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PACKING AND COVERING OF THE COMPLETE GRAPH WITH 4-CYCLES*

BY

J. SCHÖNHEIM AND A. BIALOSTOCKI

ABSTRACT. The maximal number of pairwise edge disjoint 4-cycles in the complete graph K_n and the minimal number of 4-cycles whose union is K_n are determined.

1. **Introduction.** In a survey [1] Beineke defined the general covering, respectively packing problem as follows:

The general covering (packing) problem in graph theory asks for the minimum (maximum) number of graphs with a property P, having as their union (being edge disjoint subgraphs of) a given graph G.

Solutions of these problems are known only for a few properties P, when G is arbitrary. In most cases G is supposed to be the complete graph K_n or the complete bipartite graph $K_{m,n}$. In particular Chartrand, Geller and Hedetniemi [3] solved the covering and the packing problem of the complete graph with cycles, not necessarily of equal length. Their result follows from the case when the cycles are of length 3. Then, both problems have been solved in a non-graph theoretical context, rather connected with Steiner triple systems by Fort and Hedlund [4] for the covering and by Schönheim [5] for the packing.

For the restricted problem, when a partition of the edges of K_n is possible a solution has been given if G is K_4 or K_5 by Hanani [6], if G is a 4.2^m-cycle, $m \geq 0$, by Kotzig [7] and if G is a p-cycle, p an odd prime, by Rosa [8].

The purpose of this paper is to give a complete solution to the problem of covering and packing the complete graph with 4-cycles.

In contrast to the 3-cycle case the solution does not derive from the corresponding combinatorial problem.

In the proof we will make use of Theorem 0 which is a special case of the following theorem of Beineke [2], cited also in [1]:

THEOREM. *The maximum number of $K_{r,r}$'s in a packing of $K_{m,n}$ is*

$$\min\left(\left[\frac{m}{r}\left[\frac{n}{r}\right]\right], \left[\left[\frac{m}{r}\right]\frac{n}{r}\right]\right).$$

THEOREM 0. *The complete bipartite graph $K_{m,n}$ is the union of edge disjoint 4-cycles if and only if $m \equiv n \equiv 0 (\bmod 2)$. Then the number of 4-cycles is $mn/4$.*

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As usual $[x]$ will denote the largest integer not exceeding x and $\{x\}$ the least integer not less than x.

2. **Results.** The results of the above mentioned problems are given in the following three theorems, the first being a special case of the following theorem of Kotzig [7]:

If $n\equiv 1 \pmod{8k}$ then there is a partition of the edges of K_n into $4k$-cycles, the condition being also necessary if k is a power of 2.

THEOREM 1. (*Exact covering and packing*). *The complete graph having n vertices, $n>1$, is the union of edge-disjoint 4-cycles if and only if $n\equiv 1 \pmod 8$.*

THEOREM 2. (*Packing*).

(i) *The maximum number of edge disjoint 4-cycles which are subgraphs of the complete graph having n vertices is*

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right] \quad \text{if} \quad n \not\equiv 5 \text{ or } 7 \pmod 8 \tag{1}$$

and

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]-1 \quad \textit{otherwise} \tag{2}$$

(ii) *Moreover, these 4-cycles may be chosen so that the non-packed edges form a one-factor if m is even, whereas they form a 3-cycle if $n\equiv 3 \pmod 8$, two 3-cycles having a common vertex if $n\equiv 5 \pmod 8$ and a 5-cycle if $n\equiv 7 \pmod 8$.*

THEOREM 3. (*Covering*). *The minimum number of 4-cycles whose union is the complete graph having n vertices is*

$$\left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\} \quad \textit{if} \quad n \not\equiv 3 \pmod 8 \tag{3}$$

and

$$\left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}+1 \quad \textit{otherwise.} \tag{4}$$

REMARK. Clearly, in the case of Theorem 1 the number of covering 4-cycles equals the number of packing 4-cycles and is $n(n-1)/8$.

3. **Proofs.** Vertices of a graph will be denoted by $a, b, c, \ldots$ or $1, 2, 3, \ldots$ and accordingly a 4-cycle will be denoted by $abcd$ or 1234, thus $abcd$ will mean the set $\{ab, bc, cd, da\}$ of four undirected edges.

Proof of theorem 0. Let A be a set of n vertices and B a set of m further vertices. The complete bipartite graph in which every vertex of A is connected by an edge to every vertex of B will be denoted by $K(A, B)$.

The necessity follows simply from the fact that any vertex of A has to be adjacent to two vertices of B in every 4-cycle in which x occurs and to every vertex of B

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exactly once. So $m\equiv 0(\mathrm{mod}\ 2)$. The same argument starting with a vertex of B leads to $n\equiv 0(\mathrm{mod}\ 2)$.

For the sufficiency consider a partition of the vertices of A into $n/2$ disjoint pairs $\{a_{2i-1}, a_{2i}\}_{j=1}^{n/2}$ and a similar partition $\{b_{2j-1}, b_{2j}\}_{i=1}^{m/2}$ of the vertices of B. Then $K(A, B)$ is the union of the $mn/4$ 4-cycles:

$$\{a_{2i-1}b_{2j-1}a_{2i}b_{2j}\}_{i=1}^{n/2}\ {}_{j=1}^{m/2}$$

which are edge disjoint.

Proof of theorem 1. The necessity follows from two simple arguments. First the total number $m(n-1)/2$ of edges has to be partitioned into 4-cycles, hence $m(n-1)/8$ has to be an integer i.e. $n(n-1)\equiv 0(\mathrm{mod}\ 8)$. On the other hand every vertex has to be adjacent to two vertices in every 4-cycle, hence $n\equiv 1(\mathrm{mod}\ 2)$ and therefore $n\equiv 1(\mathrm{mod}\ 8)$ is necessary.

The sufficiency will be proved by induction. For $n=9$, the following nine 4-cycles are edge disjoint and their union is K_9: 1263, 2374, 3485, 4596, 5617, 6728, 7839, 8941, 9152.

This can be seen either directly, either observing first that the four differences mod 9 taken between neighbour digits in the first cycle are all different and no two have the sum 0 (mod 9) and secondly that the other cycles are obtained from the first by cyclic shifts mod 9.

Thus, in order to start induction suppose $n=8m+1$, $m\geq 2$. Consider K_n and let x be a fixed vertex. Split the remaining vertices into two sets A and B, A containing eight arbitrary vertices and B the remaining $8(m-1)$ vertices. The complete graphs $A\cup x$ and $B\cup x$ are each the union of edge disjoint 4-cycles, the first by the initial value and the second by the hypothesis of the induction, whereas the bipartite graph $K(A, B)$ is also the union of edge disjoint 4-cycles, by Theorem 0. So every edge of K_n occurs in exactly one of the above 4-cycles.

Proof of theorem 2. Clearly there are no more packing 4-cycles then $[n/4[(n-1)/2]]$ since each vertex occurs in at most $[(n-1)/2]$ 4-cycles. This proves that (1) is an upper bound. This bound can be improved if $n\equiv 5$ or 7 (mod 8). Indeed, suppose first, for $n=8m+5$, that the above bound is attained. Then there are $8m^2+qm+2$ packing 4-cycles containing $32m^2+36m+8$ edges, i.e. all edges but two. This is impossible, because, consider a vertex v occurring only once in the two non-packed edges. Then all other edges containing v should be packed, a contradiction; since their number is odd, namely $8m+3$.

If $n=8m+7$, the assumption that bound (1) is attained leads to the same contradiction. This follows from the same argument as before, the number of packed edges being in this case $32m^2+52m+20$, i.e. all edges but one are packed.

Herewith (1) and (2) are established as upper bounds.

In order to show that these bounds are always obtained and in order to prove part (ii) of the theorem, suppose first m is even, thus $n=8k+2j$, $j=0, 1, 2, 3$. We

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have to prove that there are, in K_n, $8k^2+2(2j-1)k+(j(j-1))/2$ edge disjoint 4 cycles, as claimed in (i), such that the remaining edges form a one-factor.

The following packings correspond to the initial values of the induction we will use, namely for $n=2, 4, 6, 8$ respectively:

$$\phi, \{1234\}, \{1234, 5162, 5364\}$$

$$\{1234, 5678, 1526, 1728, 3546, 3748\} \tag{5}$$

the non-packed edges being respectively:

$$\{12\}, \{13, 24\}, \{13, 24, 56\}, \{13, 24, 57, 68\}.$$

Suppose now $n=8k+2j\geq 10$. Split the vertices of K_n into two sets A and B, A containing eight vertices and B the other $8(k-1)+2j$. By (5) and by the induction hypothesis, respectively, the complete graphs A and B may be packed in six, respectively

$$8(k-1)^2+2(2j-1)(k-1)+\frac{j(j-1)}{2},$$

4-cycles, whereas all the edges of the complete bipartite graph $K(A, B)$ may be packed, by theorem 0, in $4(4(k-1)+j)$ 4-cycles. So the only non-packed edges of K_n are in A or in B and form a one factor of K_n, while the number of packed edges sums up to $8k^2+2(2j-1)k+(j(j-1))/2$ and Theorem 2 is true for even n.

Next suppose n is odd. We will deal with the four possible cases separately, although the method used is essentially the same.

$n=8k+1$. In this case all edges are packed, by Theorem 1, the number of 4-cycles being $n(n-1)/8$, and the statement is true.

$n=8k+3$. Denote by x, a, b three fixed vertices and by S the remaining $8k$ vertices. The edges of the complete graph $S \cup x$ may be packed by Theorem 1 in $k(8k+1)$ 4-cycles, whereas the edges of the complete bipartite graph $K(S; \{a, b\})$ may be packed, by Theorem 0, in $4k$ 4-cycles. So the total number of packing 4-cycles is $8k^2+5k$, and the only non-packed edges are ab, bx, ax as claimed in (2) and in part (ii) of the theorem.

$n=8k+5$. Denote five fixed vertices by x, a, b, c, d and let S be the set of the remaining $8k$ vertices. The edges of the complete graph $S \cup x$ may be packed in $k(8k+1)$ 4-cycles, by Theorem 1. These 4-cycles together with $abcd$ and further $8k$ 4-cycles, packing the complete bipartite graph $K(S; \{a, b, c, d\}$, give $8k^2+9k+1$ packing 4-cycles. The non-packed edges are xa, xb, xc, xd, ab, cd forming two 3-cycles with common vertex x, as claimed in (2) and part (ii).

$n=8k+7$. Denote seven fixed vertices by x, 2, 3, 4, 5, 6, 7 and let S be the set of the remaining $8k$ vertices. The edges of $S \cup x$ may be packed in $k(8k+1)$ 4-cycles by Theorem 1. These 4-cycles, together with $x245$, 3462, $56x7$, $73x4$ and with further $12k$ 4-cycles, packing the edges of $K(S; \{2, 3, 4, 5, 6, 7\})$ give $8k^2+13k+4$ packing 4-cycles, the non-packed edges 25, 53, 36, 67, 72 forming a 5-cycle as stated.

This completes the proof of Theorem 2.

Proof of theorem 3. Clearly the covering is not possible with less than $\{(n/4)\{(n-1)/2\}\}$ 4-cycles, since each vertex occurs in at least $\{(n-1)/2\}$ 4-cycles. This proves that (3) is a lower bound. It can be improved if $n\equiv 3 \pmod 8$. For, suppose $n=8k+3$. Then, if the bound (3) would be attained then all edges would be covered by $8k^2+5k+1$ 4-cycles containing $32k^2+20k+5$ edges. This is only one edge more than all the edges, so only one edge can occur twice. But this is impossible since, if ab is this particular edge, then the vertex a is adjacent to b twice and to all other vertices once so totally to an odd number of vertices, a contradiction. This establishes that (3) and (4) are lower bounds.

In order to show that bounds (3) and (4) are always attained, consider first the case when n is even, say $n=8k+2j$, $j=0, 1, 2, 3$. Then by Theorem 2 there is a packing containing $[(n/4)[(n-1)/2]]$ 4-cycles and the only non-packed edges form a one-factor of $4k+j$ edges. These can be covered, clearly, with $2k+\{j/2\}$ 4-cycles. In this way the number of 4-cycles covering all edges is

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]+2k+\left\{\frac{j}{2}\right\}$$

which sums up to precisely

$$\left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}$$

as stated.

When n is odd, consider all possible cases as follows:

$n=8k+1$. The statement is true by theorem 1.

$n=8k+3$. Let a, b, c be three fixed vertices. The packing given in the proof of Theorem 2 covers all edges but ab, bc, ac. These three edges may be covered with two additional 4-cycles say $abcx$ and $acxy$, so all edges are covered with

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]+2 \quad \text{i.e.} \quad \left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}+1$$

4-cycles as stated.

$n=8k+5$. Let x, a, b, c, d be five fixed vertices. The packing given in proof of Theorem 2 covers all edges but xa, xb, ab, xc, xd, cd. These six edges may be covered with two additional 4-cycles, namely with $cxab$ and $cdxb$. So all edges are covered with

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]-1+2 \quad \text{i.e.} \quad \left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}$$

4-cycles as stated.

$n=8k+7$. Let x, 2, 3, 4, 5, 6, 7 be seven fixed edges. The packing mentioned in proof of Theorem 2 covers all edges but the edges 25, 53, 36, 67, 72. These five edges may be covered with two 4-cycles e.g. 2536 and $672x$. So all edges are covered by

$$\left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]-1+2 \quad \text{i.e.} \quad \left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}$$

4-cycles as stated.

This proves Theorem 3.

4. **Final Remark.**

The results of this paper establish two numbers:

$$M(4, n) = \begin{cases} \left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right] & \text{if } n \not\equiv 5 \text{ or } 7 \pmod 8 \\ \left[\frac{n}{4}\left[\frac{n-1}{2}\right]\right]-1 & \text{otherwise} \end{cases}$$

$$m(4, n) = \begin{cases} \left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\} & \text{if } n \not\equiv 3 \pmod 8 \\ \left\{\frac{n}{4}\left\{\frac{n-1}{2}\right\}\right\}+1 & \text{otherwise} \end{cases}$$

where $M(k, n)$ and $m(k, n)$ are respectively the k-cycle packing number and the k-cycle covering number of the complete graph K_n.

From this point of view the results in [5] and in [4] are the numbers $M(3; n)$ and $m(3; n)$.

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