z}$ is simple, $\tilde{\mathbf{T}}(\rho) = \mathbf{T} \neq \{1\}$, hence by [Ki2, Lemma 1.1] for any $\varepsilon > 0$ and $a \in A_p$ there exists a positive element $x \in A_p$, $\|x\| = 1$ with $\|xa\rho_0(x)\| < \varepsilon$. Hence the endomorphism ρ_0 on A_p satisfies (ii) in Theorem 2.1, and $A_p \times_{\rho_0} \mathbf{N}$ is purely infinite and simple, so that $A_p \times_{\rho_0} \mathbf{N}$ is isomorphic to $C^*(A_p, s_1)$. Therefore $A \times_\rho \mathbf{Z}$ has a purely infinite hereditary C^*-subalgebra and is hence purely infinite. ∎

REMARKS 3.2. (1) The above Theorem is true for purely infinite simple stable C^*-algebras. In fact, there exist a unitary $u \in M(A)$ and a projection $p \in A$ with $\mathrm{Ad}\, u \circ \alpha(p) < p$. Set $\rho = \mathrm{Ad}\, u \circ \alpha$, then $A_p \times_{\rho_0} \mathbf{N}$ ($\rho_0 = \rho|_{A_p}$) is isomorphic to a purely infinite simple full hereditary C^*-subalgebra of $A \times_\rho \mathbf{Z} \cong A \times_\alpha \mathbf{Z}$.
(2) From the Proof of Theorem 3.1, we see that every C^*-dynamical system $(A, \mathbf{Z}, \alpha)$ with a unital purely infinite simple C^*-algebra A is exterior equivalent to a dynamical system $(A, \mathbf{Z}, \beta)$ for some automorphism β of A such that there is a projection $p \in A$ with $\beta(p) < p$.

For a discrete product group $G = K \times H$, if α is an action of G on a C^*-algebra A then there is an isomorphism between two reduced crossed products

$$\phi: A \times_{\alpha r} G \longrightarrow (A \times_{\alpha|_K r} K) \times_{\beta r} H$$

where $\alpha|_K$ is the restriction of α to K and $\beta: H \longrightarrow \mathrm{Aut}(A \times_{\alpha|_K r} K)$ is an action defined by $\beta(h)(au_{(k,1)}) = \alpha_{(1,h)}(a)u_{(k,1)}$, $a \in A$ (here $u_{(k,h)}$ is the unitary such that $\alpha_{(k,h)}(a) = u_{(k,h)}au_{(k,h)}{}^*$). Then ϕ is defined by $\phi(au_{(k,h)}) = au_{(k,1)}v_h$ where v_h is the unitary with $\beta(h)(x) = v_hxv_h{}^*$, $x \in A \times_{(\alpha|_K)r} K$, $h \in H$.

COROLLARY 3.3. *Let α be an outer action of a countable abelian group G on a* *purely infinite simple unital C^*-algebra A. Then the crossed product $A \times_\alpha G$ is simple ([Ki2]) and purely infinite.*

PROOF. Assume that G is of the form $G = F \times \mathbf{Z}^n$ for some finite subgroup F. Since $\alpha|_F$ is outer we see from [Je] that $A \times_{\alpha|_F} F$ is purely infinite simple. Applying Theorem 3.1 to $(A \times_{\alpha|_F} F, \alpha_1, \mathbf{Z})$ we obtain a purely infinite simple C^*-algebra $(A \times_{\alpha|_F} F) \times_{\alpha_1} \mathbf{Z}$, where α_1 is an action induced by α for which we have

$$(A \times_{\alpha|_F} F) \times_{\alpha_1} \mathbf{Z} \cong A \times_{\alpha|_{(F\times\mathbf{Z})}} (F \times \mathbf{Z}).$$

In general the simple crossed product $A \times_\alpha G$ is the direct limit of purely infinite simple unital crossed products of the above form and therefore purely infinite. ■

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REFERENCES

[Bl] B. Blackadar, *K-Theory for Operator Algebras*, M. S. R. I. Publication Series 5, Springer-Verlag, (1986).

[BlCu] B. Blackadar and J. Cuntz, *The structure of stable algebraically simple C^*-algebras*, Amer. J. Math. **104**(1981), 813–822.

[BKR] B. Blackadar, A. Kumjian and M. Rørdam, *Approximately central matrix units and the structure of noncommutative tori*, K-Theory, **6**(1992), 267–284.

[BrPe] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131–149.

[Cl] N. Clarke, *A finite but not stably finite C^*-algebra*, Proc. Amer. Math. Soc. **96**(1976), 85–88.

[Cu 1] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57**(1977), 173–185.

[Cu 2] ———, *The structure of multiplication and addition in simple C^*-algebras*, Math. Scand. **40**(1977), 215–233.

[Cu 3] ———, *K-theory for certain C^*-algebras*, Ann. Math. **113**(1981), 181–197.

[Je] J. A. Jeong, *Purely infinite simple C^*-crossed products*, Proc. Amer. Math. Soc.; to appear.

[Ki1] A. Kishmoto, *Simple crossed products of C^*-algebras by locally compact abelian groups*, Yokohama Math. J. **28**(1980), 69–85.

[Ki2] ———, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Comm. Math. Phys. **81**(1981), 429–435.

[Ku] M. Kusuda, *Hereditary C^*-subalgebras of C^*-crossed products*, Proc. Amer. Math. Soc. **102**(1988), 90–94.

[OP] D. Olesen and G. K. Pedersen, *Applications of the Connes spectrum to C^*-dynamical systems, III*, J. Funct. Anal. **45**(1982), 357–390.

[Pa] W. Paschke, *The crossed product of a C^*-algebra by an endomorphism*, Proc. Amer. Math. Soc. **80**(1990), 113–118.

[Pe] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, New York (1990).

[Rø1] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra*, J. Funct. Anal. **100**(1991), 1–17.

[Rø2] ———, *Classification of inductive limits of Cuntz algebras*, J. Reine. Angew. Math. **440**(1993), 175–200.

[Rø3] ———, *Classification of certain infinite simple C^*-algebras*, preprint.

[Zh] S. Zhang, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras. Part I*, Pacific J. Math. **155**(1992), 169–197.

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