TRANSLATIVITY FOR STRONG BOREL SUMMABILITY

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1. Introduction. It is an obvious property of convergence that $\lim_{n\to\infty} s_n = s$ implies that $\lim_{n\to\infty} s_{n+k}$ exists and equals s for $k = -1$ (left translativity) and for $k = 1$ (right translativity). Not so for summability.

G.H. Hardy pointed out in 1903 [cf. 3, p.183 (Theorem 127), p.196] that summation by Borel's exponential means is translative to the left, but not to the right. However, for sequences $\{s_n\}$ whose rate of growth with n is restricted suitably, the Borel method is also right-translative. This was shown to be true when $s_n = O(n^K)$, K an arbitrary fixed quantity, by V. Garten [2], and under more general circumstances by J. Karamata [4] and D. Gaier[1].

Summability by Borel's exponential means (i.e., $B\text{-}\lim s_n = s$) is defined by

$$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{s_n - s}{n!} x^n = 0 . \tag{1}$$

A discussion of this and related methods is found, e.g., in [3, Chapters 8 and 9].

On this concept can be superimposed that of "strong summability" in the usual way. A sequence $\{\sigma_n\}$ will be said to be "strongly summable by Borel's exponential method, with (positive) index k, to the value σ," if

(2) $$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{|\sigma_n - \sigma|^k}{n!} x^n = 0.$$

This will be written

(3) $$S_k B\text{-}\lim \sigma_n = \sigma.$$

The $S_k B$ method can be shown to be translative both to the right and to the left. That it is translative to the left follows from the corresponding result for Borel summability [3, p.183 (Theorem 127)]. What remains then is to prove:

THEOREM. If $S_k B\text{-}\lim \sigma_n = \sigma$, then $S_k B\text{-}\lim \sigma_{n+1}$ exists and equals σ.

Two proofs will be provided: A direct elementary one, based on the law of the mean for derivatives (§ 2), and another obtained by showing that the hypothesis $S_k B\text{-}\lim \sigma_n = \sigma$ implies that $s_n = |\sigma_n - \sigma|^k = \underline{O}(n^K)$, in fact, $\underline{o}(n^{1/2})$, reducing the above Theorem to a special case of Garten's [2] (§ 3).

The main point of this note is really the direct proof, because of its simplicity and elementary character, avoiding the delicate calculations of [2] and the function-theoretic methods of [1] and [4].

2. <u>Direct Proof of the Theorem</u>. The result will follow at once from the second lemma below, itself a consequence of the mean-value theorem for derivatives.

LEMMA 1. If $G'(x) > 0$, $0 < x < \infty$; if $G'(x)$ is a non-decreasing function of x; and if

(4) $$G(x+1) - G(x) = \underline{o}(e^x) \text{ as } x\to\infty,$$

then $G'(x) = \underline{o}(e^x)$ as $x\to\infty$.

<u>Proof.</u> The mean-value theorem establishes the existence of ξ, $x < \xi < x+1$, such that $G'(\xi) = G(x+1) - G(x)$. Hence

$$0 < e^{-x} G'(x) \le e^{-x} G'(\xi) = e^{-x}\{G(x+1) - G(x)\} \to 0, \text{ as } x\to\infty.$$

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A special case of the foregoing is what is really required for the proof of the Theorem:

LEMMA 2. If $G'(x)$ is a positive non-decreasing function of x, $0 < x < \infty$, and if $G(x) = \underline{o}(e^x)$, as $x \to \infty$, then $G'(x) = \underline{o}(e^x)$ as $x \to \infty$.

Proof. It suffices to note that

$$\frac{G(x+1) - G(x)}{e^x} = e\,\frac{G(x+1)}{e^{x+1}} - \frac{G(x)}{e^x} = \underline{o}(1),$$

and apply Lemma 1.

The Theorem follows from Lemma 2 on defining

$$G(x) = \sum_{n=0}^{\infty} \frac{|\sigma_n - \sigma|^k}{n!} x^n,$$

since

$$G'(x) = \sum_{1}^{\infty} \frac{|\sigma_n - \sigma|^k}{(n-1)!} x^{n-1} = \sum_{0}^{\infty} \frac{|\sigma_{n+1} - \sigma|^k}{n!} x^n.$$

3. Reduction to Garten's Theorem. Alternatively, the Theorem of this note can be subsumed under Garten's. To this end, define $s_n = |\sigma_n - \sigma|^k$, so that $\{s_n\}$ is summable to 0 by Borel's exponential means, $s_n \geq 0$, $n = 0, 1, \ldots$.

We need

LEMMA 3. If $s_n \geq 0$ and $B\text{-}\lim s_n = 0$, then $s_n = \underline{o}(\sqrt{n})$, $n \to \infty$.

Preliminary remark. In the proof, use is made of the inequality

$$n!\, e^n n^{-n-1/2} \leq e, \quad n = 1, 2, \ldots .$$

This is an elementary result established, e.g., in the course of the proof of Lemma 16.2 of [7, p. 384], where it is shown that the

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left member decreases as n increases.

Proof of Lemma 3. Obviously, for $x > 0$,

$$\frac{s_n}{n!} x^n \leq \sum_{n=0}^{\infty} \frac{s_n}{n!} x^n, \quad n = 0, 1, 2, \ldots,$$

since $s_n \geq 0$. Putting $x = n$, it follows that

$$\frac{s_n}{n!} n^n \leq \epsilon_n e^n, \text{ where } \epsilon_n \to 0 \text{ as } n \to \infty,$$

since B-lim $s_n = 0$, i.e., $\sum_0^{\infty} (s_n/n!) x^n = \underline{o}(e^x)$.

From the preliminary remark, we have now that

$$0 \leq s_n \leq \epsilon_n e^n n! \, n^{-n} \leq \epsilon_n e\sqrt{n}.$$

This proves Lemma 3.

The Theorem of this note now follows from Garten's result, since the $S_k B$ summability of σ_n to σ is equivalent to the B-summability of s_n to 0, with $s_n \geq 0$, and Lemma 3 implies (in view of Garten's result) that B-lim s_{n+1} exists and is 0.

4. Additional Remarks. Some miscellaneous comments follow.

(a) The definition (2) of strong Borel summability does not appear in the general literature, so far as I know, but it was given in the lectures of Otto Szász at the University of Cincinnati in 1936-37 or 1937-38.

(b) The method, clearly regular, is stronger than convergence. The divergent sequence $\{\sigma_n\}$, where $\sigma_n = 1$ for $n = m^3$, $m = 0, 1, \ldots, \sigma_n = 0$ otherwise, is $S_k B$ summable to $\sigma = 0$ for all indices k.

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To see this, let $s_n = |\sigma_n - \sigma|^k = \sigma_n$. Then $s_o + \ldots + s_n = [n^{1/3}] + 1$, where $[x]$ denotes, as usual, the largest integer $\leq x$. Hence the $(C, 1)$ means of $\{s_n\}$ are

$$0 < \frac{s_o + \ldots + s_n}{n+1} = \frac{[n^{1/3}] + 1}{n+1} = \underline{O}(n^{-2/3}) = \underline{o}(n^{-1/2}).$$

The Borel summability of $\{s_n\}$ to 0 (and hence the $S_k B$ summability of $\{\sigma_n\}$ to $\sigma = 0$), then follows from a theorem of Hardy [3, p. 213, (Theorem 149)].

Another divergent sequence having this property is $\sigma_n = 1$, $n = 2^m$, $m = 0, 1, \ldots, \sigma_n = 0$ otherwise, as may be seen, e.g., from a result of G. Pólya [5], that

$$\lim_{x \to \infty} \sqrt{x}\, e^{-x} \sum_{m=0}^{\infty} \frac{x^{2^m}}{2^m!} = \frac{1}{\sqrt{2\pi}},$$

as well as from the aforementioned theorem of Hardy.

An example of an unbounded divergent sequence which is $S_k B$-summable is $\sigma_n = n^{1/3}$, $n = m^{12}$, $m = 0, 1, \ldots, \sigma_n = 0$ otherwise, $\sigma = 0$, when $k = 1$. Obvious modifications lead to analogous sequences for other values of k.

(c) In the above examples, $S_k B\text{-}\lim \sigma_n = \sigma$ implies

$$\liminf_{n \to \infty} \sigma_n = \sigma. \tag{5}$$

This is a common property of strong summability methods. To establish it for all sequences summable by $S_k B$ methods is quite straightforward.

We may write $s_n = |\sigma_n - \sigma|^k$ so that $B\text{-}\lim s_n = 0$, and suppose that $s_n > \epsilon > 0$ for all $n > N_\epsilon$. Then

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$$e^{-x}\sum_{n=0}^{\infty}\frac{s_n}{n!}x^n = e^{-x}\sum_{n=0}^{N_\epsilon}\frac{s_n}{n!}x^n + e^{-x}\sum_{N_\epsilon+1}^{\infty}\frac{s_n}{n!}x^n$$

$$> e^{-x}\sum_{n=0}^{N_\epsilon}\frac{s_n}{n!}x^n + \epsilon\, e^{-x}\sum_{N_\epsilon+1}^{\infty}\frac{x^n}{n!}$$

$$= e^{-x}\sum_{n=0}^{N_\epsilon}\frac{s_n}{n!}x^n + \epsilon - \epsilon e^{-x}\sum_{0}^{N_\epsilon}\frac{x^n}{n!}$$

$$= \epsilon + \underline{o}(1), \quad x \to \infty,$$

a contradiction.

(d) Lemma 3 can be sharpened (as is clear from the proof given) to the following:

If (i) $s_n \geq 0$, (ii) $e^{-x}\sum\frac{s_n}{n!}x^n = \omega(1/x)$, then $s_n \leq \omega(1/n)\, e\sqrt{n}$, $n = 1, 2, \ldots$; thus, $s_n \to 0$ if $w(1/n) = \underline{o}(1/\sqrt{n})$.

The inequality gives the correct order of magnitude for s_n, as may be seen from Pólya's function in (b). There

$$\omega(1/x) = \frac{1}{\sqrt{2\pi x}} + \underline{o}\left(\frac{1}{\sqrt{x}}\right) \text{ as } x \to \infty .$$

Thus, the inequality gives

$$s_n \leq \frac{e}{\sqrt{2\pi}} + \underline{o}(1),$$

which is the proper order, since infinitely many s_n equal 1 .

(e) The non-translativity of Borel's exponential method is what underlies O. Szász's example of a pair of regular summability methods T_1 and T_2 having the property that $T_1 \cdot T_2$ does not include T_1, i.e., such that the T_1 transform of the

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T_2 transform of a series need not converge even if the T_1 transform of the series converges [6, § 6, pp.81-82]. This becomes particularly clear if his example is simplified by taking, as he does, T_1 to be Borel's exponential means, but replacing his binary transformation given by $T_2(s_n) = \frac{1}{2}(s_n + s_{n+1})$ by the translation $T_2(s_n) = s_{n+1}$.

A still simpler example of the phenomenon described by Szász is provided by R.P. Agnew [Math. Reviews, vol. 15 (1954), p.26] in his report on [6].

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