ON PARTITIONS OF AN EQUILATERAL TRIANGLE

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Introduction. Let T denote a closed unit equilateral triangle. For a fixed integer n, let d_n denote the infimum of all those x for which it is possible to partition T into n subsets, each subset having a diameter not exceeding x. We recall that the diameter of a plane set A is given by

$$d(A) = \sup_{a,b \in A} \rho(a, b)$$

where $\rho(a, b)$ is the Euclidean distance between a and b.

In this note we determined d_n for some small values of n. Typical values of d_n are given in Table I. These values were obtained by three methods. As would be expected, as the value of n increases, the complexity of the argument needed to obtain d_n also increases. We begin with the simplest case.

The box-principle technique. Consider the regular arrangement of $(n + 1)(n + 2)/2$ points within T shown in Figure 1. Since any partition of T

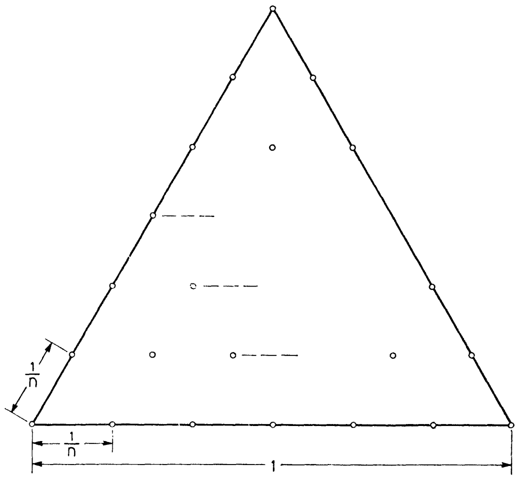

FIGURE 1

into $k < (n + 1)(n + 2)/2$ subsets must have some subset which contains at least two of these points (and consequently has diameter $\geqslant 1/n$), we see that $d_k \geqslant 1/n$. On the other hand, by joining these points in the natural way (Figure 2) we obtain a partition of T into n^2 sets of diameter $1/n$. (In this

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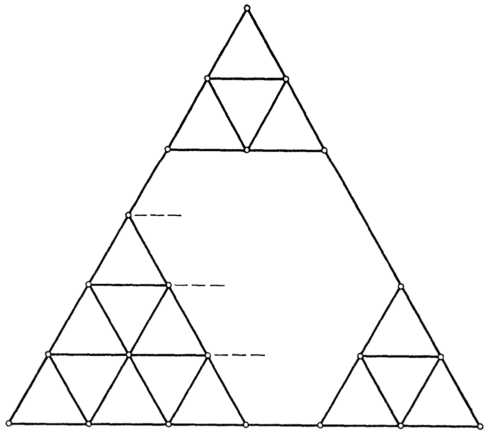

FIGURE 2

paper it will never make any difference to which of the sets a limit point of two or more sets belongs.) Thus, in general we have $d_{n^2} \leqslant 1/n$. Hence, if

(1) $$n^2 \leqslant k < (n+1)(n+2)/2,$$

then

(2) $$d_k = 1/n.$$

From this we obtain:

$$d_1 = d_2 = 1, \qquad d_4 = d_5 = \tfrac{1}{2}, \qquad d_9 = \tfrac{1}{3}.$$

Unfortunately (or fortunately, depending upon one's point of view), (1) cannot be satisfied for $n > 3$ and this simple argument yields no further values of d_n.

A slight variation of this argument gives us d_3. We consider the four points arranged in T as shown in Figure 3 (C is located at the centroid of T). Since the

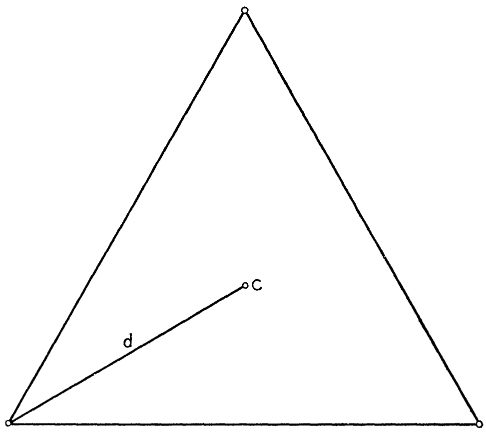

FIGURE 3

distance between any two of the four points is at least $d = 1/\sqrt{3}$, then in any partition of T into three sets, at least one of the sets must have diameter $\geqslant d$ (thus showing that $d_3 \geqslant d$). On the other hand, the partition given in Figure 4 (formed by joining C to the three side-midpoints) has no set with diameter $>d$. Consequently $d_3 = 1/\sqrt{3}$.

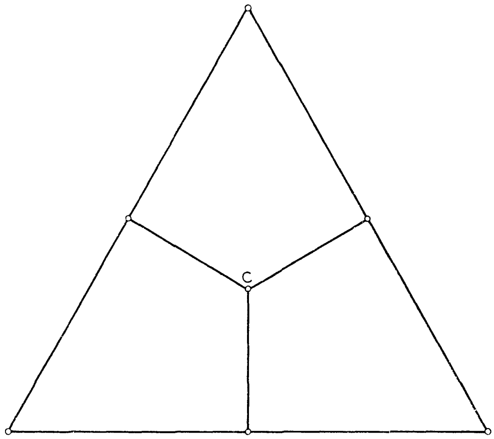

FIGURE 4

With this value of d_n the usefulness of this simple box principle is exhausted. To obtain further values of d_n, we shall have to work a little harder.

The cycling technique. We come now to a method which will eventually give us the values of d_n for $n = 6, 7, 8$, and 12. To illustrate it, we first consider the case $n = 7$.

With reference to Figure 5, we note that there exists (by continuity) a real

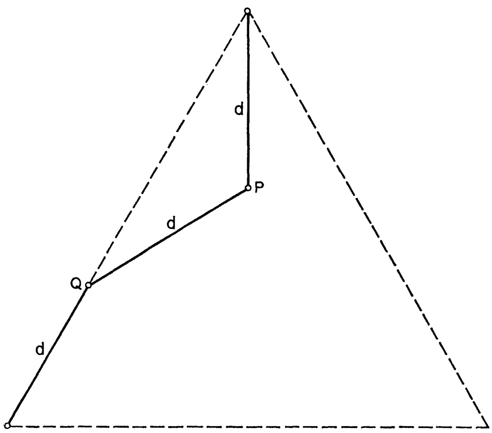

FIGURE 5

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number d such that if we go down the altitude of T a distance d (to the point P) and then go across to a side by a chord of length d (to the point Q), we are then exactly a distance d away from the nearer base vertex. (The fact that d happens to be equal to $1/(1 + \sqrt{3})$ is incidental.) Of course, P could have been located on *any* altitude and Q could have been chosen on *either* side. In Figure 6 we have drawn in all nine points we would obtain by taking all possible choices of P and

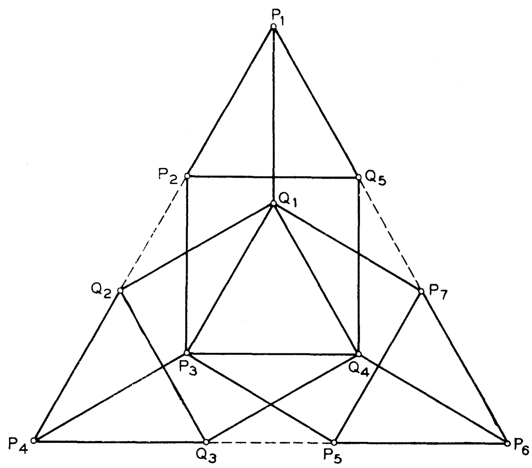

FIGURE 6

Q. The solid line segments have been drawn in between points whose mutual separation is d. The facts that $\rho(Q_1, Q_2) = d$, $\rho(P_1, P_2) = d$, etc., are immediate.

Let us now assume that it is possible to partition T into seven sets A_i, $1 \leqslant i \leqslant 7$, each of which has diameter $<d$. All the points P_i, Q_j must belong to at least one of the A_k. Since the points P_i, $1 \leqslant i \leqslant 7$, all have mutual distance $\geqslant d$, they must all belong to different A_k. We can assume without loss of generality that $P_i \in A_i$, $1 \leqslant i \leqslant 7$. Now $\rho(Q_1, P_i) \geqslant d$ for $i \neq 2$ so that we must have $Q_1 \in A_2$. Similarly we find that $Q_2 \in A_3$, since $\rho(Q_2, P_i)$, $i = 1, 4, 5, 6, 7$ and $\rho(Q_2, Q_1)$ are all $\geqslant d$. Continuing in this way, we deduce that $Q_3 \in A_5$ and $Q_4 \in A_7$. We now note that each A_i contains a point which is a distance $\geqslant d$ from Q_5 and hence Q_5 cannot belong to any A_i. This is a contradiction. Hence we must have $d_7 \geqslant d$.

Now we wish to show that $d_7 \leqslant d$. Figure 7 gives us the surprising conclusion that $d_6 \leqslant d$. Therefore we can conclude that $d_6 = d_7 = d$.

To handle the case $n = 8$, a similar though slightly more complex argument is used. For this we let d now denote that unique distance determined by Figure 8. In this figure, the point P lies on the altitude of T. As before, by reflection and rotation, this configuration of points determines 24 points (counting the three vertices) in T shown in Figure 9. (The lines indicating the

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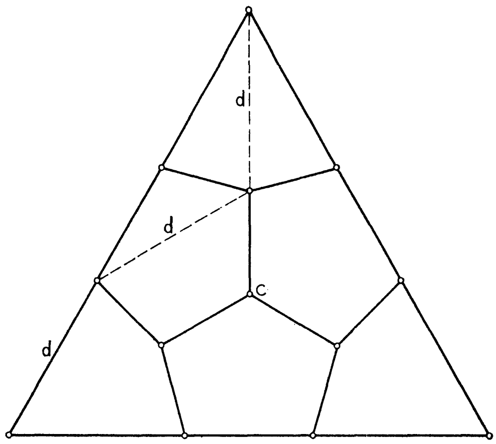

FIGURE 7

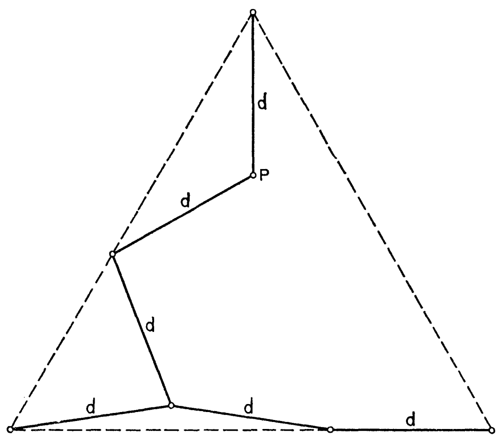

FIGURE 8

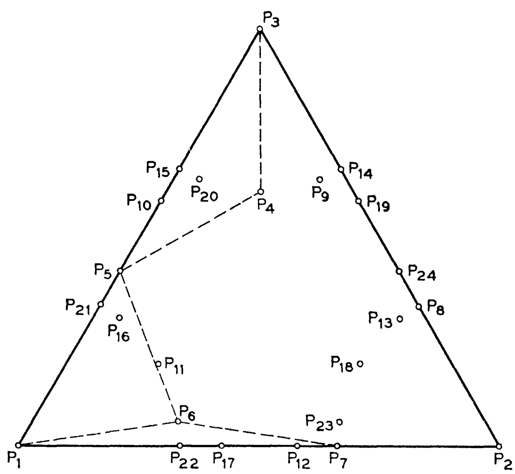

FIGURE 9

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interpoint distances of d have been omitted.) The points have been labelled so that $\rho(P_k, P_{k+1}) = d$ for $k \geqslant 3$ while $\rho(P_k, P_{k+1}) = 1$ for $k = 1, 2$.

Let us now assume that it is possible to partition T into eight sets, A_i, $1 \leqslant i \leqslant 8$, each having diameter $<d$. We can suppose without loss of generality that $P_i \in A_i$ for $1 \leqslant i \leqslant 8$. As in the case of $n = 7$, it now follows by geometric arguments that we must have:

$$\begin{array}{llll} P_9 \in A_4, & P_{10} \in A_5, & P_{11} \in A_6, & P_{12} \in A_7, \\ P_{13} \in A_8, & P_{14} \in A_4, & P_{15} \in A_5, & P_{16} \in A_6, \\ P_{17} \in A_7, & P_{18} \in A_8, & P_{19} \in A_4, & P_{20} \in A_5, \\ P_{21} \in A_6, & P_{22} \in A_7, & P_{23} \in A_8, & P_{24} \in \,? \;. \end{array}$$

Since P_{24} is now too far from some point in each of the A_i, we have obtained a contradiction. We conclude that some A_i must have diameter $\geqslant d$. Hence $d_8 \geqslant d$.

A partition of T for which the maximum diameter of any set is $\leqslant d$ is given in Figure 10. (The reader will have no difficulty in verifying that all the sets have

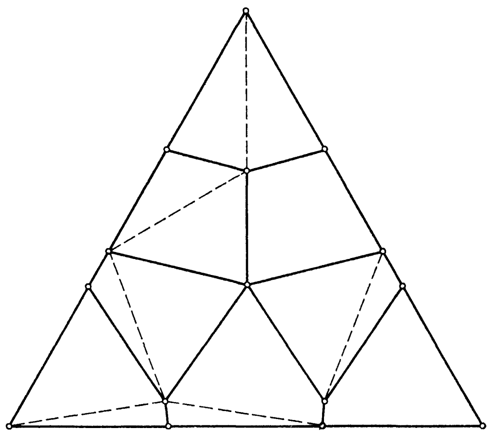

FIGURE 10

diameter d.) Since d is readily determined to be equal to

$$\frac{2}{1 + \sqrt{3} + \sqrt{(6\sqrt{3})}},$$

we have proved that

$$d_8 = \frac{2}{1 + \sqrt{3} + \sqrt{(6\sqrt{3})}}.$$

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The final value of d_n we obtain by this technique is d_{12}. As in the two preceding cases we first define d to be that distance determined by Figure 11. The

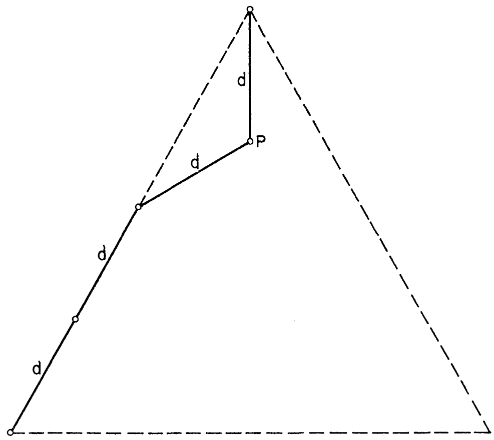

FIGURE 11

point P lies on the altitude of T. It happens that $d = 2 - \sqrt{3}$, which is just the d that satisfies Figure 12 (where the points Q and Q' lie on the altitude of T).

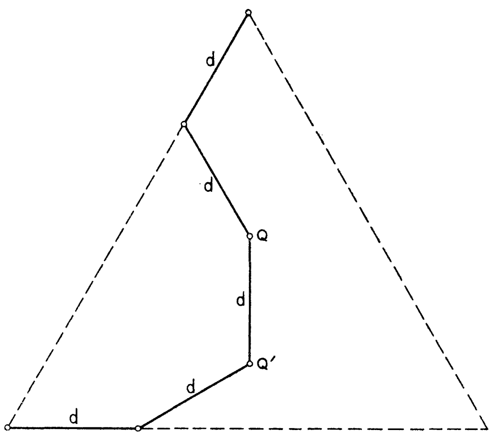

FIGURE 12

Generating all the points in Figure 11 and Figure 12, we obtain Figure 13, which has 24 of these points (including the three vertices). The point C is the centroid of T. As before, the points have been labelled so that $\rho(P_k, P_{k+1}) \geqslant d$ for $1 \leqslant k \leqslant 23$.

Suppose now that T can be decomposed into 12 sets A_i, $1 \leqslant i \leqslant 12$, each having diameter $<d$. We can assume without loss of generality that $P_i \in A_i$,

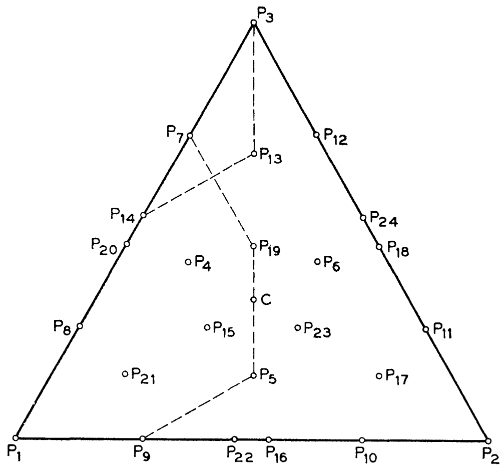

FIGURE 13

for $1 \leqslant i \leqslant 12$. There are two possibilities for P_{13}, namely, $P_{13} \in A_7$ and $P_{13} \in A_{12}$. However, because of the symmetry of the configuration, we can assume that $P_{13} \in A_7$. The remainder of the argument is straightforward:

$$P_{14} \in A_4, \quad P_{15} \in A_5, \quad P_{16} \in A_{10}, \quad P_{17} \in A_{11}, \quad P_{18} \in A_6,$$
$$P_{19} \in A_4, \quad P_{20} \in A_8, \quad P_{21} \in A_9, \quad P_{22} \in A_5, \quad C \in ?.$$

It is now easy to see that the point C is a distance $\geqslant d$ from some point in each of the A_i, $1 \leqslant i \leqslant 12$, which is a contradiction. Hence, we have $d_{12} \geqslant d$. Figure 14 shows a partition for which d is the maximum set diameter. This concludes the proof that $d_{12} = 2 - \sqrt{3}$.

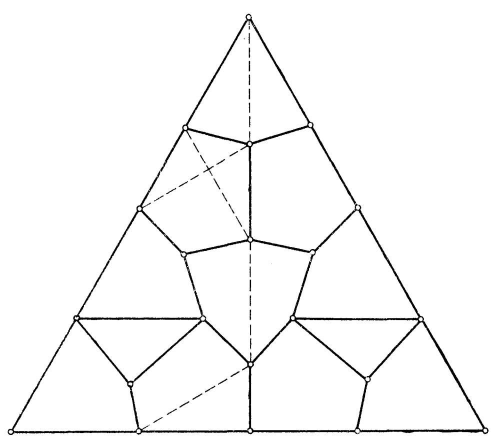

FIGURE 14

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The inscribed polygon lemma. Before we can determine any other values of d_n, we first need a lemma. It deals with the number of sets to which the *boundary* of T can belong.

Consider the figure $T(d)$ shown in Figure 15. For a fixed $d \leqslant \frac{1}{4}$, $T(d)$ is

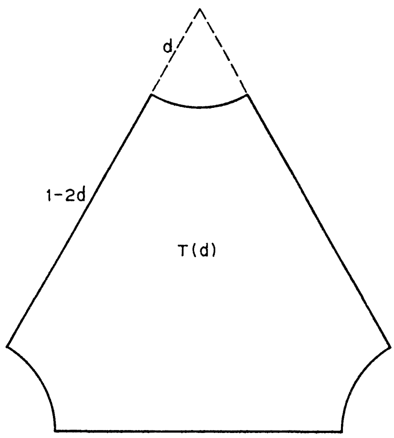

FIGURE 15

formed from T by deleting circular sectors of radius d from each of the three vertices of T. For an arbitrary point P on the upper arc of $T(d)$, let us determine the points $Q_1, \ldots, Q_n, Q'_1, \ldots, Q'_{n'}, R, R', S$, and S' on the boundary of $T(d)$ as shown in Figure 16. These points are determined simply by starting at P and marking off consecutive chords of length d along the boundary of $T(d)$ until the

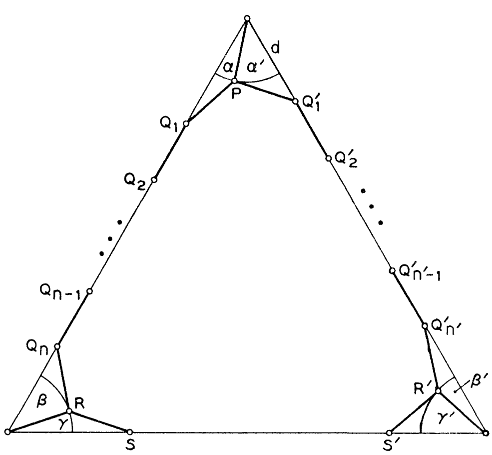

FIGURE 16

base of $T(d)$ is reached. We assume that $0 \leqslant \beta, \beta' \leqslant \pi/3$. From the figure we have the immediate relations:

$$\alpha + \alpha' = \pi/6, \tag{3}$$

$$2d \cos \alpha + (n - 1)d + 2d \cos \beta = 1, \tag{4}$$

$$2d \cos \alpha' + (n' - 1)d + 2d \cos \beta' = 1. \tag{4'}$$

An easy calculation shows that the distance $\rho(S, S')$ between the points S and S' is given by

$$\begin{aligned} \rho(S, S') &= 1 - 2d(\cos \gamma + \cos \gamma') \\ &= 1 - 2d\{\cos(\pi/3 - \beta) + \cos(\pi/3 - \beta')\} \\ &= 1 - d(\cos \beta + \cos \beta') - \sqrt{3}\, d(\sin \beta + \sin \beta') \\ &= d\{\cos \alpha + \cos \alpha' - \sqrt{3} \sin \beta - \sqrt{3} \sin \beta' + \tfrac{1}{2}(n + n') - 1\}. \end{aligned} \tag{5}$$

By (3), (4), and (4′), we can regard $\rho(S, S')$ as a function of α and write it as $F(\alpha)$. We now state a lemma.

LEMMA. *Suppose* $n = n'$; *let* I *denote the set of values of* α *for which* $0 \leqslant \alpha \leqslant \pi/3$ *and* $0 \leqslant \beta, \beta' \leqslant \pi/3$. *Then* $F(\alpha)$ *assumes a unique minimum on* I *for* $\alpha = \pi/6$.

Proof. It is easily seen that I is an interval centred at $\pi/6$. The derivative of F on I is given by:

$$F'(\alpha) = d\left(-\sin \alpha + \sin \alpha' - \sqrt{3} \cos \beta \frac{d\beta}{d\alpha} - \sqrt{3} \cos \beta' \frac{d\beta'}{d\alpha}\right). \tag{6}$$

Since (3), (4), and (4′) imply

$$\frac{d\beta}{d\alpha} = -\frac{\sin \alpha}{\sin \beta}, \qquad \frac{d\beta'}{d\alpha} = \frac{\sin \alpha'}{\sin \beta'},$$

we have

$$F'(\alpha) = d\{\sin \alpha(\sqrt{3} \cot \beta - 1) - \sin \alpha'(\sqrt{3} \cot \beta' - 1)\}. \tag{7}$$

Now, $0 \leqslant \beta, \beta' \leqslant \pi/3$ imply

$$\sqrt{3} \cot \beta - 1 \geqslant 0, \qquad \sqrt{3} \cot \beta' - 1 \geqslant 0,$$

while $0 \leqslant \alpha \leqslant \pi/3$ implies

$$0 \leqslant \sin \alpha, \qquad 0 \leqslant \sin \alpha'.$$

Also we note that $\alpha \geqslant \alpha'$ implies $\beta \leqslant \beta'$ and hence $\cot \beta \geqslant \cot \beta'$ (for $\alpha \in I$). Therefore, for $\alpha \in I$,

$$F'(\alpha) \begin{cases} <0, & \alpha < \pi/6, \\ =0, & \alpha = \pi/6, \\ >0, & \alpha > \pi/6. \end{cases}$$

This establishes the lemma.

If it happens that $\rho(S, S')$ is an integral multiple of d, then we can think of these chords together with new ones drawn between S and S' as forming an inscribed polygon with sides of length d. An immediate consequence of the preceding lemma is the following lemma.

INSCRIBED POLYGON LEMMA. *Suppose that for an integer m there is an m-gon M with sides of length d inscribed in $T(d)$ such that some vertex of M bisects an arc of $T(d)$. Then any m-gon with sides of length d inscribed in $T(d)$ has a vertex which bisects some arc of $T(d)$.*

Applications of the Inscribed Polygon Lemma. We are now in a position to determine the remaining values of d_n which we consider in this paper, i.e., d_{10}, d_{11}, d_{13}, and d_{15}. The arguments used for each of these cases are quite similar and so only those for d_{11} and d_{15} will be presented.

We first consider d_{11}. Let d denote the real number for which starting from an arc bisector of $T(d)$ the consecutive chords of length d form an inscribed 7-gon; cf. Figure 17.

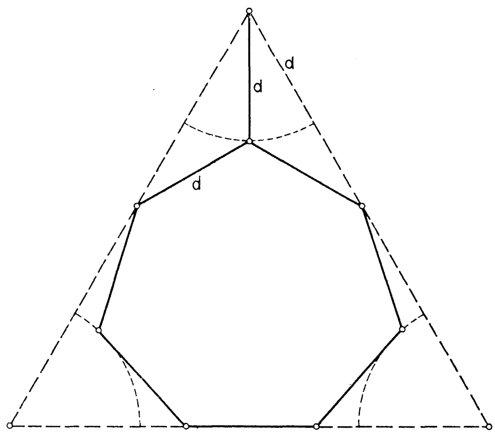

FIGURE 17

Suppose now that $d_{11} < d$. By the Inscribed Polygon Lemma, the boundary of $T(d)$ must contain points which belong to at least eight of the eleven sets into which T is partitioned. Since the vertices of T also belong to three of the sets (distinct from the sets which contain boundary points of $T(d)$), all eleven sets contain either a vertex of T or a boundary point of $T(d)$. But the centroid of T is a distance of

$$\frac{1}{2\sqrt{3}} = 0.2887\ldots > 0.2712\ldots = \frac{3}{3\sqrt{3} + \sqrt{(24 + 6\sqrt{3})}} = d$$

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from the boundary of $T(d)$ so that we reach a contradiction. Hence we must have $d_{11} \geqslant d$. In Figure 18, we give a decomposition of T into eleven sets each having diameter $\leqslant d$. This completes the proof that

$$d_{11} = \frac{3}{3\sqrt{3} + \sqrt{(24 + 6\sqrt{3})}}\,.$$

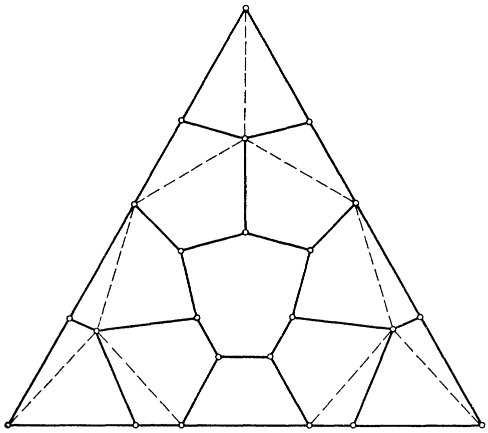

FIGURE 18

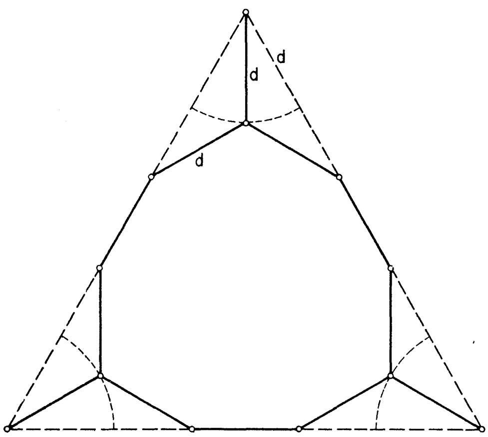

FIGURE 19

We conclude this section by showing that $d_{15} = 1/(1 + 2\sqrt{3})$. To do this, we now let d denote the unique length such that consecutive chords starting from an arc bisector of $T(d)$ form an inscribed 9-gon; cf. Figure 19.

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Assume that we can partition T into 15 sets each having diameter $<d$. Consider the three points A, B, C shown in Figure 20. The points A, B, C are

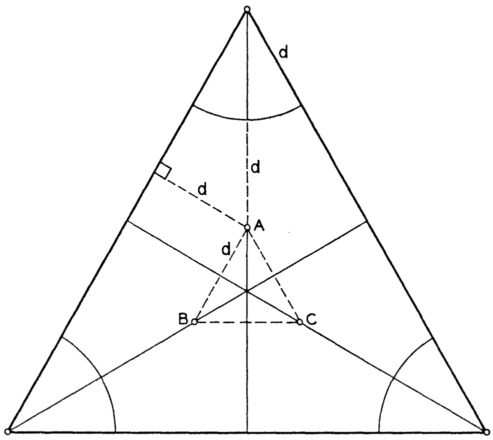

FIGURE 20

located on the altitudes of T at a distance $2d$ from the corresponding vertices. An elementary computation shows that

$$\rho(A, B) = \rho(B, C) = \rho(C, A) = d$$

and that these points are a distance d from the boundary of $T(d)$. By the Inscribed Polygon Lemma, the boundary of $T(d)$ must contain points which belong to at least 10 of the 15 sets. Hence the boundary of $T(d)$ together with the three vertices of T contain points which belong to at least 13 of the 15 sets. This means that the points A, B, C belong to at most two of the 15 sets, which is impossible. Thus, we must have $d_{15} \geqslant d$. Figure 21 gives an elegant partition of T into 15 sets each having diameter $d_{15} = 1/(1 + 2\sqrt{3})$.

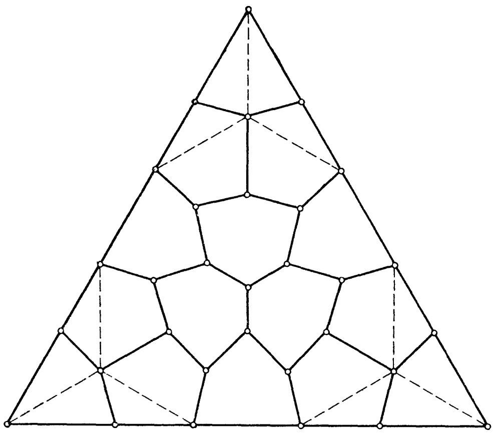

FIGURE 21

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We point out in passing that the omitted arguments for d_{10} and d_{13} correspond to similar arguments using inscribed 6-gons and 8-gons respectively.

Concluding remarks. It may be noted that as n increases, more of the sets in optimal decompositions of T are "interior" sets, i.e., contain no boundary points of T. The presence of these sets makes it much more difficult to determine d_n by the techniques demonstrated thus far. An extension of these ideas can be used to obtain d_{14} but the argument is somewhat involved and will not be presented here.

It might be natural to conjecture that as n increases, the sets of an optimal decomposition of T tend to look more like regular hexagons of diameter d_n. This would give

$$d_n \sim 2/\sqrt{(6n)}.$$

However, this conclusion is unwarranted since one notes that there are a continuum of ways to partition the plane into two classes of congruent sets of diameter 1 that their average area is $\frac{1}{2}\sqrt{6}$. An example is shown in Figure 22 (the dashed lines correspond to unit diameters).

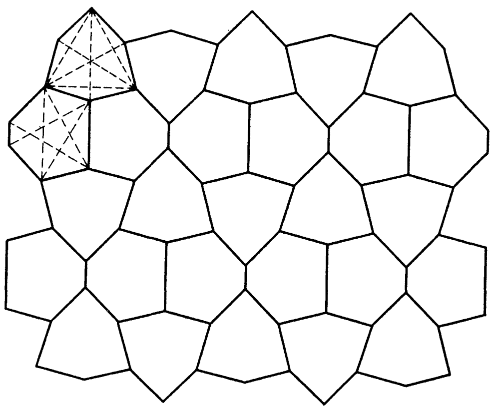

FIGURE 22

In Figure 23, we compare d_n with $2/\sqrt{(6n)}$. It is tempting to conjecture that $d_n > 2/\sqrt{(6n)}$. If it were true that $d_n \sim 2/\sqrt{6n})$, then it would follow that $d_n \geqslant 2/\sqrt{(6n)}$ for all n. On the other hand, it is easy to show (by covering T with regular hexagons) that

$$d_n \leqslant \frac{4}{\sqrt{(24n+3)} - 3\sqrt{3}}$$

for all n.† It follows from a theorem of Blaschke **(1)** that there exists a partition of T into n sets each of diameter $\leqslant d_n$. Hence, for all $\epsilon > 0$, if n is sufficiently large, then $d_n - 2/\sqrt{(6n)} < \epsilon$. Another question which arises is that of the

†Results of Lenz **(3, 4)** show that $\sqrt{[(\sqrt{3})/n\pi]} < d_n < (\sqrt{2})/[\sqrt{n}]$ for $n \geqslant 2$.

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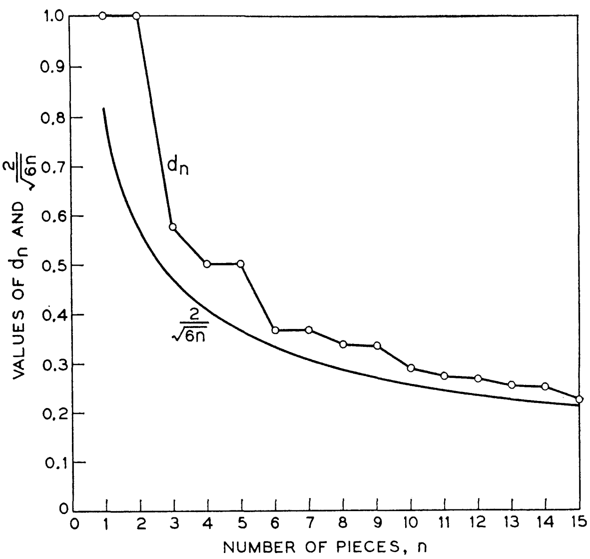

FIGURE 23

TABLE I

n	d_n
1	1
2	1
3	$1/\surd 3$
4	$\frac{1}{2}$
5	$\frac{1}{2}$
6	$1/(1+\surd 3)$
7	$1/(1+\surd 3)$
8	$2/\{1+\surd 3+\surd(6\surd 3)\}$
9	$\frac{1}{3}$
10	$1/2\surd 3$
11	$1/\{3\surd 3+\surd(24+6\surd 3)\}$
12	$1/(2+\surd 3)$
13	$3/\{3+3\surd 3+\surd(24-6\surd 3)\}$
14	—
15	$1/(1+2\surd 3)$

eventual strict monotonicity of d_n. We note from Table I that $d_n = d_{n+1}$ for $n = 1, 4$, and 6. Is this phenomenon typical?

We mention in conclusion that the preceding ideas can obviously be applied to regions other than equilateral triangles, although as far as the author knows, the extent to which this has been done is limited; cf. **(25, 6)**.

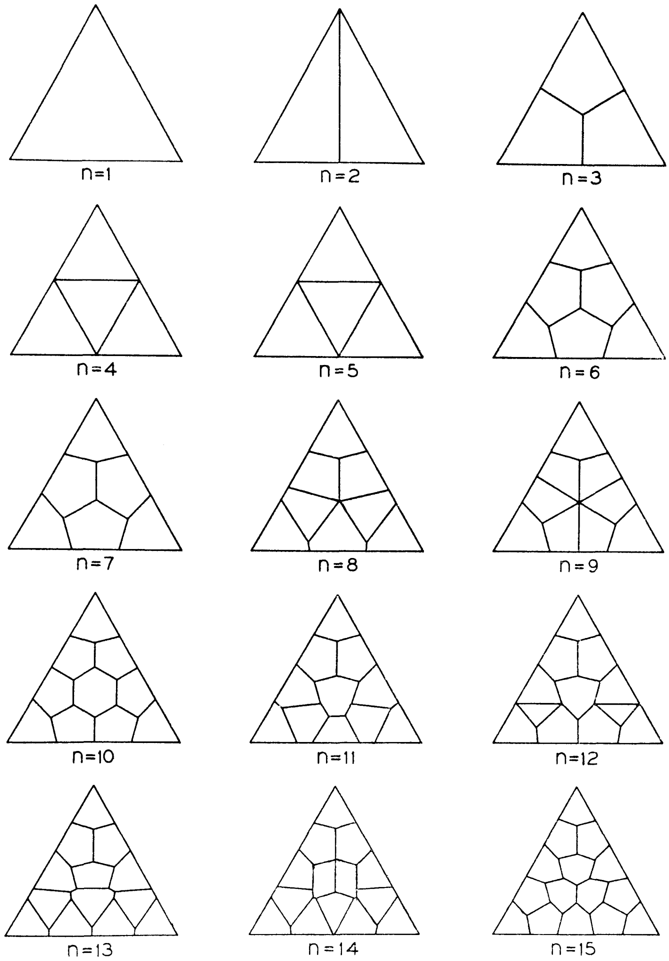

FIGURE 24. Optimal partitions of T.

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