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# An inhomogeneous Dirichlet theorem via shrinking targets

Dmitry Kleinbock and Nick Wadleigh

Abstract

We give an integrability criterion on a real-valued non-increasing function $\psi$ guaranteeing that for almost all (or almost no) pairs $(A, \mathbf{b})$, where $A$ is a real $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, the system

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(T), \quad \|\mathbf{q}\|^n < T,$$

is solvable in $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$ for all sufficiently large $T$. The proof consists of a reduction to a shrinking target problem on the space of grids in $\mathbb{R}^{m+n}$. We also comment on the homogeneous counterpart to this problem, whose $m = n = 1$ case was recently solved, but whose general case remains open.

## 1. Introduction and motivation

### 1.1 Homogeneous Diophantine approximation

Fix positive integers $m, n$. Let $M_{m,n}$ denote the space of real $m \times n$ matrices. The starting point for the present paper is the following theorem, proved by Dirichlet in 1842.

Theorem 1.1 (Dirichlet's theorem). *For any $A \in M_{m,n}$ and $T > 1$, there exist $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ such that*

$$\|A\mathbf{q} - \mathbf{p}\|^m \leqslant \frac{1}{T} \quad \textit{and} \quad \|\mathbf{q}\|^n < T. \tag{1.1}$$

Here and hereafter $\|\cdot\|$ stands for the supremum norm on $\mathbb{R}^k$, $k \in \mathbb{N}$. Informally speaking, a matrix $A$ represents a vector-valued function $\mathbf{q} \mapsto A\mathbf{q}$, and the above theorem asserts that one can choose a not-so-large non-zero integer vector $\mathbf{q}$ so that the output of that function is close to an integer vector. In the case $m = n = 1$ the theorem just asserts that for any real number $\alpha$ and $T > 1$, one of the first $T$ multiples of $\alpha$ lies within $1/T$ of an integer. Theorem 1.1 is the archetypal *uniform* Diophantine approximation result, so called because it guarantees a non-trivial integer solution for *all* $T$. A weaker form of approximation (sometimes called *asymptotic* approximation; see, for example, [Wal12, KL18]) guarantees that such a system is solvable for an *unbounded* set of $T$. For instance, Theorem 1.1 implies that (1.1) is solvable for an unbounded set of $T$, *a fortiori*. The following corollary, which follows trivially from this weaker statement, is the archetypal asymptotic result.

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Corollary 1.2. *For any $A \in M_{m,n}$ there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\|A\mathbf{q} - \mathbf{p}\|^m < \frac{1}{\|\mathbf{q}\|^n} \quad \textit{for some } \mathbf{p} \in \mathbb{Z}^m. \tag{1.2}$$

Together the aforementioned results initiate the *metric theory of Diophantine approximation*, a field concerned with understanding sets of $A \in M_{m,n}$ which admit improvements to Theorem 1.1 and Corollary 1.2. This paper has been motivated by an observation that the sensible 'first questions' about the asymptotic set-up were settled long ago, while the analogous questions about the uniform set-up remain open. Let us start by reviewing what is known in the asymptotic set-up.

For a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, let us define $W_{m,n}(\psi)$, the set of $\psi$-*approximable* matrices, to be the set of $A \in M_{m,n}$ for which there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that[1]

$$\|A\mathbf{q} - \mathbf{p}\|^m \leqslant \psi(\|\mathbf{q}\|^n) \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m. \tag{1.3}$$

Throughout the paper we use the notation $\psi_a(x) := x^{-a}$. Thus Corollary 1.2 asserts that $W_{m,n}(\psi_1) = M_{m,n}$, and in the above definition we have simply replaced $\psi_1(\|\mathbf{q}\|^n)$ in (1.2) with $\psi(\|\mathbf{q}\|^n)$. Precise conditions for the Lebesgue measure of $W_{m,n}(\psi)$ to be zero or full are given in the following theorem.

Theorem 1.3 (Khintchine–Groshev theorem [Gro38]). *Given a non-increasing*[2] *$\psi$, the set $W_{m,n}(\psi)$ has zero (respectively, full) measure if and only if the series $\sum_k \psi(k)$ converges (respectively, diverges).*

See [Spr79] or [BDV06] for details, and also [KM99] for an alternative proof using dynamics on the space of lattices.

Questions related to similarly improving Theorem 1.1 were first addressed in two seminal papers [DS70, DS69] by Davenport and Schmidt. However no zero–one law analogous to Theorem 1.3 has yet been proved in the set-up of uniform approximation for general $m, n \in \mathbb{N}$. Let us introduce the following definition: for a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$, where $T_0 > 1$ is fixed, say that $A \in M_{m,n}$ is $\psi$-*Dirichlet*, or $A \in D_{m,n}(\psi)$, if the system

$$\|A\mathbf{q} - \mathbf{p}\|^m < \psi(T) \quad \text{and} \quad \|\mathbf{q}\|^n < T \tag{1.4}$$

has a non-trivial integer solution for all large enough $T$. In other words, we have replaced $\psi_1(T)$ in (1.1) with $\psi(T)$, demanded the existence of non-trivial integer solutions for all $T$ except those belonging to a bounded set, and sharpened one of the inequalities in (1.1). The latter change, in particular, implies the following observation: for non-increasing $\psi$, membership in $D_{m,n}(\psi)$ depends only on the solvability of the system (1.4) at integer values of $T$. (To show this it suffices to replace $T$ with $\lceil T \rceil$ and use the monotonicity of $\psi$.)

It is not difficult to see that $D_{1,1}(\psi_1) = \mathbb{R}$, and that for general $m, n$, almost every matrix is $\psi_1$-Dirichlet. In contrast, it was proved in [DS69] for $\min(m, n) = 1$, and in [KW08] for the general case, that for any $c < 1$, the set $D_{m,n}(c\psi_1)$ of $c\psi_1$-Dirichlet matrices has Lebesgue measure zero. This naturally motivates the following question.

[1] This definition essentially coincides with the one given in [KM99] but differs slightly from other sources, such as [BDV06, § 13], where the inequality $\|A\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|\psi(\|\mathbf{q}\|)$ is used instead of (1.3).

[2] The monotonicity condition can be removed unless $m = n = 1$.



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*Question* 1.4. What is a necessary and sufficient condition on a non-increasing function $\psi$ (presumably expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $D_{m,n}(\psi)$ has zero or full measure?

In [KW18] we give an answer to this question for $m = n = 1$, but in general Question 1.4 seems to be much harder than its counterpart for the sets $W_{m,n}(\psi)$, answered by Theorem 1.3. We comment later in the paper on the reason for this difficulty, but the main subject of this paper is different: we take up an analogous inhomogeneous approximation problem, describe the analogues of the statements and concepts discussed in this section, and then show how an inhomogeneous analogue of Question 1.4 admits a complete solution based on a correspondence between Diophantine approximation and dynamics on homogeneous spaces.

### 1.2 Inhomogeneous approximation: the main result

The theory of inhomogeneous Diophantine approximation starts when one replaces the values of a system of linear forms $A\mathbf{q}$ by those of a system of *affine forms* $\mathbf{q} \mapsto A\mathbf{q} + \mathbf{b}$, where $A \in M_{m,n}$ and $\mathbf{b} \in \mathbb{R}^m$. Consider a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$ and, following the definition of the set $D_{m,n}(\psi)$, let us say that a pair $(A, \mathbf{b}) \in M_{m,n} \times \mathbb{R}^m$ is $\psi$-*Dirichlet* if there exist $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(T), \quad \|\mathbf{q}\|^n < T, \tag{1.5}$$

whenever $T$ is large enough. (Note that in this set-up there is no need to single out the case $\mathbf{q} = 0$.) Denote the set of $\psi$-Dirichlet pairs by $\widehat{D}_{m,n}(\psi)$. Note that, as is the case with $D_{m,n}(\psi)$, membership in $\widehat{D}_{m,n}(\psi)$ depends only on the solubility of these inequalities at integer values of $T$, provided $\psi$ is non-increasing. Hence without loss of generality one can assume $\psi$ to be continuous.

Let us start with the simplest case: $\psi \equiv c$ is a constant function, or $\psi = c\psi_0$ in our notation. It is a trivial consequence of Dirichlet's theorem that whenever $c > 0$,

$$\|A\mathbf{q} - \mathbf{p}\|^m < c, \quad \|\mathbf{q}\|^n < T,$$

is solvable in $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ whenever $T > c^{-1}$. By contrast, it is clear that one cannot always solve

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < c, \quad \|\mathbf{q}\|^n < T,$$

for $c \leqslant 1/2^m$; for example, take $A$ to be an integer matrix and take $\mathbf{b}$ with coordinates in $\mathbb{Z} + \frac{1}{2}$. However, it follows from Kronecker's theorem [Cas57, § 3.5] that, for a given $A \in M_{m,n}$, there exist $\mathbf{b} \in \mathbb{R}^m$ and $c > 0$ such that $(A, \mathbf{b}) \notin \widehat{D}_{m,n}(c\psi_0)$, which amounts to saying that $A\mathbb{Z}^n$ is not dense in $\mathbb{R}^m/\mathbb{Z}^m$, only if $A^t(\mathbb{Z}^m \smallsetminus \{0\})$ contains an integer vector. The set of such $A$ has measure zero since it is the union over $\mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p} \in \mathbb{Z}^m \smallsetminus \{0\}$ of the sets $\{A : A^t\mathbf{p} = \mathbf{q}\}$. Thus for every $c > 0$, $\widehat{D}_{m,n}(c\psi_0)$ has full measure.

Once $\psi$ is allowed to decay to zero, the sets $\widehat{D}_{m,n}(\psi)$ become smaller. In particular, using dynamics on the space of grids in $\mathbb{R}^{m+n}$, one can easily prove (see Proposition 2.3 below) that $\widehat{D}_{m,n}(C\psi_1)$ is null for any $C > 0$. Thus one can naturally ask the following inhomogeneous analogue of Question 1.4.

*Question* 1.5. What is a necessary and sufficient condition on a non-increasing function $\psi$ (presumably expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $\widehat{D}_{m,n}(\psi)$ has zero or full measure?

The remainder of this work will be given to a proof of the following answer.

Theorem 1.6. *Given a non-increasing $\psi$, the set $\widehat{D}_{m,n}(\psi)$ has zero (respectively, full) measure if and only if the series*

$$\sum_j \frac{1}{\psi(j)j^2} \tag{1.6}$$

*diverges (respectively, converges).*

Note that this immediately gives results such as:

- $\widehat{D}_{m,n}(C\psi_a)$ has zero (respectively, full) measure if $a \geqslant 1$ (respectively, $a < 1$);
- for $\psi(T) = C(\log T)^b \psi_1(T)$, $\widehat{D}_{m,n}(\psi)$ has zero (respectively, full) measure if $b \leqslant 1$ (respectively, $b > 1$).

Our argument is based on a correspondence between Diophantine approximation and homogeneous dynamics. In the next section we introduce the space of grids in $\mathbb{R}^{m+n}$ and reduce the aforementioned inhomogeneous approximation problem to a shrinking target phenomenon for a flow on that space. We present a warm-up problem, Proposition 2.3, that demonstrates the usefulness of the reduction to dynamics and introduces several key ideas to be used later. This is followed by the statement of the main dynamical result, Theorem 3.6, which we prove in the two subsequent sections. The last section contains some concluding remarks, in particular a discussion of Question 1.4 and other open questions.

## 2. Dynamics on the space of grids: a warm-up

Fix $k \in \mathbb{N}$ and let

$$G_k := \mathrm{SL}_k(\mathbb{R}) \quad \text{and} \quad \widehat{G}_k := \mathrm{ASL}_k(\mathbb{R}) = G_k \ltimes \mathbb{R}^k;$$

the latter is the group of volume-preserving affine transformations of $\mathbb{R}^k$. Also put

$$\Gamma_k := \mathrm{SL}_k(\mathbb{Z}) \quad \text{and} \quad \widehat{\Gamma}_k := \mathrm{ASL}_k(\mathbb{Z}) = \Gamma_k \ltimes \mathbb{Z}^k.$$

Elements of $\widehat{G}_k$ will be denoted by $\langle g, \mathbf{w}\rangle$ where $g \in G_k$ and $\mathbf{w} \in \mathbb{R}^k$; that is, $\langle g, \mathbf{w}\rangle$ is the affine transformation $\mathbf{x} \mapsto g\mathbf{x} + \mathbf{w}$. Denote by $\widehat{X}_k$ the space of translates of unimodular lattices in $\mathbb{R}^k$; elements of $\widehat{X}_k$ will be referred to as *unimodular grids*. Clearly $\widehat{X}_k$ is canonically identified with $\widehat{G}_k/\widehat{\Gamma}_k$ via

$$\langle g, \mathbf{w}\rangle\widehat{\Gamma}_k \in \widehat{G}_k/\widehat{\Gamma}_k \quad \longleftrightarrow \quad g\mathbb{Z}^k + \mathbf{w} \in \widehat{X}_k.$$

Similarly, $X_k := G_k/\Gamma_k$ is identified with the space of unimodular lattices in $\mathbb{R}^k$ (i.e. unimodular grids containing the zero vector). Note that $\widehat{\Gamma}_k$ (respectively, $\Gamma_k$) is a lattice in $\widehat{G}_k$ (respectively, $G_k$). We will denote by $\widehat{\mu}$ (respectively $\mu$) the normalized Haar measures on $\widehat{X}_k$ and $X_k$, respectively.

Now fix $m, n \in \mathbb{N}$ with $m + n = k$, and for $t \in \mathbb{R}$ let

$$g_t := \mathrm{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n}), \tag{2.1}$$

where there are $m$ copies of $e^{t/m}$ and $n$ copies of $e^{-t/n}$. The so-called *expanding horospherical subgroup* of $\widehat{G}_k$ with respect to $\{g_t : t > 0\}$ is given by

$$H := \{u_{A,\mathbf{b}} : A \in M_{m,n}, \mathbf{b} \in \mathbb{R}^m\}, \quad \text{where } u_{A,\mathbf{b}} := \left\langle \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \right\rangle. \tag{2.2}$$



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On the other hand,

$$\tilde{H} := \left\{ \left\langle \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix} \right\rangle \,\middle|\, \begin{matrix} P \in M_{m,m},\ Q \in M_{n,n},\ \det(P)\det(Q) = 1 \\ R \in M_{n,m},\ \mathbf{d} \in \mathbb{R}^n \end{matrix} \right\} \tag{2.3}$$

is a subgroup of $\widehat{G}_k$ complementary to $H$ which is *non-expanding* with respect to conjugation by $g_t$, $t \geqslant 0$: it is easy to see that

$$g_t \left\langle \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix} \right\rangle g_{-t} = \left\langle \begin{pmatrix} P & 0 \\ e^{-((m+n)/mn)t} R & Q \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-t/n}\mathbf{d} \end{pmatrix} \right\rangle. \tag{2.4}$$

Let us also denote

$$\Lambda_{A,\mathbf{b}} := u_{A,\mathbf{b}} \mathbb{Z}^k = \left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{b} - \mathbf{p} \\ \mathbf{q} \end{pmatrix} : \mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n \right\}. \tag{2.5}$$

The reduction of Diophantine properties of $(A, \mathbf{b})$ to the behavior of the $g_t$-trajectory of $\Lambda_{A,\mathbf{b}}$ described below mimics the classical Dani correspondence for homogeneous Diophantine approximation [Dan85, KM99] and dates back to [Kle99] (see also more recent papers [Sha11, ET11, GV18]). The crucial role is played by a function $\Delta : \widehat{X}_k \to [-\infty, \infty)$ given by

$$\Delta(\Lambda) := \log \inf_{\mathbf{v} \in \Lambda} \|\mathbf{v}\|. \tag{2.6}$$

Note that $\Delta(\Lambda) = -\infty$ if and only if $\Lambda \ni 0$. Also it is easy to see that $\Delta$ is uniformly continuous outside of the set where it takes small values.

LEMMA 2.1. *For any $z \in \mathbb{R}$, $\Delta$ is uniformly continuous on the set $\Delta^{-1}([z, \infty))$. That is, for any $z \in \mathbb{R}$ and any $\varepsilon > 0$ there exists a neighborhood $U$ of the identity in $\widehat{G}_k$ such that whenever $\Delta(\Lambda) \geqslant z$ and $g \in U$, one has $|\Delta(\Lambda) - \Delta(g\Lambda)| < \varepsilon$.*

*Proof.* Let $c > 1$, $z \in \mathbb{R}$. Choose $\delta > 0$ so that

$$c^{-1}\|\mathbf{v}\| \leqslant \|\mathbf{v} + \mathbf{w}\| \leqslant c\|\mathbf{v}\|$$

whenever $\|\mathbf{w}\| \leqslant \delta$ and $\log \|\mathbf{v}\| \geqslant z - \log c$. Then if $\log \|\mathbf{v}\| \geqslant z$, $\|\mathbf{w}\| < \delta$ and the operator norms of both $g$ and $g^{-1}$ are not greater than $c$ (the latter two conditions define an open neighborhood $U$ of the identity in $\widehat{G}$ such that $\langle g, \mathbf{w} \rangle \in U$), we have

$$\frac{\|\mathbf{v}\|}{c^2} \leqslant \frac{\|g\mathbf{v}\|}{c} \leqslant \|g\mathbf{v} + \mathbf{w}\| \leqslant c \cdot \|g\mathbf{v}\| \leqslant c^2 \cdot \|\mathbf{v}\|.$$

Thus if $\Delta(\Lambda) \geqslant z$ and $\langle g, \mathbf{w} \rangle \in U$, we have

$$\Delta(\Lambda) - 2\log c \leqslant \Delta(g\Lambda + \mathbf{w}) \leqslant \Delta(\Lambda) + 2\log c.$$

Since $c > 1$ is arbitrary, $\Delta$ is uniformly continuous on $\Delta^{-1}([z, \infty))$. □

Another important feature of $\Delta$ is that it is unbounded from above; indeed, the grid

$$\operatorname{diag}\big(1, \ldots, 1, \tfrac{1}{4}e^{-z}, 4e^{z}\big)\mathbb{Z}^k + (0, \ldots, 0, 2e^{z})$$

is disjoint from the ball centered at 0 of radius $e^z$. Consequently, sets $\Delta^{-1}([z, \infty))$ have non-empty interior for all $z \in \mathbb{R}$.

Let us now describe a basic special case of the correspondence between inhomogeneous improvement of Dirichlet's theorem and dynamics on $\widehat{X}_k$. The next lemma is essentially an inhomogeneous analogue of [KW08, Proposition 2.1].



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Lemma 2.2. *Let $C > 0$ and put $z = (\log C)/(m+n)$. Then $(A, \mathbf{b}) \in \widehat{D}_{m,n}(C\psi_1)$ if and only if $\Delta(g_t\Lambda_{A,\mathbf{b}}) < z$ for all large enough $t > 0$.*

*Proof.* For $T > 1$, put $\psi(T) = C\psi_1(T) = C/T$, and define

$$t := \log T - \frac{n}{m+n}\log C \quad \Longleftrightarrow \quad T = C^{m/(m+n)}e^t.$$

Then $\psi(T) = C^{n/(m+n)}e^{-t}$, and the system (1.5) can be written as

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < C^{n/(m+n)}e^{-t}, \quad \|\mathbf{q}\|^n < C^{m/(m+n)}e^t,$$

which is the same as

$$e^{t/m}\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\| < C^{1/(m+n)}, \quad e^{-t/n}\|\mathbf{q}\| < C^{1/(m+n)}.$$

In view of (2.1), (2.5) and (2.6), the solvability of (1.5) in $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ is equivalent to

$$\Delta(g_t\Lambda_{A,\mathbf{b}}) < \frac{\log C}{m+n} = z,$$

and the conclusion follows. □

We will use the above lemma and the ergodicity of the $g_t$-action on $\widehat{X}_k$ to compute the Lebesgue measure of $\widehat{D}_{m,n}(C\psi_1)$. The proof contains a Fubini theorem argument (following [KM99, Theorem 8.7] and dating back to [Dan85]) used to pass from an almost-everywhere statement for lattices to an almost-everywhere statement for pairs in $M_{m,n} \times \mathbb{R}^m$. We will refer to this argument twice more in the sequel.

Proposition 2.3. *For any $m, n \in \mathbb{N}$ and any $C > 0$, the set $\widehat{D}_{m,n}(C\psi_1)$ has Lebesgue measure zero.*

*Proof.* Suppose $U$ is a subset of $M_{m,n} \times \mathbb{R}^m$ ($\cong H$ as in (2.2)) of positive Lebesgue measure such that $\Delta(g_t\Lambda_{A,\mathbf{b}}) < (\log C)/(m+n)$ for any $(A, \mathbf{b}) \in U$ and all large enough $t$. Then there exists a neighborhood $V$ of identity in $\tilde{H}$ as in (2.3) such that for all $g \in V$, $(A, \mathbf{b}) \in U$ and all large enough $t$,

$$\Delta(g_t g\Lambda_{A,\mathbf{b}}) = \Delta(g_t g g_t^{-1} g_t\Lambda_{A,\mathbf{b}}) < \frac{\log C}{m+n} + 1. \tag{2.7}$$

Indeed, one can use Lemma 2.1 and (2.4) to choose $V$ such that if (2.7) does not hold for $g \in V$, then $|\Delta(g_t g g_t^{-1} g_t\Lambda_{A,\mathbf{b}}) - \Delta(g_t\Lambda_{A,\mathbf{b}})| < 1$. But since the product map $\tilde{H} \times H \to \widehat{G}_k$ is a local diffeomorphism, $V \times U$ is mapped onto a set of positive measure. It follows that $\Delta(g_t\Lambda) < (\log C)/(m+n) + 1$ for all large enough $t$ and for a set of lattices $\Lambda$ of positive Haar measure in $\widehat{X}_k$.

On the other hand, from Moore's ergodicity theorem [Moo66] together with the ergodicity criterion of Brezin and Moore (see [BM81, Theorem 6.1] or [Mar91, Theorem 6]) it follows that every unbounded subgroup of $G_k$, in particular, $\{g_t : t \in \mathbb{R}\}$ as above, acts ergodically on $\widehat{X}_k$. Since for any $C > 0$ the set $\Delta^{-1}([(\log C)/(m+n) + 1, \infty))$ has a non-empty interior, it follows that $\mu$-almost every $\Lambda \in \widehat{X}_k$ must visit any such set at unbounded times under the action of $g_t$, a contradiction. □



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## 3. A correspondence between Dirichlet improvability and dynamics

Lemma 2.2 relates the complement of $\widehat{D}_{m,n}(C\psi_0)$ to the set of grids visiting certain 'target' subsets of $\widehat{X}_k$ at unbounded times under the diagonal flow $g_t$. This is the special case where the target does not change with the time parameter $t$. For general non-increasing $\psi$, we get a family of 'shrinking targets' $\Delta^{-1}([z_\psi(t),\infty))$ (which in fact are shrinking only in a weak sense; see Remark 3.3), where $z_\psi$ is gotten by the following change of variables, known as the Dani correspondence.

Lemma 3.1 (See [KM99, Lemma 8.3]). *Let positive integers $m, n$ and $T_0 \in \mathbb{R}_+$ be given. Suppose $\psi : [T_0,\infty) \to \mathbb{R}_+$ is a continuous, non-increasing function. Then there exists a unique continuous function*

$$z = z_\psi : [t_0,\infty) \to \mathbb{R},$$

*where $t_0 := (m/(m+n))\log T_0 - (n/(m+n))\log\psi(T_0)$, such that:*

(i) *the function $t \mapsto t + nz(t)$ is strictly increasing and unbounded;*
(ii) *the function $t \mapsto t - mz(t)$ is non-decreasing;*
(iii) *$\psi(e^{t+nz(t)}) = e^{-t+mz(t)}$ for all $t \geqslant t_0$.*

*Remark* 3.2. The function $z$ of Lemma 3.1 differs from the function $r$ of [KM99, Lemma 8.3] by a minus sign. This reflects the difference between the asymptotic and uniform approximation problems.

*Remark* 3.3. For future reference, we point out that properties (1) and (2) of Lemma 3.1 imply that any $z = z_\psi$ does not oscillate too wildly. Namely,

$$z(s) - \frac{1}{m} \leqslant z(u) \leqslant z(s) + \frac{1}{n} \quad \text{whenever } s \leqslant u \leqslant s+1.$$

Now we can state a general version of the correspondence between the improvability of the inhomogeneous Dirichlet theorem and dynamics on $\widehat{X}_k$, generalizing the first paragraph of the proof of Theorem 2.3.

Lemma 3.4. *Let $\psi : [T_0,\infty) \to \mathbb{R}_+$ be a non-increasing continuous function, and let $z = z_\psi$ be the function associated to $\psi$ by Lemma 3.1. The pair $(A,\mathbf{b})$ is in $\widehat{D}_{m,n}(\psi)$ if and only if $\Delta(g_t\Lambda_{A,b}) < z_\psi(t)$ for all sufficiently large $t$.*

*Proof.* We argue as in the proof of Lemma 2.2. Since $t \mapsto t+nz(t)$ is increasing and unbounded, $(A,\mathbf{b}) \in \widehat{D}_{m,n}(\psi)$ if and only if for all large enough $t$ we have

$$\|A\mathbf{q}+\mathbf{b}-\mathbf{p}\|^m < \psi(e^{t+nz(t)}) = e^{-t+mz(t)}, \quad \|\mathbf{q}\|^n < e^{t+nz(t)},$$

for some $\mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p} \in \mathbb{Z}^m$. This is the same as the solvability of

$$e^{t/m}\|A\mathbf{q}+\mathbf{b}-\mathbf{p}\| < e^{z(t)}, \quad e^{-t/n}\|\mathbf{q}\| < e^{z(t)},$$

which is the same as $\Delta(g_t\Lambda_{A,\mathbf{b}}) < z_\psi(t)$. □

Thus a pair fails to be $\psi$-Dirichlet if and only if the associated grid visits the 'target' $\Delta^{-1}([z_\psi(t),\infty))$ at unbounded times $t$ under the flow $g_t$. This is known as a 'shrinking target phenomenon'. Our next goal is to recast condition (1.6) using the function $z_\psi$.



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Lemma 3.5. *Let $\psi : [T_0, \infty) \to \mathbb{R}_+, T_0 \geqslant 0$, be a non-increasing continuous function, and $z = z_\psi$ the function associated to $\psi$ by Lemma 3.1. Then we have*

$$\sum_{j=\lceil T_0 \rceil}^{\infty} \frac{1}{j^2 \psi(j)} < \infty \quad \textit{if and only if} \quad \sum_{t=\lceil t_0 \rceil}^{\infty} e^{-(m+n)z(t)} < \infty.$$

*Proof.* We follow the lines of the proof of [KM99, Lemma 8.3]. Using the monotonicity of $\psi$ and Remark 3.3, we may replace the sums with integrals

$$\int_{T_0}^{\infty} x^{-2}\psi(x)^{-1}\, dx \quad \text{and} \quad \int_{t_0}^{\infty} e^{-(m+n)z(t)}\, dt,$$

respectively. Define

$$P := -\log \circ\, \psi \circ \exp : [T_0, \infty) \to \mathbb{R} \quad \text{and} \quad \lambda(t) := t + nz(t).$$

Since $\psi(e^\lambda) = e^{-P(\lambda)}$, we have

$$\int_{T_0}^{\infty} x^{-2}\psi(x)^{-1}\, dx = \int_{\log T_0}^{\infty} \psi(e^\lambda)^{-1} e^{-\lambda}\, d\lambda = \int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda.$$

Using $P(\lambda(t)) = t - mz(t)$, we also have

$$\begin{aligned}\int_{t_0}^{\infty} e^{-(m+n)z(t)}\, dt &= \int_{\log T_0}^{\infty} e^{-(m+n)z(m\lambda/(m+n)+nP(\lambda)/(m+n))}\, d\left[\frac{m}{m+n}\lambda + \frac{n}{m+n}P(\lambda)\right] \\ &= \frac{m}{m+n}\int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda + \frac{n}{m+n}\int_{\log T_0}^{\infty} e^{-\lambda}e^{P(\lambda)}\, dP(\lambda) \\ &= \frac{m}{m+n}\int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda + \frac{n}{m+n}\int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda \\ &\quad + \frac{n}{m+n}\Big(\lim_{\lambda\to\infty} e^{P(\lambda)-\lambda} - 1\Big),\end{aligned}$$

where we integrated by parts in the last line. Since all these quantities (aside from the constant $-1$) are positive, the convergence of $\int_{t_0}^{\infty} e^{-(m+n)z(t)}\, dt$ implies the convergence of $\int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda$. Conversely, suppose $\int_{\log T_0}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda$ converges, yet $\int_{t_0}^{\infty} e^{-(m+n)z(t)}\, dt$ diverges. Then since $u \mapsto \int_{t_0}^{u} e^{(m+n)z(t)}\, dt$ is increasing in $u$, and the first two terms of the sum above converge, we must have $e^{P(\lambda)-\lambda}$ eventually increasing in $\lambda$ (recall that $\lambda$ is an increasing and unbounded function). But this contradicts the convergence of $\int_{\log(T_0)}^{\infty} e^{P(\lambda)-\lambda}\, d\lambda$. □

Now we are ready to reduce Theorem 1.6 to the following statement concerning dynamics on $\widehat{X}_k$.

Theorem 3.6. *Fix $k \in \mathbb{N}$ and let $\{g_t : t \in \mathbb{R}\}$ be a diagonalizable unbounded one-parameter subgroup of $G_k$. Also take an arbitrary sequence $\{z(t) : t \in \mathbb{N}\}$ of real numbers. Then the set*

$$\{\Lambda \in \widehat{X}_k : \Delta(g_t\Lambda) \geqslant z(t) \textit{ for infinitely many } t \in \mathbb{N}\} \tag{3.1}$$

*is null (respectively, conull) if the sum*

$$\sum_{t=1}^{\infty} e^{-kz(t)} \tag{3.2}$$

*converges (respectively, diverges).*

*Proof of Theorem 1.6 assuming Theorem 3.6.* Suppose that the series (1.6) converges, and take $z(t) = z_\psi(t)$, the function associated to $\psi$ by Lemma 3.1. In view of Lemma 3.5, the series (3.2) converges as well. In particular, it follows that $z(t) \geqslant 0$ for all large enough $t \in \mathbb{N}$, and also that $\sum_{t=1}^\infty e^{-k(z(t)-C)} < \infty$ for any $C > 0$. Take $g_t$ as in (2.1); Theorem 3.6 then implies that

$$\widehat{\mu}\big(\{\Lambda \in \widehat{X}_k : \Delta(g_t\Lambda) \geqslant z(t) - C \text{ for infinitely many } t \in \mathbb{N}\}\big) = 0. \tag{3.3}$$

Suppose that the Lebesgue measure of $\widehat{D}_{m,n}(\psi)^c$ is positive. Lemma 3.4 asserts that there exists a set $U$ of positive measure consisting of pairs $(A, \mathbf{b})$ for which $\Delta(g_t\Lambda_{A,\mathbf{b}}) \geqslant z(t)$ for an unbounded set of $t \geqslant 0$. Then, using $z(t) \geqslant 0$ and Lemma 2.1, we can replace $t$ with its integer part:

$$\Delta(g_{\lfloor t\rfloor}\Lambda_{A,\mathbf{b}}) = \Delta(g_{(\lfloor t\rfloor - t)}g_t\Lambda_{A,\mathbf{b}}) \geqslant \Delta(g_t\Lambda_{A,\mathbf{b}}) - c \geqslant z(t) - c \geqslant z(\lfloor t\rfloor) - c - 1/m,$$

where $c$ is a positive constant and the last inequality follows from Remark 3.3. Therefore we get $\Delta(g_t\Lambda_{A,\mathbf{b}}) \geqslant z(t) - c - 1/m$ for an unbounded set of $t \in \mathbb{N}$ as long as $(A, \mathbf{b}) \in U$.

Now recall the groups $H$ and $\tilde{H}$ from Equations (2.2) and (2.3). As in the proof of Proposition 2.3, we may identify $U$ with a subset of $H$ and, using the uniform continuity of $\Delta$ (Lemma 2.1), find a neighborhood of identity $V \subset \tilde{H}$ such that, for all $g \in V$ and $(A, \mathbf{b}) \in U$,

$$\Delta(g_t g\Lambda_{A,\mathbf{b}}) = \Delta(g_t g g_t^{-1} g_t\Lambda_{A,\mathbf{b}}) \geqslant \Delta(g_t\Lambda_{A,\mathbf{b}}) - 1$$

for all $t \geqslant 0$, hence $\Delta(g_t g\Lambda_{A,\mathbf{b}}) \geqslant z(t) - 1 - c - 1/m$ for an unbounded set of $t \in \mathbb{N}$. Since the product map $\tilde{H} \times H \to \widehat{G}_k$ is a local diffeomorphism, the image of $V \times U$ is a set of positive measure in $G_k$, contradicting (3.3).

The proof of the divergence case proceeds along the same lines. If (1.6) diverges, by Lemma 3.5 so does (3.2). Define $z'(t) := \max(z(t), 0)$; then we have $\sum_{t=1}^\infty e^{-k(z'(t))} = \infty$ as well, therefore $\sum_{t=1}^\infty e^{-k(z'(t)+C)} = \infty$ for any $C > 0$. In view of Theorem 3.6,

$$\text{the set } \{\Lambda \in \widehat{X}_k : \Delta(g_t\Lambda) \geqslant z'(t) + C \text{ for infinitely many } t \in \mathbb{N}\} \text{ has full measure.} \tag{3.4}$$

Now assume that the set $\widehat{D}_{m,n}(\psi)$ has positive measure. Then using Lemma 3.4, one can choose a set $U$ of positive measure consisting of pairs $(A, \mathbf{b})$ for which

$$\Delta(g_t\Lambda_{A,\mathbf{b}}) < z(t) \leqslant z'(t)$$

for all large enough $t$. Then, as before, using Lemma 2.1 with $z = 0$ and (2.4), one finds a neighborhood of identity $V \subset \tilde{H}$ such that for all $g \in V$ and $(A, \mathbf{b}) \in U$,

$$\Delta(g_t g\Lambda_{A,\mathbf{b}}) = \Delta(g_t g g_t^{-1} g_t\Lambda_{A,\mathbf{b}}) < \max(\Delta(g_t\Lambda_{A,\mathbf{b}}), 0) + 1$$

for all $t \geqslant 0$; hence $\Delta(g_t g\Lambda_{A,\mathbf{b}}) < z'(t) + 1$ for all large enough $t$. Again using the local product structure of $\widehat{G}_k$, one concludes that the image of $V \times U$ in $\widehat{X}_k$ is a set of positive measure, contradicting (3.4). □

We are now left with the task of proving Theorem 3.6. The proof will have two ingredients. In the next section we will establish a dynamical Borel–Cantelli lemma (Theorem 4.4) showing that the limsup set (3.1) is null or conull according to the convergence or divergence of the series

$$\sum_{t=1}^\infty \widehat{\mu}(\{\Lambda \in \widehat{X}_k : \Delta(\Lambda) \geqslant z(t)\}). \tag{3.5}$$

The proof is based on the methods of [KM99, KM18]; namely, it uses the exponential mixing of the $g_t$-action on $\widehat{X}_k$, as well as the so-called DL property of $\Delta$. The latter will be established in § 6. Moreover, there we will relate (3.2) and (3.5) by showing that the summands in (3.5) are equal to $e^{-kz(t)}$ up to a constant (Theorem 4.6).



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## 4. A general dynamical Borel–Cantelli lemma and exponential mixing

In this section we let $G$ be a Lie group and $\Gamma$ a lattice in $G$. Denote by $X$ the homogeneous space $G/\Gamma$ and by $\mu$ the $G$-invariant probability measure on $X$. In what follows, $\|\cdot\|_p$ will stand for the $L^p$-norm. Fix a basis $\{Y_1,\ldots,Y_n\}$ for the Lie algebra $\mathfrak{g}$ of $G$, and, given a smooth function $h \in C^\infty(X)$ and $\ell \in \mathbb{Z}_+$, define the '$L^2$, *order $\ell$' Sobolev norm* $\|h\|_{2,\ell}$ of $h$ by

$$\|h\|_{2,\ell} \stackrel{\text{def}}{=} \sum_{|\alpha|\leqslant \ell} \|D^\alpha h\|_2,$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $D^\alpha$ is a differential operator of order $|\alpha|$ which is a monomial in $Y_1,\ldots,Y_n$, namely $D^\alpha = Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}$. This definition depends on the basis; however, a change of basis would only distort $\|h\|_{2,\ell}$ by a bounded factor. We also let

$$C_2^\infty(X) = \{h \in C^\infty(X) : \|h\|_{2,\ell} < \infty \text{ for any } \ell \in \mathbb{Z}_+\}.$$

Fix a right-invariant Riemannian metric on $G$ and the corresponding metric 'dist' on $X$. For $g \in G$, let us denote by $\|g\|$ the distance between $g \in G$ and the identity element of $G$. Note that $\|g\| = \|g^{-1}\|$ due to the right-invariance of the metric.

Definition 4.1. Let $L$ be a subgroup of $G$. Say that the $L$-action on $X$ is *exponentially mixing* if there exist $\gamma, E > 0$ and $\ell \in \mathbb{Z}_+$ such that for any $\varphi, \psi \in C_2^\infty(X)$ and for any $g \in L$ one has

$$\left|\langle g\varphi,\psi\rangle - \int_X \varphi\, d\mu \int_X \psi\, d\mu\right| \leqslant E e^{-\gamma\|g\|}\|\varphi\|_{2,\ell}\|\psi\|_{2,\ell}. \tag{EM}$$

Here $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(X,\mu)$.

We also need two more definitions from [KM99, KM18].

Definition 4.2. A sequence of elements $\{f_t : t \in \mathbb{N}\}$ of elements of $G$ is called *exponentially divergent* if

$$\sup_{t\in\mathbb{N}} \sum_{s=1}^\infty e^{-\gamma\|f_s f_t^{-1}\|} < \infty \quad \forall\, \gamma > 0. \tag{4.1}$$

Now let $\Delta$ be a real-valued function on $X$, and for $z \in \mathbb{R}$ denote

$$\Phi_\Delta(z) \stackrel{\text{def}}{=} \mu(\Delta^{-1}([z,\infty))).$$

Definition 4.3. Say that $\Delta$ is *DL* (an abbreviation for 'distance-like') if there exists $z_0 \in \mathbb{R}$ such that $\Phi_\Delta(z_0) > 0$ and

(a) $\Delta$ is uniformly continuous on $\Delta^{-1}([z_0,\infty))$; that is, for all $\varepsilon > 0$ there exists a neighborhood $U$ of identity in $G$ such that for any $x \in X$ with $\Delta(x) \geqslant z_0$,

$$g \in U \quad \Longrightarrow \quad |\Delta(x) - \Delta(gx)| < \varepsilon;$$

(b) the function $\Phi_\Delta$ does not decrease very fast; more precisely,

$$\exists\, c, \delta > 0 \text{ such that } \Phi_\Delta(z) \geqslant c\Phi_\Delta(z-\delta) \quad \forall\, z \geqslant z_0. \tag{4.2}$$

The next theorem is a direct consequence of [KM18, Theorem 1.3].



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Theorem 4.4. *Suppose that the action of a subgroup $L \subset G$ on $X$ is exponentially mixing. Let $\{f_t : t \in \mathbb{N}\}$ be a sequence of elements of $L$ satisfying (4.1), and let $\Delta$ be a DL function on $X$. Also let $\{z(t) : t \in \mathbb{N}\}$ be a sequence of real numbers. Then the set*

$$\{\Lambda \in X : \Delta(g_t\Lambda) \geqslant z(t) \text{ for infinitely many } t \in \mathbb{N}\} \tag{4.3}$$

*is null (respectively, conull) if the sum*

$$\sum_{t=1}^{\infty} \Phi_\Delta(z(t)) \tag{4.4}$$

*converges (respectively, diverges).*

*Proof.* The convergence case is immediate from the classical Borel–Cantelli lemma. The divergence case is established in [KM99, KM18] for $L = G$, but the argument applies verbatim if $G$ is replaced by a subgroup. □

From now on we will take $k \geqslant 2$ and consider the case $G = \widehat{G}_k$, $\Gamma = \widehat{\Gamma}_k$, $X = \widehat{X}_k$, and $L = G_k$, with notation as in the previous section. Then we have the following theorem.

Theorem 4.5. *The $G_k$-action on $\widehat{X}_k = \widehat{G}_k/\widehat{\Gamma}_k$ is exponentially mixing.*

*Proof.* According to [KM99, Theorem 3.4], exponential mixing holds whenever the regular representation of $G_k$ on the space $L^2_0(\widehat{X}_k)$ (functions in $L^2(\widehat{X}_k)$ with integral zero) is isolated in the Fell topology from the trivial representation. This is immediate if $k > 2$ since in this case $G_k$ has Property (T).

If $k = 2$, let us write $L^2_0(\widehat{X}_2)$ as a direct sum of two spaces: functions invariant under the action of $\mathbb{R}^2$ by translations, and its orthogonal complement. The first representation is isomorphic to the regular representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2_0(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}))$, which is isolated from the trivial representation by [KM99, Theorem 1.12]. As for the second component, one can use [HT92, Theorem V.3.3.1] (see also [GGN18, Theorem 4.3]) which asserts that for any unitary representation $(\rho, V)$ of $\mathrm{ASL}_2(\mathbb{R})$ with no non-zero vectors fixed by $\mathbb{R}^2$, the restriction of $\rho$ to $\mathrm{SL}_2(\mathbb{R})$ is *tempered*, that is, there exists a dense set of vectors in $V$ whose matrix coefficients are in $L^{2+\varepsilon}$ for any $\varepsilon > 0$. Exponential mixing thus follows from [KS94, Theorem 3.1], which establishes exponential decay of matrix coefficients of strongly $L^p$ irreducible unitary representations of connected semisimple centerfree Lie groups. See also the preprint [Edw13] for more precise estimates. □

Now let $\Delta$ be the function on $\widehat{X}_k$ defined by (2.6). In the next section we will establish the following two-sided estimate for the measure of super-level sets of $\Delta$.

Theorem 4.6. *For any $k \geqslant 2$ there exist $c, C > 0$ such that*

$$ce^{-kz} \leqslant \Phi_\Delta(z) \leqslant Ce^{-kz} \quad \text{for all } z \geqslant 0. \tag{4.5}$$

This is all one needs to settle Theorem 3.6.

*Proof of Theorem 3.6 modulo Theorem 4.6.* Let $\{g_t : t \in \mathbb{R}\}$ be a diagonalizable unbounded one-parameter subgroup of $G_k$. By Theorem 4.5, the action of $G_k$ on $\widehat{X}_k$ is exponentially mixing. Observe also that one has $\|g_t\| \geqslant \alpha t$ for some $\alpha > 0$, which immediately implies (4.1). It is easy to see that (4.5) implies (4.2) with $z_0 = 0$, and part (a) of Definition 4.3 is given by Lemma 2.1. The conditions of Theorem 4.4 are therefore met, and Theorem 3.6 follows. □

## 5. $\boldsymbol{\Delta}$ is distance-like: a warm-up

For the rest of the paper we keep the notation

$$G = G_k, \quad \widehat{G} = \widehat{G}_k = G_k \ltimes \mathbb{R}^k, \quad X = X_k = G_k/\Gamma_k, \quad \widehat{X} = \widehat{X}_k = \widehat{G}_k/\widehat{\Gamma}_k,$$

and let $\mu$ (respectively, $\widehat{\mu}$) be the Haar probability measure on $X$ (respectively, $\widehat{X}$). We denote by $\mu_G$ and $\mu_{\widehat{G}}$ the left-invariant Haar measures on $G$ and $\widehat{G}$, respectively, which are locally pushed forward to $\mu$ and $\widehat{\mu}$.

Recall that

$$\Phi_\Delta(z) = \widehat{\mu}(\{\Lambda \in \widehat{X} : \Delta(\Lambda) \geqslant z\}) = \widehat{\mu}(\{\Lambda \in X : \Lambda \cap B(0, e^z) = \emptyset\}),$$

where for $\mathbf{v} \in \mathbb{R}^k$ and $r \geqslant 0$ we let $B(\mathbf{v}, r)$ be the open ball in $\mathbb{R}^k$ centered at $\mathbf{v}$ of radius $r$ with respect to the supremum norm. It will be convenient to write

$$S_r := \Delta^{-1}([\log r, \infty)) = \{\Lambda \in \widehat{X} : B(0, r) \cap \Lambda = \emptyset\}.$$

Our goal is thus to prove that

$$cr^{-k} \leqslant \widehat{\mu}(S_r) \leqslant Cr^{-k} \quad \text{for all } r \geqslant 1, \tag{5.1}$$

where $c, C$ are constants dependent only on $k$.

First let us discuss the upper bound. It is in fact a special case of a recent result due to Athreya, namely a random Minkowski-type theorem for the space of grids [Ath15, Theorem 1].

Proposition 5.1 (Athreya). *For a measurable $E \subset \mathbb{R}^k$,*

$$\widehat{\mu}(\{\Lambda \in \widehat{X} : \Lambda \cap E = \emptyset\}) \leqslant \frac{1}{1 + \lambda(E)}.$$

Here and hereafter $\lambda$ stands for Lebesgue measure on $\mathbb{R}^k$. Taking $E = B(0, r)$ shows that $\widehat{\mu}(S_r) < 2^{-k} r^{-k}$. Thus it only remains to establish a lower bound in (5.1).

There exists an obvious projection, $\pi : \widehat{X} \to X$, making $\widehat{X}$ into a $\mathbb{T}^k$-bundle over $X$ ($\pi$ simply translates one of the vectors in a grid to the origin). It is easy to see that $\mu_{\widehat{G}}$ is the product of $\mu_G$ and $\lambda$. Therefore one has the following Fubini formula:

$$\widehat{\mu}(S_r) = \int_X Q(\Lambda, r)\, d\mu(\Lambda), \quad \text{where } Q(\Lambda, r) := \lambda(S_r \cap \pi^{-1}(\Lambda)). \tag{5.2}$$

Here, for $\Lambda \in X$, $\pi^{-1}(\Lambda)$ is identified with $\mathbb{R}^k/\Lambda$ via

$$[\mathbf{v}] \in \mathbb{R}^k/\Lambda \longleftrightarrow \Lambda - \mathbf{v}, \tag{5.3}$$

and, in the hope that it will not cause any confusion, we will let $\lambda$ stand for the normalized Haar measure on $\mathbb{R}^k/\Lambda$ for any $\Lambda \in X$. Writing $\rho_\Lambda$ for the projection $\mathbb{R}^k \to \mathbb{R}^k/\Lambda$, we have

$$\begin{aligned} S(r) \cap \pi^{-1}(\Lambda) &= \{[\mathbf{v}] \in \mathbb{R}^k/\Lambda : B(0, r) \cap (\Lambda - \mathbf{v}) = \emptyset\} \\ &= \{[\mathbf{v}] \in \mathbb{R}^k/\Lambda : B(\mathbf{v}, r) \cap \Lambda = \emptyset\} = \rho_\Lambda\Big(\mathbb{R}^k \smallsetminus \bigcup_{\mathbf{v} \in \Lambda} B(\mathbf{v}, r)\Big), \end{aligned} \tag{5.4}$$

so that $Q(\Lambda, r)$ is the area of a region in a fundamental domain (parallelepiped) in $\mathbb{R}^k$ for $\Lambda$ consisting of points which are farther than $r$ from its vertices, that is, from all points of $\Lambda$.



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Recall that $\mathrm{SL}_2(\mathbb{R})$ double-covers the unit tangent bundle of the hyperbolic upper-half plane, $\mathbb{H}^2$. Since the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ has a convenient fundamental domain, there are convenient coordinates for a set of full measure in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. This enables us to give a rather tidy proof for the two-dimensional case of (5.1), handling both bounds simultaneously without using Proposition 5.1. This proof also illustrates the main idea necessary to proving the lower bound in the general case. We therefore start with a separate, redundant proof of the two-dimensional case.

*Proof of (5.1) for $k=2$.* For fixed $r$, consider the map $(\kappa, n, a) \mapsto Q(\kappa n a\mathbb{Z}^2, r)$, whose domain is $K \times N \times A$, the Iwasawa decomposition for $G = \mathrm{SL}_2(\mathbb{R})$. (Here $K$, $N$, $A$ are the groups of orthogonal, upper-triangular unipotent, and diagonal matrices respectively.) We first show that a change of $\kappa$ does not significantly change the value of $Q$. Indeed, since rotation perturbs the sup norm by no more than a factor of $\sqrt{2}$, for any $\kappa \in K$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ we have

$$\|\kappa\mathbf{x} - \kappa\mathbf{y}\| \geqslant \sqrt{2}r \implies \|\mathbf{x} - \mathbf{y}\| \geqslant r \implies \|\kappa\mathbf{x} - \kappa\mathbf{y}\| \geqslant r/\sqrt{2},$$

hence

$$\rho_\Lambda\left(\mathbb{R}^2 \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\kappa\mathbf{v}, \sqrt{2}r)\right) \subset \rho_\Lambda\left(\mathbb{R}^2 \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\mathbf{v}, r)\right) \subset \rho_\Lambda\left(\mathbb{R}^2 \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\kappa\mathbf{v}, r/\sqrt{2})\right).$$

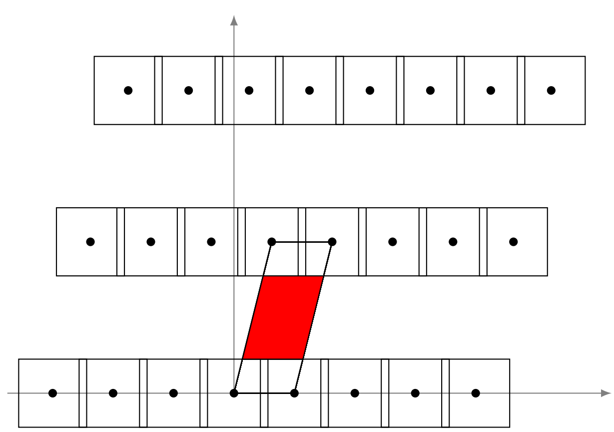
By (5.4), this implies

$$Q(\kappa\Lambda, \sqrt{2}r) \leqslant Q(\Lambda, r) \leqslant Q(\kappa\Lambda, r/\sqrt{2}). \tag{5.5}$$

Let $a = \mathrm{diag}(\alpha, \alpha^{-1})$. If

$$2r > \alpha, \tag{5.6}$$

then the lattice $na\mathbb{Z}^2$ consists of horizontal rows of vectors, each closer than $2r$ to its horizontal neighbors. Thus the boxes making up the union $\bigcup_{\mathbf{v}\in na\mathbb{Z}^2} B(\mathbf{v}, r)$ overlap in the horizontal direction, creating horizontal strips. Thus, by (5.4), $Q(na\mathbb{Z}^2, r)$ is just the area of the fundamental parallelogram $na(I \times I)$ minus the strips on top and bottom, as in the following figure.



This smaller parallelogram has area $\alpha(\alpha^{-1} - 2r)$, provided it is non-empty, that is, provided $2r \leqslant \alpha^{-1}$. Thus from (5.5), if $\Lambda = \kappa n a\mathbb{Z}^2$, where $a = \mathrm{diag}(\alpha, \alpha^{-1})$, we have

$$Q(\Lambda, r) \leqslant Q(\kappa^{-1}\Lambda, r/\sqrt{2}) = Q(na\mathbb{Z}^2, r/\sqrt{2}).$$

Then if $\sqrt{2}r > \alpha$ (so that (5.6) holds for $na\mathbb{Z}^2$, after adjusting $r$ as above), we have

$$\begin{aligned} \alpha(\alpha^{-1} - 2\sqrt{2}r) \leqslant Q(\Lambda, r) \leqslant \alpha(\alpha^{-1} - \sqrt{2}r) \quad &\text{if } \sqrt{2}r \leqslant \alpha^{-1}, \\ Q(\Lambda, r) = 0 \quad &\text{if } \sqrt{2}r \geqslant \alpha^{-1}. \end{aligned} \tag{5.7}$$

We now identify $X_2$ with $\mathrm{SL}_2(\mathbb{Z})\backslash T^1(\mathbb{H}^2)$ via $g\mathbb{Z}^2 \mapsto g^{-1}(i,i)$ (here, the matrix $g^{-1}$ acts on $(i,i)$ as a fractional-linear transformation). Recall that

$$F := T^1\{|z| \geqslant 1, |\mathrm{Re}(z)| \leqslant 1/2\}$$

is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $T^1(\mathbb{H}^2)$. Under the correspondence $g^{-1}(i,i) \mapsto g\mathbb{Z}^2$, a point $(z,\theta) \in F$ maps to a lattice $\Lambda = \kappa n a \mathbb{Z}^2$ with $\sqrt{3}/2 \leqslant \mathrm{Im}(z) = \alpha^{-2}$. Thus if

$$\sqrt{3}r^2 > 1 \tag{5.8}$$

(which ensures that the condition $\sqrt{2}r > \alpha$ for (5.7) is met) then the estimates (5.7) hold with $\mathrm{Im}(z) = \alpha^{-2}$ for any lattice $\Lambda_{(z,\theta)}$, $(z,\theta) \in F$. Writing $y = \mathrm{Im}(z)$, the estimates become

$$\begin{aligned} 1 - \frac{2\sqrt{2}r}{\sqrt{y}} \leqslant Q(\Lambda_{(z,\theta)}, r) \leqslant 1 - \frac{\sqrt{2}r}{\sqrt{y}} \quad &\text{if } \sqrt{2}r \leqslant \sqrt{y}, \\ Q(\Lambda_{(z,\theta)}, r) = 0 \quad &\text{if } \sqrt{2}r \geqslant \sqrt{y}. \end{aligned} \tag{5.9}$$

Since the Haar measure on $X_2$ corresponds to the hyperbolic measure $(1/y^2)\,dx\,dy\,d\theta$ on $F$, we have

$$\int_{X_2'} Q(\Lambda, r)\, d\mu'(\Lambda) = \int_F Q(\Lambda_{(x+iy,\theta)}, r) \cdot \frac{1}{y^2}\, dx\, dy\, d\theta.$$

Finally, since $r$ is large,[3] $Q(\Lambda_{(z,\theta)}, r)$ vanishes in the region of $F$ between the line $y = 2r^2$ and the arc of the unit circle, permitting us to integrate over an unbounded rectangular region. The estimates (5.9) give

$$2\pi \int_{8r^2}^{\infty} \left(1 - \frac{2\sqrt{2}r}{\sqrt{y}}\right) \frac{dy}{y^2} \leqslant \int_{X_2} Q(\Lambda, r)\, d\mu(\Lambda) \leqslant 2\pi \int_{2r^2}^{\infty} \left(1 - \frac{\sqrt{2}r}{\sqrt{y}}\right) \frac{dy}{y^2},$$

where the $2\pi$ comes from integrating a constant function over the $\theta$ factor. Computing these integrals gives

$$\frac{\pi}{12r^2} \leqslant \widehat{\mu}(S_r) \leqslant \frac{\pi}{3r^2} \quad \text{whenever } r \geqslant 3^{-1/4}.$$

This proves (5.1). □

## 6. Completion of the proof of Theorem 4.6

We now set up the proof of the general case ($k \geqslant 2$) with some notation and remarks on Siegel sets. Then the proof will be given following two lemmas generalizing some statements from the proof of the two-dimensional case.

As before, we wish to write $Q(\Lambda, r)$ introduced in (5.2) in terms of the coordinates of the Iwasawa decomposition of a representative $g \in G$ for $\Lambda = g\mathbb{Z}^k$. We will assume $g$ lies in a subset of a particular Siegel set. Specifically, for elements of $G$ of the form

$$n = \begin{bmatrix} 1 & \nu_{1,1} & \nu_{1,2} & \cdots & \nu_{1,k-1} \\ 0 & 1 & \nu_{2,1} & \cdots & \nu_{2,k-2} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \nu_{k-1,1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_k \end{bmatrix}, \tag{6.1}$$

[3] Specifically we need $\sqrt{2}r \geqslant 1$, which is already covered by (5.8).

and for $d, e \in \mathbb{R}$, $c \in \mathbb{R}_+$, define

$$\begin{aligned} A_c &:= \{a \in A : a_{j+1} \geqslant c a_j > 0 \ \ (j = 1, \dots, k-1)\}, \\ N_{e,d} &:= \{n \in N : e \leqslant \nu_{i,j} \leqslant d \ (1 \leqslant i, j \leqslant k-1)\}. \end{aligned}$$

Also write $K$ for $\mathrm{SO}\,(k)$. It is known that $KA_{1/2}N_{-1,0}$ is a 'coarse fundamental domain' for $\Gamma_k$ in $G_k$ (see [Mor15, § 19.4(ii), following Remark 7.3.4][4]). That is, $KA_{1/2}N_{-1,0}$ contains a fundamental domain for the right-action of $\Gamma_k$ on $G_k$, and it is covered by finitely many $\Gamma_k$-translates of that domain. Therefore $KA_1N_{-1,0}$ is contained in a coarse fundamental domain, and since we are interested in a lower bound for $\int_X Q(\Lambda, r)\, d\mu(\Lambda)$, it will suffice to bound the integral

$$\int_{KA_1N_{-1,0}} Q(g\mathbb{Z}^k, r)\, d\mu_G(g) \tag{6.2}$$

from below.

For the purpose of the lower bound it will suffice to restrict ourselves to the subset of $KA_1N_{-1,0}$ with $a$ satisfying

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{k-1} < 2r \leqslant a_k; \tag{6.3}$$

as we will show, the integral over this set contains the highest-order term of (6.2) as a function of $r$.

Lemma 6.1. *Suppose $a$ and $n$ are as in (6.1), and assume that $a$ satisfies (6.3). Then*

$$Q(na\mathbb{Z}^k, r) = 1 - 2ra_1 \dots a_{k-1}. \tag{6.4}$$

*Proof.* The proof follows that of the two-dimensional case. Write $\Lambda = na\mathbb{Z}^k$ and let $\rho_\Lambda : \mathbb{R}^k \to \mathbb{R}^k/\Lambda$ be the projection. Using (5.4), one can write

$$Q(\Lambda, r) = \lambda\Bigg(\rho_\Lambda\Bigg(\mathbb{R}^k \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\mathbf{v}, r)\Bigg)\Bigg) = \lambda\Bigg(naI^k \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\mathbf{v}, r)\Bigg), \tag{6.5}$$

where $I^k = [0,1] \times \cdots \times [0,1]$, and $\lambda$, as before, stands for both the normalized volume on $\pi^{-1}(\Lambda)$ and Lebesgue measure on $\mathbb{R}^k$. Equation (6.3) implies

$$\bigcup_{\mathbf{v}\in\Lambda} B(\mathbf{v}, r) = \mathbb{R}^{k-1} \times \bigcup_{\ell\in\mathbb{Z}} (\ell a_k - r, \ell a_k + r),$$

so that the measure of $naI^k \smallsetminus \bigcup_{\mathbf{v}\in\Lambda} B(\mathbf{v}, r)$ is the measure of a parallelepiped of dimensions $a_1, a_2, \dots, a_{k-1}$ and $a_k - 2r$, precisely as in the two-dimensional case. In fact the figure used in the proof of the two-dimensional case is still illustrative: just replace the squares with hypercubes, let the $y$-axis stand for the $a_k$-axis, and let the $x$-axis stand for the hyperplane $a_k = 0$. This yields (6.4). □

The next lemma will allow us to disregard the factor $K$ when estimating the integral (6.2).

Lemma 6.2. *For $\kappa \in K$ and $\Lambda \in X$,*

$$Q(\Lambda, k^{1/2} r) \leqslant Q(\kappa\Lambda, r) \leqslant Q(\Lambda, k^{-1/2} r).$$

[4] Our definition is that of [Moo66] post-composed with $g \mapsto g^{-1}$, since our action is on the right.

*Proof.* If $P \subset \mathbb{R}^k$ is a fundamental parallelipiped for the action of $\Lambda$ on $\mathbb{R}^k$, (6.5) gives

$$Q(\kappa\Lambda, r) = \lambda\biggl(\kappa P \smallsetminus \bigcup_{\mathbf{v}\in\Lambda}\{B(0,r)+\kappa\mathbf{v}\}\biggr) = \lambda\biggl(P \smallsetminus \bigcup_{\mathbf{v}\in\Lambda}\{\kappa^{-1}B(0,r)+\mathbf{v}\}\biggr).$$

But

$$B(0, rk^{-1/2}) \subset \kappa^{-1}B(0,r) \subset B(0, rk^{1/2}),$$

so the result follows from another application of (6.5). □

Now we are ready to write down the proof of (5.1) for the $(k>2)$-dimensional case.

*Proof of (5.1) for $k>2$.* Let $da, dn, d\kappa$ denote Haar measures on $A$, $N$, and $K$. Define

$$\eta : A \to \mathbb{R}, \quad a = \mathrm{diag}(a_1, \ldots, a_k) \mapsto \prod_{i<j} \frac{a_i}{a_j}.$$

Then the Iwasawa decomposition identifies $\mu_G$ with the product measure $\eta(a)\, d\kappa\, da\, dn$ (cf. [BM00, V.2.4]). Recall that we aim to bound the integral (6.2) from below. Let us write $n^a = ana^{-1}$ for $n \in N$, $a \in A$. By decomposing $\mu_G$ as above and restricting the domain of integration, we have

$$\begin{aligned}\int_{KA_1N_{-1,0}} Q(g\mathbb{Z}^k, r)\, d\mu_G(g) &= \int_{KA_1N_{-1,0}} Q(\kappa an\mathbb{Z}^k, r)\, d\kappa\, da\, dn\\ &= \int_{KA_1N_{-1,0}} Q(\kappa n^a a\mathbb{Z}^k, r)\, d\kappa\, da\, dn\\ &\geqslant \int_K \int_{N_{-1,0}} \int_{\{a\in A_1 : a_{k-1}\leqslant 2r\sqrt{k}\leqslant a_k\}} Q(\kappa n^a a\mathbb{Z}^k, r)\eta(a)\, da\, dn\, d\kappa.\end{aligned}$$

By Lemma 6.2, the latter integral is not smaller than

$$\int_K \int_{N_{-1,0}} \int_{\{a\in A_1 : a_{k-1}\leqslant 2r\sqrt{k}\leqslant a_k\}} Q(n^a a\mathbb{Z}^k, k^{1/2}r)\eta(a)\, da\, dn\, d\kappa,$$

and by Lemma 6.1 this is the same as

$$\int_K \int_{N_{-1,0}} \int_{\{a\in A_1 : a_{k-1}\leqslant 2r\sqrt{k}\leqslant a_k\}} (1 - 2rk^{1/2}a_1 \ldots a_{k-1})\eta(a)\, da\, dn\, d\kappa.$$

Since this integrand depends only on $a$, and the other factors have finite measure, it suffices to consider

$$\int_{\{a\in A_1 : a_{k-1}\leqslant 2r\sqrt{k}\leqslant a_k\}} (1 - 2rk^{1/2}a_1 \ldots a_{k-1})\eta(a)\, da.$$

Finally we identify $da$ with Lebesgue measure (up to a constant) on $\mathbb{R}^{k-1}$ via

$$\mathrm{diag}(a_1, \ldots, a_k) \mapsto (\log(a_1), \log(a_2), \ldots, \log(a_{k-1})),$$

see [BM00, V.2.3].[5] We are therefore left with the integral

$$\int_{b_1\leqslant b_2\leqslant\cdots\leqslant b_{k-1}\leqslant\log(2r\sqrt{k})\leqslant -\sum_{i=1}^{k-1} b_i} \biggl(1 - 2rk^{1/2}\exp\biggl[\sum_{i=1}^{k-1} b_i\biggr]\biggr) \exp\biggl[\sum_{i<j}(b_i - b_j)\biggr]\, d\lambda,$$

[5] Our identification is theirs composed with a linear isomorphism of $\mathbb{R}^{k-1}$.

where the $b_k$ occurring in the exponent of the second factor of the integrand must be understood to stand for $-\sum_{i=1}^{k-1} b_i$.

Now the challenge is not the integrand (which consists of nice exponential functions) but the domain of integration. Thankfully we only have to integrate over a piece of it, since we are interested in a lower bound. The piece we will consider is the following set:

$$\left\{(b_1,\ldots,b_{k-1}) : b_i \leqslant b_{i+1} \leqslant \frac{-\log(2r\sqrt{k})}{k-1}\ (1 \leqslant i \leqslant k-2)\right\}. \tag{6.6}$$

This set is clearly contained in the domain of integration above. Reordering the variables $x_i := b_{k-i}$, and using the identity $\sum_{i<j} x_i - x_j = \sum_{i=1}^{k-1} 2ix_i$, we can compute the integral of $Q(a\mathbb{Z}^k, \sqrt{k}r)$ over (6.6) as an iterated integral:

$$\int_{-\infty}^{-\log(2r\sqrt{k})/(k-1)} \int_{x_{k-1}}^{-\log(2r\sqrt{k})/(k-1)} \cdots \int_{x_2}^{-\log(2r\sqrt{k})/(k-1)} \times \left(e^{\sum_{i=1}^{k-1} 2ix_i} - 2re^{\sum_{i=1}^{k-1}(2i+1)x_i}\right) dx_1\, dx_2 \cdots dx_{k-1}. \tag{6.7}$$

It is easily seen by induction that for $2 \leqslant \ell \leqslant k-1$,

$$\int_{x_\ell}^{-\log(2r\sqrt{k})/(k-1)} \int_{x_{\ell-1}}^{-\log(2r\sqrt{k})/(k-1)} \cdots \int_{x_2}^{-\log(2r\sqrt{k})/(k-1)} \times \left(e^{\sum_{i=1}^{k-1} 2ix_i} - 2re^{\sum_{i=1}^{k-1}(2i+1)x_i}\right) dx_1\, dx_2 \cdots dx_{\ell-1}$$

is a sum of terms of the form

$$c(2r\sqrt{k})^{-m/(k-1)} e^{\sum_{i=\ell}^{k-1} p_i x_i}$$

where $c > 0$, $p_i$ are positive integers, and $m + \sum_{i=\ell}^{k-1} p_i = k(k-1)$. Indeed,

$$\begin{aligned}
&\int_{x_{\ell+1}}^{-\log(2r\sqrt{k})/(k-1)} c(2r\sqrt{k})^{-m/(k-1)} e^{\sum_{i=\ell}^{k-1} p_i x_i}\, dx_\ell \\
&= \frac{c}{p_\ell}(2r\sqrt{k})^{-(m+p_\ell)/(k-1)} \exp\left[\sum_{i=\ell+1}^{k-1} p_i x_i\right] \\
&\quad - \frac{c}{p_\ell}(2r\sqrt{k})^{-m/(k-1)} \exp\left[(p_\ell + p_{\ell+1})x_{\ell+1} + \sum_{i=\ell+2}^{k-1} p_i x_i\right],
\end{aligned}$$

so that we have only to notice that

$$(m + p_\ell) + \sum_{i=\ell+1}^{k-1} p_i = m + \left[(p_\ell + p_{\ell+1}) + \sum_{i=\ell+2}^{k-1} p_i\right] = m + \sum_{i=\ell}^{k-1} p_i = k(k-1)$$

from the induction hypothesis. Thus (6.7) is a sum of terms of the form

$$\begin{aligned}
\int_{-\infty}^{-\log(2r\sqrt{k})/(k-1)} c(2r\sqrt{k})^{-m/(k-1)} e^{p_{k-1}x_{k-1}}\, dx_{k-1} &= \frac{c}{p_{k-1}}(2r\sqrt{k})^{-(m+p_{k-1})/(k-1)} \\
&= \frac{c}{p_{k-1}}(2r\sqrt{k})^{-k(k-1)/(k-1)} \\
&= \frac{c}{p_{k-1}}(2r\sqrt{k})^{-k},
\end{aligned}$$

where we have used $m + p_{k-1} = k(k-1)$. Since the integral is positive, the sum of the coefficients must be positive, and the integral grows no more slowly than some multiple of $r^{-k}$. □



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## 7. Concluding remarks and open questions

### 7.1 The homogeneous problem

Here we return to the homogeneous case and discuss the approach to Question 1.4 suggested by the foregoing argument. Recall that $X = X_k = \mathrm{SL}_k(\mathbb{R})/\mathrm{SL}_k(\mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^k$. Define

$$\Delta_0 : X_k \to \mathbb{R}, \quad \Lambda \mapsto \log \inf_{\mathbf{v}\in\Lambda\smallsetminus 0} \|\mathbf{v}\|,$$

and for $A \in M_{m,n}$, define

$$\Lambda_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n} \in X_k,$$

where $k = m+n$. If we restrict the flow $g_t$ to $X_k$, it is not difficult to show[6] the following homogeneous version of Lemma 3.4.

Proposition 7.1. *Fix positive integers $m, n$, and let $\psi : [t_0, \infty) \to (0,1)$ be continuous and non-increasing. Let $z = z_\psi$ be as in Lemma 3.1. Then $A \in D_{m,n}(\psi)$ if and only if*

$$\Delta_0(g_s\Lambda_A) < z_\psi(s)$$

*for all sufficiently large $s$.*

This way Question 1.4 reduces to a shrinking target problem for the flow $(X, g_t)$, where the targets are super-level sets $\Delta_0^{-1}([z,\infty))$. But the family of super-level sets of $\Delta_0$ differs in important ways from the family of super-level sets of $\Delta$. In particular, by Minkowski's theorem, $\Delta_0^{-1}[z,\infty)$ is empty for $z > 0$. Hence the problem reduces to the case where the values $z_\psi(t)$ accumulate at 0, so that the targets shrink to the set $\Delta_0^{-1}(0)$. The latter set is a union of finitely many compact submanifolds of $X$ whose structure is explicitly described by the Hajós–Minkowski theorem (see [Cas71, § XI.1.3] or [Sha10, Theorem 2.3]). In particular, the function $\Delta_0$ is not DL, and Theorem 4.4 is not applicable. Other approaches to shrinking target problems on homogeneous spaces [Kel17, KY19, KZ18, Mau06] also do not seem to be directly applicable.

On the other hand, the one-dimensional case ($m = n = 1$) has been completely settled in [KW18]. In particular, the following zero–one law has been established.

Theorem 7.2 [KW18, Theorem 1.8]. *Let $\psi : [t_0,\infty) \to \mathbb{R}_+$ be non-increasing, and suppose the function $t \mapsto t\psi(t)$ is non-decreasing and*

$$t\psi(t) < 1 \quad \textit{for all } t \geqslant t_0. \tag{7.1}$$

*Then if*

$$\sum_i \frac{-\log(1-i\psi(i))(1-i\psi(i))}{i} = \infty \quad (\textit{respectively} < \infty), \tag{7.2}$$

*then the Lebesgue measure of $D_{1,1}(\psi)$ (respectively, of $D_{1,1}(\psi)^c$) is zero.*

The proof is based on the observation that the condition $\alpha \in D_{1,1}(\psi)$ can be explicitly described in terms of the continued fraction expansion of $\alpha$. However, this phenomenon is inherently one-dimensional, and new ideas are needed to settle the general case.

[6] See [KW18, Proposition 4.5], though notice that the function used there differs from $\Delta_0$ by a minus sign.

### 7.2 Hausdorff dimension

A sequel [HKWW18] to the paper [KW18] computes the Hausdorff dimension of limsup sets $D_{1,1}(\psi)^c$, and, more generally, establishes zero–infinity laws for the Hausdorff measure of those sets. For example, it is proved there that

$$\dim(D(\psi)^c) = \frac{2}{2+\tau} \quad \text{when } \psi(t) = \frac{1-at^{-\tau}}{t} \ (a>0, \tau>0).$$

One can ask similar questions for higher-dimensional versions, both in homogeneous and inhomogeneous settings. Even the $m=n=1$ case of the inhomogeneous problem is open.

### 7.3 Singly versus doubly metric problems

The main result of the present paper computes Lebesgue measure of the set $\widehat{D}_{m,n}(\psi) \subset M_{m,n} \times \mathbb{R}^m$. As often happens in inhomogeneous Diophantine problems, one can fix either $A$ or $\mathbf{b}$ and ask for the Lebesgue (or Hausdorff) measure of the corresponding slices of $\widehat{D}_{m,n}(\psi)$. It seems plausible that the convergence/divergence of the same series (1.6) is responsible for a full/zero measure dichotomy for slices

$$\{A \in M_{m,n} : (A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)\}$$

for any fixed $\mathbf{b} \notin \mathbb{Z}^m$. On the other hand, the Lebesgue measure of the set

$$\{\mathbf{b} \in \mathbb{R}^m : (A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)\}$$

for a fixed $A \in M_{m,n}$ seems to depend heavily on Diophantine properties of $A$. For example, if $A$ has rational entries, then $(A, \mathbf{b})$ is not in $\widehat{D}_{m,n}(\psi)$ whenever $\mathbf{b} \notin \mathbb{Q}^m$ and $\psi(T) \to 0$ as $T \to \infty$. And on the other end of the approximation spectrum, if $A$ is badly approximable it is easy to see that there exists $C > 0$ such that for all $\mathbf{b} \in \mathbb{R}^m$, $(A, \mathbf{b})$ belongs to the (null) set $\widehat{D}_{m,n}(C\psi_1)$. Indeed, by the classical Dani correspondence, $A$ is badly approximable if and only if the trajectory $\{g_t\Lambda_A : t > 0\}$ is bounded in $X_k$, which is the case if and only if $\{g_t\Lambda_{A,\mathbf{b}} : t > 0\}$ is bounded in $\widehat{X}_k$ for any $\mathbf{b} \in \mathbb{R}^m$. Thus the claim follows in view of Lemma 2.2. It would be interesting to describe, for a given arbitrary non-increasing function $\psi$, explicit Diophantine conditions on $A \in M_{m,n}$ guaranteeing that $(A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)$ for all (or almost all) $\mathbf{b} \in \mathbb{R}^m$.

### 7.4 Eventually always hitting

Finally, let us connect our results on improving the inhomogeneous Dirichlet theorem with a shrinking target property introduced recently by Kelmer [Kel17]. We start by setting some notation. Let $\alpha$ be a measure-preserving $\mathbb{Z}^n$-action on a probability space $(Y, \nu)$. For any $N \in \mathbb{N}$ denote

$$D_N := \{\mathbf{q} \in \mathbb{Z}^n : \|\mathbf{q}\| \leqslant N\}$$

(here, as before, $\|\cdot\|$ stands for the supremum norm). Then, given a nested family $\mathcal{B} = \{B_N : N \in \mathbb{N}\}$ of subsets of $Y$, let us say that the $\alpha$-orbit of a point $x \in Y$ *eventually always hits* $\mathcal{B}$ if $\alpha(D_N)x \cap B_N \neq \emptyset$ for all sufficiently large $N \in \mathbb{N}$. Following [Kel17], denote by $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ the set of points of $Y$ with $\alpha$-orbits eventually always hitting $\mathcal{B}$. This is a liminf set with a rather complicated structure. In [Kel17] sufficient conditions for sets $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ to be of full measure were found for unipotent and diagonalizable actions $\alpha$ on hyperbolic manifolds. Namely, it was shown[7]

[7] Note that Kelmer considered the eventually always hitting property for forward orbits, that is, with sets $D_N^+ := \{\mathbf{q} \in \mathbb{Z}^n : q_i \geqslant 0, \|\mathbf{q}\| \leqslant N\}$ in place of $D_N$.

(see [Kel17, Theorem 22 and Proposition 24]) that for rotation-invariant monotonically shrinking families $\mathcal{B}$, $\nu(\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B}))=1$ if the series

$$\sum_j \frac{1}{2^{nj}\nu(B_{2^j})} \tag{7.3}$$

converges. See also [KY19] for some extensions to actions on homogeneous spaces of semisimple Lie groups. However, to the best of the authors' knowledge, there are no non-trivial examples of measure-preserving systems for which necessary and sufficient conditions for sets $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ to be of full measure exist in the literature.

Now, given $A\in M_{m,n}$, take $Y=\mathbb{T}^m$ with normalized Lebesgue measure $\nu$ and consider the $\mathbb{Z}^n$-action

$$\mathbf{x}\mapsto \alpha(\mathbf{q})\mathbf{x} := \mathbf{x}+A\mathbf{q} \mod \mathbb{Z}^m \tag{7.4}$$

on $Y$ (generated by $n$ independent rotations of $\mathbb{T}^m$ by the column vectors of $A$). Also fix $\mathbf{y}\in Y$ and a non-increasing sequence $\{r(N):N\in\mathbb{N}\}$ of positive numbers, and consider the family $\mathcal{B}$ of open balls

$$B_N := \{\mathbf{x}\in\mathbb{T}^m : \|\mathbf{x}-\mathbf{y}\| < r(N)\}. \tag{7.5}$$

Then it is easy to see that $\mathbf{x}\in\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ if and only if for all sufficiently large $N\in\mathbb{N}$ there exist $\mathbf{q}\in\mathbb{Z}^n$ and $\mathbf{p}\in\mathbb{Z}^m$ such that

$$\|\mathbf{q}\| < N+1 \quad\text{and}\quad \|\mathbf{x}+A\mathbf{q}-\mathbf{p}-\mathbf{y}\| < r(N). \tag{7.6}$$

Here and hereafter $\alpha$ and $A$ are related via (7.4). A connection to the improvement of the inhomogeneous Dirichlet theorem is now straightforward. Indeed, from Theorem 1.6 one can derive the following corollary.

Corollary 7.3. *Fix $\mathbf{y}\in\mathbb{T}^m$ and let $\mathcal{B}=\{B_N:N\in\mathbb{N}\}$ be as in (7.5), where $\{r(N):N\in\mathbb{N}\}$ is a non-increasing sequence of positive numbers. Then for Lebesgue-almost every $A\in M_{m,n}$ the set $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ has zero (respectively, full) measure provided the sum (7.3) diverges (respectively, converges).*

*Proof.* Extend $r(\cdot)$ to a non-increasing continuous function on $\mathbb{R}_+$ in an arbitrary way (for example, piecewise linearly). Then, similarly to the observation made after (1.4), one can notice that $\mathbf{x}\in\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ if and only if the system (7.6) is solvable in integers $\mathbf{p},\mathbf{q}$ for all sufficiently large $N\in\mathbb{R}_+$. The latter happens if and only if the pair $(A,\mathbf{x}-\mathbf{y})$ belongs to $\widehat{D}_{m,n}(\psi)$, where

$$\psi(T) := r(T^{1/n}-1)^m.$$

In view of Theorem 1.6, the divergence of the sum

$$\begin{aligned}\sum_j \frac{1}{\psi(j)j^2} &= \sum_j \frac{1}{r(j^{1/n}-1)^m j^2} \asymp \int \frac{dx}{r(x^{1/n}-1)^m x^2} \asymp \int \frac{(y+1)^{n-1}\,dy}{r(y)^m (y+1)^{2n}}\\ &\asymp \int \frac{dy}{r(y)^m y^{n+1}} \asymp \int \frac{2^z\,dz}{r(2^z)^m 2^{z(n+1)}} \asymp \sum_j \frac{1}{r(2^j)^m 2^{nj}} \asymp \sum_j \frac{1}{2^{nj}\nu(B_{2^j})}\end{aligned}$$

implies that $\widehat{D}_{m,n}(\psi)$ has measure zero. Hence for almost every $A$ the set $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ is null. Similarly, the convergence of (7.3) implies that $\widehat{D}_{m,n}(\psi)$ is conull. Thus for Lebesgue-generic $A$ the set $\mathcal{A}^{\alpha}_{\mathbf{ah}}(\mathcal{B})$ has full measure. □



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